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SUBMAXIMAL IDEALS OF WEAK IDEMPOTENT RINGS

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Abstract. We study the structure of submaximal ideals in weak idempotent rings. We establish certain relations among semiprime, primary, radical and submaximal ideals. We prove that every submaximal ideal of a commutative WIR with unity is either semiprime or primary. Also we obtain that the product of two submaximal ideals is not submaximal and the product of two maximal ideals is submaximal. Further we prove that the intersection of all submaximal ideals of the product ring of two WIRs is the nilradical. Finally we prove that the inverse image of a submaximal ideal is submaximal under epimorphism and also $S^{-1}R$ is a field if and only if $R \cong \mathbb{Z}_2$.

1. Introduction

It was A.L. Foster [**1**] who introduced the notion of Boolean like ring as a generalization of Boolean ring. A Boolean like ring (BLR, for short) is a commutative ring with unity of characterstic 2 and $ab(1+a)(1+b) = 0$ for all $a,\ b$ in the ring. V. Swaminathan [**3, 4, 5**] made an extensive study on the class of Boolean like rings. K. Venkateswarlu, D. Wasihun, T. Abebaw and Y. Yitayew [**6**] introduced the notion of a weak idempotent ring (WIR, for short) as a generalization of Boolean like ring which is a ring with characteristic 2 and $a^4 = a^2$ for every element a in the ring. A study was made on certain characterizations of WIR and their ideals namely completely prime and left (right) completely primary ideals. Also obtained that if WIR is commutative ring, completely prime ideal is prime and left (right) completely primary ideal is primary. K. Venkateswarlu, D. Wasihun and T. Abebaw [**7**], made a further study on the structure of weak idempotent rings. An

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ideal of a ring is called semiprime if its radical is the ideal itself and an ideal I of a WIR R is called submaximal if I is covered by a maximal ideal of R i.e. there exists a maximal ideal M of R such that $I \subsetneq M$ and for any ideal J of R such that $I \subseteq J \subseteq M$, then $J = I$ or $J = M$. This paper is focused to investigate a further study on submaximal ideals of weak idempotent rings which is in continuation to the paper [**7**].

2. Preliminaries

We recall certain results concerning WIRs and their properties from [**6**] and [**7**].

REMARK 2.1. In a WIR R, for all $a \in R$,

(1) $a^n = a,\ a^2$ or a^3 for any positive integer n.
(2) If $0 \neq a$ is a nilpotent element, then $a^2 = 0$.
(3) $a = a^2 + (a^2 + a)$, where a^2 is an idempotent and $a^2 + a$ is a nilpotent.

REMARK 2.2. 0 is the only both nilpotent and idempotent element and every element a of R is the sum of a nilpotent and an idempotent elements. But the fact is that the representation is not unique as in BLRs.

REMARK 2.3. If the ring is a commutative WIR, then the representation of each element as the sum of an idempotent and a nilpotent element is unique and we use the notation a_B for the idempotent a^2 and a_N for the nilpotent $a + a^2$ of the unique representation.

THEOREM 2.1. *Every non-zero non-unit element in a WIR with unity is a zero-divisor.*

Notation: For any ring R, R_B stands for the set of all idempotent elements and N stands for the set of all nilpotent elements.

THEOREM 2.2. *Let R be a commutative WIR. Then R_B is isomorphic with R/N.*

THEOREM 2.3. *Every completely prime ideal of a WIR with unity is a maximal ideal.*

THEOREM 2.4. *In a WIR R with unity, a left completely primary ideal I is completely prime if and only if the nilradical of R is a subset of I.*

THEOREM 2.5. *In a commutative WIR R with unity, let I be an ideal of R and $x \in R$ such that $x \notin I$.*

(1) *If $x_B \notin I$, then there exists a maximal ideal J of R such that $I \subseteq J$ and $x \notin J$.*
(2) *If $x_N \notin I$, then there exists a primary ideal P of R such that $I \subseteq P$ and $x \notin P$.*

THEOREM 2.6. *Let I be an ideal of a commutative WIR R with unity. Then the following statements are equivalent.*

1. *I is semiprime*
2. *The nilradical N of R is contained in I*
3. *R/I is a Boolean ring*

THEOREM 2.7. *Let I be an ideal of a commutative WIR R with unity. Then I is contained in at least two maximal ideals of R if and only if I is not primary.*

THEOREM 2.8. *The intersection of any two distinct maximal ideals of a commutative WIR with unity is submaximal and it is covered by both of the maximal ideals. Further, there exists no other maximal ideal containing it.*

REMARK 2.4. In the ring $\mathbb{Z}_2 \times \mathbb{Z}_2$, the ideals $M_1 = \{(0,0),(0,1)\}$ and $M_2 = \{(0,0),(1,0)\}$ are the only maximal ideals of the ring.

LEMMA 2.1. *Every four element Boolean ring R has exactly two maximal ideals.*

PROOF. In [**2**] [Theorem 39], every finite Boolean ring with 2^k elements is isomorphic to the direct sum of k copies of the field $\mathbb{Z}_2$. Hence $R \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. By Remark 2.4, R has exactly two maximal ideals. □

3. Submaximal ideals

We begin with the following

THEOREM 3.1. *Every maximal ideal that contains a submaximal ideal I of a commutative WIR R with unity is a cover of I.*

PROOF. Let I be a submaximal ideal of R. Then there exists a maximal ideal J that covers I. Suppose J' is a maximal ideal that contains I. Suppose $J \neq J'$. Then, by Theorem 2.8, $J \cap J'$ is a submaximal ideal of R and covered by only J and J'. Since I is contained by both J and J', $I \subseteq J \cap J'$ and hence $I = J \cap J'$. Thus, J' covers I. □

THEOREM 3.2. *Every submaximal ideal I of a commutative WIR R with unity is covered by at most two maximal ideals.*

PROOF. Let I be a submaximal ideal of R and a subset of two distinct maximal ideals of R. Since I is a submaximal ideal, there exists a maximal ideal J of R such that I is covered by J. By the assumption, there exists a maximal ideal J_1 of R such that $J_1 \neq J$ and $I \subset J_1$ that is $I \subseteq J_1 \cap J \subseteq J$. As I is covered by J, either $J_1 \cap J = I$ or $J_1 \cap J = J$. If $J_1 \cap J = J$, then we get $J = J_1$ since both ideals are maximal. Thus $J_1 \cap J \neq J$ and hence $J_1 \cap J = I$. By Theorem 2.8, I is covered by J_1 and J and they are the only maximal ideals of R containing I. □

THEOREM 3.3. *A submaximal ideal I of a commutative WIR R with unity is semiprime if and only if it is covered by two distinct maximal ideals.*

PROOF. Let I be semiprime. Assume I is covered by only one maximal ideal J. Thus, R/I is a local WIR and so I is primary. In this case $N \nsubseteq I$ since $N \subseteq I$ implies that I is maximal by Theorem 2.4. By Theorem 2.6, I is not semiprime that contradicts the assumption. Hence, I is covered by two distinct maximal ideals. Conversely, suppose I is covered by two distinct maximal ideals. Then I is the intersection of the two maximal ideals. Since the ring R is commutative, the nilradical N of R is contained by I. By Theorem 2.6, I is semiprime. □

Corollary 3.1. *Every submaximal ideal of a commutative WIR R with unity covered by a unique maximal ideal is primary.*

Proof. Suppose a submaximal ideal I is covered by a unique maximal ideal M. Then R/I is local. Hence, by Theorem 2.7, I is primary. □

Remark 3.1. In [**5**], V. Swaminathan has proved that there exists a primary ideal P of BLR which is a maximal ideal in the poset of all ideals of the ring not containing a particular nilpotent element n and for every other nilpotent $n_1 \notin P$, $n + n_1 \in P$ and also P is a submaximal ideal of R. In [**6**], it was also proved that there exists a primary ideal P of the ring R which is a maximal ideal in the poset of all ideals of R not containing a particular nilpotent element. However for every other nilpotent element $n_1 \notin P$, $n + n_1 \in P$ does not hold and also P need not be submaximal.

We clarify the above remark in the following

Example 3.1. Consider the Quaternion ring R over the field $\mathbb{Z}_2$. Clearly R is a commutative WIR with unity and $N = \{0, 1+i, 1+j, 1+k, i+j, i+k, j+k, 1+i+j+k\}$. Further $Q = \{0\}$, $Q_1 = \{0, 1+i+j+k\}$, $Q_2 = \{0, 1+i, j+k, 1+i+j+k\}$, $Q_3 = \{0, 1+j, i+k, 1+i+j+k\}$, $Q_4 = \{0, 1+k, i+j, 1+i+j+k\}$ and N are all possible proper ideals of R and also they are primary. It can be seen that Q is the maximal primary ideal that does not contain a nilpotent element $1+i+j+k$. Also observe that $(1+i)+(1+i+j+k) = j+k \notin Q$ and also it is not submaximal ideal of R.

Theorem 3.4. *If all submaximal ideals of a commutative WIR R with unity are not semiprime, then R is local.*

Proof. Let all submaximal ideals of R be not semiprime. Assume that R is not local. Hence R has at least two distinct maximal ideals say J_1 and J_2. By Theorem 2.8, $J_1 \cap J_2$ is a submaximal ideal of R. By Theorem 3.3, $J_1 \cap J_2$ is semiprime which is a contradiction to the assumption. Hence, R is local. □

Theorem 3.5. *If the submaximal ideal of a commutative WIR R with unity is not semiprime, then its radical is a maximal ideal.*

Proof. Let submaximal ideal J of R be not semiprime. Then, by Theorem 3.3, there exists only one maximal ideal M that covers J. Let $a \in R$ and $a \notin M$. Assume that $a^n \in J$ for some $n \in \mathbb{N}$. Then $a \in r(J)$ and $a^n \in M$. Since every maximal ideal of R is prime, hence $a \in M$ which is a contradiction. Thus, $a^n \notin J$ for all $n \in \mathbb{N}$. Hence, $r(J) \neq R$. Since J is not semiprime, we have $r(J) = M$ □

Corollary 3.2. *The nilradical N of a commutative WIR R with unity is a submaximal ideal if and only if R has exactly four idempotent elements.*

Proof. Suppose R has exactly four idempotent elements. By Theorem 2.2 , $R/N \cong R_B$. By Lemma 2.1, R/N has exactly two maximal ideals. Thus N is contained by exactly two maximal ideals and hence N is the intersection of two maximal ideals. By Theorem 2.8, N is a submaximal ideal of R. Conversely, suppose N is a submaximal ideal of R. Since N is semiprime, by Theorem 3.3, N is covered by two distinct maximal ideals. Thus, R/N is four element Boolean ring. Therefore, R has exactly four idempotent elements. □

4. Product of submaximal ideals

Throughout this section, R and R' are commutative WIRs with unity.

Theorem 4.1. *The following holds*

1. *If S and S' are submaximal ideals of R and R', respectively, then $S \times S'$ is not a submaximal ideal of $R \times R'$.*
2. *If M and M' are maximal ideals of R and R', respectively, then $M \times M'$ is a submaximal ideal of $R \times R'$.*

Proof. (1) Let S be a submaximal ideal of R and S' be a submaximal ideal of R' . Then there exist maximal ideals M and M' of R and R' that cover S and S', respectively. Since $S \times S' \subsetneq M \times M' \subsetneq M \times R' \subsetneq R \times R'$, $S \times S'$ is not a submaximal ideal of $R \times R'$.

(2) $M \times M' \subsetneq M \times R' \subsetneq R \times R'$ and $M \times R'$ is a maximal ideal of $R \times R'$. If $J \times J'$ is an ideal of $R \times R'$ and $M \times M' \subseteq J \times J' \subseteq M \times R'$, then J is an ideal of R and J' is an ideal of R' and $M \subseteq J \subseteq M$ and $M' \subseteq J' \subseteq R'$. Thus, $M = J$ and $M' = J'$ or $(J' = R')$. Hence $M \times M' = J \times J'$ or $J \times J' = M \times R'$. Therefore, $M \times M'$ is a submaximal ideal of $R \times R'$. □

Theorem 4.2. *If J is a submaximal ideal of $R \times R'$, then*

$$P_1(J) = \{a \in R/(a, a') \in J \textit{ for some } a' \in R'\}$$

and

$$P_2(J) = \{a' \in R'/(a, a') \in J \textit{ for some } a \in R\}$$

are maximal ideals of R and R', respectively.

Proof. Let M be an ideal of R such that $P_1(J) \subseteq M \subseteq R$. Suppose $P_1(J) \neq M$ and $M \neq R$. Then $P_1(J) \times P_2(J) \subsetneq M \times P_2(J) \subsetneq R \times P_2(J) \subsetneq R \times R'$. But $J \subseteq P_1(J) \times P_2(J)$ and this contradicts the submaximality of J. Thus either $P_1(J) = M$ or $M = R$. Hence $P_1(J)$ is a maximal ideal of R and similarly $P_2(J)$ is a maximal ideal of R'. □

Theorem 4.3. *Every submaximal ideal of $R \times R'$ is semiprime.*

Proof. Let J be a submaximal ideal of $R \times R'$. By Theorem 4.2, $P_1(J)$ and $P_2(J)$ are maximal ideals of R and R', respectively. Thus $J \subsetneq R \times P_2(J)$ and $J \subsetneq P_1(J) \times R'$ that is J is covered by two distinct maximal ideals of $R \times R'$. By Theorem 3.3, J is semiprime. □

Theorem 4.4. *The intersection of all submaximal ideals of $R \times R'$ is the nilradical.*

Proof. Let $\{J_j\}$ be the set of all submaximal ideals, $\{M_i\}$ be the set of all maximal ideals and K be the nilradical of $R \times R'$. By Theorem 2.3, every prime ideal of $R \times R'$ is maximal and hence $\cap M_i$ is the nilradical of $R \times R'$. By Theorem 4.3, $\{J_j\}$ is semiprime for all j. Hence $K \subseteq J_j$ for all j by Theorem 2.6. Thus $\cap M_i = K \subseteq \cap J_j \subseteq \cap M_i$. Hence, $K = \cap J_j$. □

5. Further results on submaximal ideals

Let us consider the ring $\mathbb{Z}_4$. Clearly it is not WIR but it has a subring namely $S = \{0, 2\}$ having the property that $a^4 = a^2$ and $a + a = 0$ for every $a \in S$. This leads us to define weak idempotent subring in any arbitrary ring .

Definition 5.1. A subring S of an arbitrary ring R is a weak idempotent subring if S is of characteristic 2 and satisfies $a^4 = a^2$ for every $a \in S$.

Theorem 5.1. *Let R be a commutative WIR with unity, R' be any arbitrary ring (not necessarily be WIR) and $f : R \to R'$ be a homomorphism. Then $f(R)$ is a weak idempotent subring of R'.*

Proof. Clearly $f(R)$ is a subring of R'. Let $b \in f(R)$. Then for some $a \in R,\ f(a) = b$. Thus

$$b + b = f(a) + f(a) = f(a + a) = f(0) = 0 \text{ and}$$
$$b^4 = f^4(a) = f(a^4) = f(a^2) = f^2(a) = b^2.$$

Hence $f(R)$ is a weak idempotent subring of R'. □

Theorem 5.2. *Let R and R' be commutative WIRs with unity and $f : R \to R'$ be an epimorphism. Then*

(1) *If J is a submaximal ideal of R and $ker(f) \subseteq J$, then $f(J)$ is a submaximal ideal of R'.*
(2) *If J' is a submaximal ideal of R', then $f^{-1}(J')$ is a submaximal ideal of R.*

Proof. (1) Let J be a submaximal ideal of R and $ker(f) \subseteq J$. Then there exists a maximal ideal M such that $J \subsetneq M \subsetneq R$. Let $a \in M \setminus J$. Then $f(a) \in f(M)$. Suppose $f(a) \in f(J)$. This implies that there exists $b \in J$ such that $f(a) = f(b)$. Thus, $f(a - b) = 0$ and $a - b \in J$. Since $b \in J$, $a \in J$ which is a contradiction. Hence $f(a) \notin f(J)$ that is $f(J) \subsetneq f(M)$ and similarly $f(M) \subsetneq R'$. Let S be an ideal of R' and $f(J) \subseteq S \subseteq f(M)$. Assume $f(J) \neq S$ and $S \neq f(M)$. Let $s \in S \setminus f(J)$. This implies that there exists $x \in R$ such that $f(x) = s$. Thus $x \in f^{-1}(S) \setminus J$ and hence $J \subsetneq f^{-1}(S)$. Similarly, $f^{-1}(S) \subsetneq M$ which is a contradiction to the submaximality of J. Thus $f(J) = S$ or $S = f(M)$.

(2) It is clear that $f^{-1}(J')$ and $f^{-1}(M')$ are ideals of R. Suppose $f^{-1}(M') \subseteq S \subseteq R$. If $S \neq R$ and $f^{-1}(M') \neq S$, by the above discussion $M' \neq f(S) \neq R'$ which is a contradiction. Thus, $f^{-1}(M')$ is a maximal ideal of R. Similarly there exists

no proper ideal between $f^{-1}(J')$ and $f^{-1}(M')$. Hence, $f^{-1}(J')$ is a submaximal ideal of R. $\square$

In a WIR R, the multiplicative subset S of R consists of elements which are not zero-divisors has only unit elements (Theorem 2.1). Thus any non-trivial ideal I of R does not have common element with the multiplicative subset S that is $S^{-1}I \neq S^{-1}R$.

THEOREM 5.3. *Let R be any arbitrary commutative ring with unity and S is the multiplicative subset of R consists of elements which are not zero-divisors including the unity 1. Then the ring R is a WIR if and only if $S^{-1}R$ is a WIR.*

PROOF. Suppose R is a WIR. For every

$$\tfrac{a}{s} \in S^{-1}R,\ \tfrac{a}{s} + \tfrac{a}{s} = \tfrac{(a+a)}{s} = \tfrac{0}{s} = 0 \text{ and } (\tfrac{a}{s})^4 = \tfrac{a^4}{s^4} = \tfrac{a^2}{s^2} = (\tfrac{a}{s})^2.$$

Hence, $S^{-1}R$ is a WIR. Conversely, suppose $S^{-1}R$ is a WIR. For every $a \in R$, $\frac{a}{s} \in S^{-1}R$. Thus, $\frac{a}{s} + \frac{a}{s} = 0$ which implies that $s_1 s(a + a) = 0$ for some $s_1 \in S$. Since $s_1 s$ is not a zero-divisor, $a + a = 0$. $\frac{a}{s} \in S^{-1}R$ for every $a \in R$ and $s \in S$. By the assumption, $(\frac{a}{s})^4 = (\frac{a}{s})^2$ which implies $\frac{a^4}{s^4} = \frac{a^2}{s^2}$. Thus, $s_1 s^2(a^4 - a^2 s^2) = 0$ for some $s_1 \in S$. Since $s_1 s^2$ is not a zero-divisor, $a^4 - a^2 s^2 = 0$. $\frac{s}{1}$ is not a zero-divisor in $S^{-1}R$ since $s \in S$. Thus $(\frac{s}{1})^2 = 1$ and hence $a^4 = a^2$. Therefore, R is a WIR. $\square$

THEOREM 5.4. *Let S be a multiplicative subset of a commutative WIR R with unity consisting of elements which are not zero-divisors. Then I is a submaximal ideal of R if and only if $S^{-1}I$ is a submaximal ideal of $S^{-1}R$.*

PROOF. Since S that consists of elements which are not zero-divisors implies it consists of units, it is known that φ_s is an isomorphism. Thus the theorem holds. $\square$

COROLLARY 5.1. *$S^{-1}R$ is a field if and only if $R \cong \mathbb{Z}_2$.*

PROOF. Suppose $S^{-1}R$ is a field. For every $(0 \neq)a \in R$, $\frac{a}{s}$ is unit. Thus $(\frac{a}{s})^2 = 1$ which implies $a^2 = 1$. Hence 0 is the only nilpotent element of R. Hence R is a Boolean ring with all non-zero elements are units. Thus $R \cong \mathbb{Z}_2$. The proof of the converse is obvious. $\square$

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