BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., Vol. **11**(1)(2021), 159-168
DOI: 10.7251/BIMVI2101159L

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

EXISTENCE RESULTS FOR INTEGRAL BOUNDARY VALUE PROBLEMS OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH TWO NONLINEAR TERMS IN BANACH SPACES

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Abstract. In this paper, we study the existence of solutions for a boundary value problem of fractional differential equations with integral boundary conditions by using measure of noncompactness combined with fixed point theorem of Mönch. An example is given to illustrate our results.

1. Introduction

Measure of non compactness combined with one of fixed point theorems, as Darbo [**16**] Sadovski [**25**], Mönch [**23**] is an important and efficacy tool in study of differential or integral equations. Kuratowski [**21**] introduced the concept of measure of noncompactness, which played an important role in fixed point theory. Gohberg [**18**] gave an other measure called Hausdorff measure later Darbo [**16**] used Kuratowski's measure of noncompactness to generalize the Schauder theorem of fixed point. After, that many authors studied and solved some problems by using measure of noncompactness in study of different kind problems, as differential equations, integral equations and integro-differential equations, see [**1, 11, 12, 19, 27**].

On other hand, fractional calculus is one of important tool to study many problems and phenomenons from fields of science and engineering, as in physics, chemistry, hydrology, biophysics, thermodynamics, blood flow problems, statistical

2010 *Mathematics Subject Classification.* Primary 34A08; Secondary 34B15.
Key words and phrases. Fractional differential equations, Riemann-Liouville fractional derivatives, measure of noncompactness, fixed point, existence, Banach space.
Communicated by Ilkay Yaslan Karaca.

mechanics and control theory. Recently, it has known a significant development in fractional differential and integral equations, for example see [**1**], [**4**]-[**9**], [**12**]-[**15**], [**17**], [**19**], [**22**], [**24**], [**26**]-[**31**]. In recent years, many authors used the technique of non compactness measure to study existence of solutions to nonlinear integral equations of order fractional and fractional differential equations, and there are some results in literary.

In this paper, we concentrate on the existence of solutions for the boundary value problem of a fractional differential equation with integral boundary conditions of the form

$$\begin{cases} D^{\alpha}x(t) - f(t, x(t)) = D^{\alpha-1}g(t, x(t)), \ t \in (0,1), \\ x(0) = 0, \ x(1) = \int_0^1 g(s, x(s))\, ds, \end{cases} \tag{1.1}$$

where D^{α} is the standard Riemann-Liouville fractional derivative of order $1 < \alpha \leqslant 2$, $f, g : [0,1] \times E \to E$ are given functions satisfying some assumptions that will be specified later, and E be a Banach space with the norm $\|.\|$. In the case $E = \mathbb{R}$, Xu and Sun in [**30**] investigated the existence and uniqueness of a positive solution of (1.1) by using the method of the upper and lower solutions and the Schauder and Banach fixed point theorems. Then, the existence results obtained here extend the main results in [**30**].

2. Preliminaries

In this section we present some basic definitions, notations and results of fractional calculus which are used throughout this paper.

Let $J = [0,1]$. By $C(J,E)$ we denote the Banach space of all continuous functions from J into E with the norm

$$\|x\|_{\infty} = \sup\{\|x(t)\| : t \in J\}.$$

Let $L^1(J,E)$ be the Banach space of measurable functions $x : J \to E$ that are Lebesgue integrable with norm

$$\|x\|_{L^1} = \int_J \|x(t)\|\, dt.$$

And $AC(J,E)$ be the space of absolutely continuous valued functions on J, and set

$$AC^n(J) = \left\{x : J \to E : x, x', x'', \quad , x^{n-1} \in C(J,E) \text{ and } x^{n-1} \in AC(J,E)\right\}.$$

Moreover, for a given set V of function $v : J \to E$, let us denote by

$$V(t) = \{v(t) : v \in V\}, \ t \in J,$$

and

$$V(J) = \{v(t) : v \in V, \ t \in J\}.$$

Definition 2.1 ([**20**]). The fractional integral of order $\alpha > 0$ of a function $x : J \to E$ is given by

$$I^{\alpha}x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\, ds,$$

provided the right side is pointwise defined on J.

Definition 2.2 ([**20**]). For a function $x \in AC^n(J)$, the Riemann-Liouville fractional order derivative of order α of x, is defined by

$$D^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} x(s)\, ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$denotes the integer part of real number α.

Lemma 2.1 ([**20**]). *The solution of linear fractional differential equation*

$$D^\alpha x(t) = 0,$$

is given by

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + c_3 t^{\alpha-3} + ... + c_n t^{\alpha-n},\ c_i \in \mathbb{R},\ i = 1, 2, ..., n,$$

where $n = [\alpha] + 1$ *and* $[\alpha]$ *denotes the integer part of real number* α.

Now let us recall some fundamental facts of the notion of Kuratowski measure of noncompactness.

Definition 2.3 ([**3, 10**]). Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\mu : \Omega_E \to [0, \infty)$ defined by

$$\mu(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^n B_i \text{ and } diam(B_i) \leqslant \epsilon\}, \text{ here } B \in \Omega_E.$$

The measure of noncompactness satisfies some important properties

(a) $\mu(B) = 0 \Leftrightarrow \overline{B}$ is compact (B is relatively compact),

(b) $\mu(B) = \mu(\overline{B})$,

(c) $A \subset B \Rightarrow \mu(A) \leqslant \mu(B)$,

(d) $\mu(A + B) \leqslant \mu(A) + \mu(B)$,

(e) $\mu(cB) = |c|\mu(B),\ c \in \mathbb{R}$,

(f) $\mu(convB) = \mu(B)$.

Here $\overline{B}$ and $convB$ denote the closure and the convex hull of the bounded set B, respectively. The details of μ and its properties can be found in [**3, 10**].

Definition 2.4. A map $f : J \times E \to E$ is said to be Caratheodory if

(i) $t \to f(t, x)$ is measurable for each $x \in E$.

(ii) $x \to f(t, x)$ is continuous for almost all $t \in J$

To prove the existence of solutions of (1.1), we need the following results.

Theorem 2.1 ([**2**]). *Let* D *be a bounded, closed and convex subset of a Banach space such that* $0 \in D$, *and let* N *be a continuous mapping of* D *into itself. If the implication*

$$V = \overline{conv} N(V) \ \text{ or } V = N(V) \cup \{0\} \Rightarrow \mu(V) = 0,$$

holds for every V *of* D, *then* N *has a fixed point.*

LEMMA 2.2 ([**28**]). *Let D be a bounded, closed and convex subset of the Banach space $C(J,E)$. Let G be a continuous function on $J \times J$ and f a function from $J \times E \to E$, which satisfies the Carathéodory conditions, and assume there exists $p \in L^1(J, \mathbb{R}^+)$ such that, for each $t \in J$ and each bounded set $B \subset E$, we have*

$$\lim_{h \to 0^+} \mu\left(f\left(J_{t,h} \times B\right)\right) \leqslant p(t)\,\mu(B)\,, \text{ here } J_{t,h} = [t-h, t] \cap J.$$

If V is an equicontinuous subset of D, then

$$\mu\left(\left\{\int_J G(s,t)\, f(s, y(s))\, ds : y \in V\right\}\right) \leqslant \int_J \|G(s,t)\|\, p(s)\, \mu(V(s))\, ds.$$

3. Existence results

Let us start by defining what we mean by a solution of the problem (1.1).

DEFINITION 3.1. A function $x \in AC^2(J,E)$ is said to be a solution of problem (1.1) if x satisfies the equation $D^\alpha x(t) - f(t, x(t)) = D^{\alpha-1} g(t, x(t))$ on J and the conditions $x(0) = 0$, $x(1) = \int_0^1 g(s, x(s))\, ds$.

For the existence of solutions for the problem (1.1), we need the following auxiliary lemma.

LEMMA 3.1. *The function x solves the problem (1.1) if and only if it is a solution of the integral equation*

$$x(t) = \int_0^1 G(t,s)\, f(s, x(s))\, ds + \int_0^t g(s, x(s))\, ds, \; t \in J,$$

where G is the Green function given by

$$G(t,s) = \begin{cases} \frac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leqslant s \leqslant t \leqslant 1, \\ \frac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \leqslant t \leqslant s \leqslant 1. \end{cases}$$

PROOF. From Lemma 2.1, applying the Riemann-Liouville fractional integral I^α on both sides of (1.1), we have

$$\begin{aligned} x(t) - c_1 t^{\alpha-1} - c_2 t^{\alpha-2} + I^\alpha f(t, x(t)) &= I^\alpha \left(I^{\alpha-1} D_{0^+}^{\alpha-1} g(t, x(t))\right) \\ &= I^\alpha \left(g(t, x(t)) - c_3 t^{\alpha-2}\right). \end{aligned}$$

That is,

$$\begin{aligned} x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))\, ds \\ + \int_0^t g(s, x(s))\, ds - \frac{c_3}{\alpha-1} t^{\alpha-1}. \end{aligned}$$

By the boundary conditions $x(0) = 0$, $x(1) = \int_0^1 g(s, x(s))\, ds$, one has $c_2 = 0$ and

$$c_1 = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} f(s, x(s))\, ds + \frac{c_3}{\alpha-1}.$$

Therefore

$$\begin{aligned} x(t) &= \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} (1-s)^{\alpha-1} f(s, x(s))\, ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))\, ds + \int_0^t g(s, x(s))\, ds \\ &= \int_0^1 G(t,s) f(s, x(s))\, ds + \int_0^t g(s, x(s))\, ds. \end{aligned}$$

This process is reversible. The proof is complete. □

In the following, we prove existence results for the boundary value problem (1.1) by using a Mönch of fixed point theorems.

The following assumptions will be used in our main results

(H1) The functions $f, g : J \times E \to E$ satisfy the Caratheodory conditions.

(H2) There exist $p_f, p_g \in L^1(J, \mathbb{R}^+) \cap C(J, \mathbb{R}^+)$ such that

$$\begin{aligned} \|f(t,x)\| &\leqslant p_f(t) \|x\|, \text{ for } t \in J \text{ and each } x \in E, \\ \|g(t,x)\| &\leqslant p_g(t) \|x\|, \text{ for } t \in J \text{ and each } x \in E. \end{aligned}$$

(H3) For each $t \in J$ and each bounded set $B \subset E$, we have

$$\begin{aligned} \lim_{h \to 0^+} \mu(f(J_{t,h} \times B)) &\leqslant p_f(t) \mu(B), \text{ here } J_{t,h} = [t-h, t] \cap J, \\ \lim_{h \to 0^+} \mu(g(J_{t,h} \times B)) &\leqslant p_g(t) \mu(B), \text{ here } J_{t,h} = [t-h, t] \cap J. \end{aligned}$$

Theorem 3.1. *Assume that the assumptions (H1)-(H3) hold. If*

$$\frac{2}{\Gamma(\alpha+1)} \|p_f\|_\infty + \|p_g\|_\infty < 1, \tag{3.1}$$

then the boundary value problem (1.1) has at least one solution.

Proof. We transform the problem (1.1) into a fixed point problem by defining an operator $N : C(J,E) \to C(J,E)$ as

$$\begin{aligned} (Nx)(t) &= \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} (1-s)^{\alpha-1} f(s, x(s))\, ds \\ &\quad - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, x(s))\, ds + \int_0^t g(s, x(s))\, ds, \end{aligned}$$

Clearly, the fixed points of operator N are solutions of the problem (1.1). Let $R > 0$ and consider the set

$$D_R = \{x \in C(J,E) : \|x\|_\infty \leqslant R\}.$$

Clearly, the subset D_R is closed, bounded, and convex. We will show that N satisfies the assumptions of Theorem 2.1. The proof will be given in three steps.

Step 1. N maps D_R into itself.

For each $x \in D_R$, by (H2) and (3.1) we have for each $t \in J$

$$\begin{aligned}
&\|(Nx)(t)\| \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} (1-s)^{\alpha-1} \|f(s, x(s))\| \, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x(s))\| \, ds + \int_0^t \|g(s, x(s))\| \, ds \\
&\leqslant R \left(\frac{2}{\Gamma(\alpha+1)} \|p_f\|_\infty + \|p_g\|_\infty \right) \\
&\leqslant R.
\end{aligned}$$

Step 2. $N(D_R)$ is bounded and equicontinuous.

By Step 1, we have $N(D_R) = \{Nx : x \in D_R\} \subset D_R$. Thus, for each $x \in D_R$, we have $\|Nx\|_\infty \leqslant R$, which means that ND_R is bounded. For the equicontinuity of $N(D_R)$. Let $t_1, t_2 \in J$, $t_1 < t_2$ and $x \in D_R$. Then

$$\begin{aligned}
&\|(Nx)(t_2) - (Nx)(t_1)\| \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left| (t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1} \right| \|f(s, x(s))\| \, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} \|f(s, x(s))\| \, ds + \int_{t_1}^{t_2} \|g(s, x(s))\| \, ds \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \left((t_2-s)^{\alpha-1} - (t_1-s)^{\alpha-1} \right) p_f(s) \|x(s)\| \, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} p_f(s) \|x(s)\| \, ds + \int_{t_1}^{t_2} p_g(s) \|x(s)\| \, ds \\
&\leqslant \frac{\|p_f\|_\infty R}{\Gamma(\alpha+1)} (t_2^\alpha - t_1^\alpha) + \|p_g\|_\infty R (t_2 - t_1).
\end{aligned}$$

As $t_1 \to t_2$, the right-hand side of the above inequality tends to zero.

Step 3. N is continuous.

Let $\{x_n\}$ be sequence such that $x_n \to x$ in $C(J, E)$. Then, for each $t \in J$

$$\begin{aligned}
&\|(Nx_n)(t) - (Nx)(t)\| \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} (1-s)^{\alpha-1} \|f(s, x_n(s)) - f(s, x(s))\| \, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|f(s, x_n(s)) - f(s, x(s))\| \, ds \\
&+ \int_0^t \|g(s, x_n(s)) - g(s, x(s))\| \, ds.
\end{aligned}$$

Since f and g are Caratheodory functions, the Lebesgue dominated convergence theorem implies that

$$\|(Nx_n)(t) - (Nx)(t)\| \to 0 \text{ as } n \to \infty.$$

This shows that (Nx_n) converges pointwise to Nx on J. Moreover, the sequence (Nx_n) is equicontinuous by a similar proof of Step 2. Therefore (Nx_n) converges uniformly to Nx and hence N is continuous.

Now let V be a subset of D_R such that $V \subset \overline{conv}\left((NV) \cup \{0\}\right)$. V is bounded and equicontinuous, and therefore the function $v \to v(t) = \mu(v(t))$ is continuous on J. By assumption (H3), Lemma 2.2 and the properties of the measure μ we have for each $t \in J$

$$\begin{aligned}
v(t) &\leqslant \mu\left((NV)(t) \cup \{0\}\right) \leqslant \mu\left((NV)(t)\right) \\
&\leqslant \frac{1}{\Gamma(\alpha)} \int_0^1 t^{\alpha-1} (1-s)^{\alpha-1} p_f(s)\, \mu(v(s))\, ds \\
&+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} p_f(s)\, \mu(v(s))\, ds + \int_0^t p_g(s)\, \mu(v(s))\, ds \\
&\leqslant \|v\|_\infty \left(\frac{2}{\Gamma(\alpha+1)} \|p_f\|_\infty + \|p_g\|_\infty \right).
\end{aligned}$$

This means that

$$\|v\|_\infty \left(1 - \left[\frac{2}{\Gamma(\alpha+1)} \|p_f\|_\infty + \|p_g\|_\infty \right] \right) \leqslant 0.$$

By (3.1), it follows that $\|v\|_\infty = 0$, that is $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in E. In view of the Ascoli-Arzela theorem, V is relatively compact in D_R. Applying now Theorem 2.1, we conclude that N has a fixed point, which is a solution of the problem (1.1). $\square$

4. Example

As an application of our results, we consider the following boundary value problem of a fractional differential equation

$$\text{(4.1)} \qquad \begin{cases} D^{\frac{3}{2}} x(t) - \frac{1}{3+\exp(t)} x(t) = D^{\frac{1}{2}} \frac{1}{5+\exp(t^2)} x(t), \ t \in J = [0,1], \\ x(0) = 0, \ x(1) = \int_0^1 \frac{1}{5+\exp(s^2)} x(s)\, ds. \end{cases}$$

Let

$$E = l^1 = \left\{ x = (x_1, x_2, ..., x_n, ...) : \sum_{n=1}^{\infty} |x_n| < \infty \right\},$$

equipped with the norm

$$\|x\|_E = \sum_{n=1}^{\infty} |x_n|.$$

Set

$$x = (x_1, x_2, ..., x_n, ...), \ f = (f_1, f_2, ..., f_n, ...), \ g = (g_1, g_2, ..., g_n, ...),$$

and

$$f_n(t, x_n) = \frac{1}{3 + \exp(t)} x_n, \ t \in J,$$
$$g_n(t, x_n) = \frac{1}{5 + \exp(t^2)} x_n, \ t \in J.$$

For each x_n and $t \in J$, we have

$$|f_n(t, x_n)| \leqslant \frac{1}{3 + \exp(t)} |x_n|, \tag{4.2}$$

and

$$|g_n(t, x_n)| \leqslant \frac{1}{5 + \exp(t^2)} |x_n|. \tag{4.3}$$

Hence conditions (H1) and (H2) are satisfied with

$$p_f(t) = \tfrac{1}{3+\exp(t)} \text{ and } p_g(t) = \tfrac{1}{5+\exp(t^2)}.$$

By (4.2) and (4.3), for any bounded set $B \subset l^1$, we have

$$\mu(f(t, B)) \leqslant \frac{1}{3 + \exp(t)} \mu(B) \ \text{ for each } t \in J,$$
$$\mu(g(t, B)) \leqslant \frac{1}{5 + \exp(t^2)} \mu(B) \ \text{ for each } t \in J.$$

Hence (H3) is satisfied. The condition

$$\frac{2}{\Gamma(\alpha + 1)} \|p_f\|_\infty + \|p_g\|_\infty \simeq 0.54 < 1,$$

is satisfied with

$$\|p_f\|_\infty = \tfrac{1}{4}, \ \|p_g\|_\infty = \tfrac{1}{6} \text{ and } \alpha = \tfrac{3}{2}.$$

Consequently, Theorem 3.1 implies that the problem (4.1) has a solution defined on J.

Acknowledgements. The authors are grateful to the referees for their valuable comments which have led to improvement of the presentation.

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Received by editors 13.07.2020; Revised version 04.09.2020; Available online 14.09.2020.

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