BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., Vol. **11**(1)(2021), 135-146
DOI: 10.7251/BIMVI2101135A

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

SUM OF POWERS OF NORMALIZED SIGNLESS LAPLACIAN EIGENVALUES AND RANDIĆ (NORMALIZED) INCIDENCE ENERGY OF GRAPHS

Ş. Burcu Bozkurt Altındağ

Abstract. Let G be a simple connected graph and let α be a real number. The graph invariant $\sigma_\alpha(G)$ introduced as the sum of the αth powers of the normalized signless Laplacian eigenvalues of G generalizes Randić (normalized) incidence energy. In this paper, we obtain some bounds on $\sigma_\alpha(G)$. As the special case of some of these bounds, we present new and better results on Randić (normalized) incidence energy.

1. Introduction

Let $G = (V, E)$ be a finite, simple and connected graph with n vertices and m edges, where $|V| = n$ and $|E| = m$. Let $v_1, v_2, \ldots, v_n$ be the vertices of G. For $1 \leqslant i \leqslant n$, let d_i denote the degree of the vertex v_i. If any two vertices v_i and v_j of G are adjacent then, we use the notation $i \sim j$. Let Δ and δ denote the maximum and minimum vertex degrees of G, respectively.

Let $A(G)$ be the $(0,1)$-adjacency matrix of a graph G. Denote by $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ the eigenvalues of $A(G)$. These eigenvalues are called as the eigenvalues of G [**11**]. Then, the energy of G was defined by Gutman as [**16**]:

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i| .$$

This concept is used in Hückel molecular orbital theory in order to estimate the total π-electron energy of a molecule [**17, 27**]. There exists a vast literature on

2010 *Mathematics Subject Classification.* 05C50; 05C90; 15A42.

Key words and phrases. Normalized signless Laplacian eigenvalues, Randic (normalized) incidence energy, Laplacian incidence energy, bound.

Communicated by Igor Milovanović.

graph energy in both chemistry and mathematics. For survey and details, see the recent papers [**14, 18**] and the monograph [**22**] with the list of its references.

Nikiforov [**32**] extended the definition of graph energy to the energy of any matrix M in the following way. The energy of the matrix M, denoted by $E(M)$, is defined as the sum of its singular values [**32**]. For a graph G, it is obvious that $E(A(G)) = E(G)$. In line with Nikiforov's definition [**32**], the incidence energy, pertaining to incidence matrix, of a graph was defined in [**21**]. For details on this graph invariant, see [**19, 20, 21**].

Let $D(G)$ denote the diagonal matrix of vertex degrees of G. The Laplacian and signless Laplacian matrices of G are defined as $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$, respectively [**12, 28**]. Since G is a connected graph, $D(G)$ is non-singular, then the normalized Laplacian and normalized signless Laplacian matrices of G are, respectively, defined as [**9**]

$$\mathcal{L}(G) = D(G)^{-1/2} L(G) D(G)^{-1/2} = I_n - R(G) \tag{1.1}$$

and

$$\mathcal{L}^+(G) = D(G)^{-1/2} Q(G) D(G)^{-1/2} = I_n + R(G) \tag{1.2}$$

where I_n is the $n \times n$ unit matrix and $R(G)$ is the Randić matrix of G with eigenvalues $\rho_1 = 1 \geqslant \rho_2 \geqslant \cdots \geqslant \rho_n$ [**2, 11, 25**]. Denote by $\gamma_1 \geqslant \gamma_2 \geqslant \cdots \geqslant \gamma_n = 0$ and $\gamma_1^+ \geqslant \gamma_2^+ \geqslant \cdots \geqslant \gamma_n^+$ the eigenvalues of $\mathcal{L}(G)$ and $\mathcal{L}^+(G)$ (or the normalized Laplacian and normalized signless Laplacian eigenvalues of G), respectively. For more information on these eigenvalues, see [**9**].

From (1.1) and (1.2), the eigenvalues of $\mathcal{L}(G)$ and $\mathcal{L}^+(G)$ are, respectively, of the form [**15, 25**]

$$\gamma_i = 1 - \rho_{n-i+1} \text{ and } \gamma_i^+ = 1 + \rho_i, \text{ for } i = 1, 2, \ldots, n. \tag{1.3}$$

By analogy with the Laplacian energy-like invariant defined in [**26**], the Laplacian incidence energy of G was introduced as [**35**]

$$LIE = LIE(G) = \sum_{i=1}^{n-1} \sqrt{\gamma_i}.$$

Some basic properties and the upper and lower bounds of LIE may be found in [**29, 30, 33, 35**].

In an analogous manner with the incidence energy [**19**], the Randić incidence energy was defined as [**15**]

$$I_R E = I_R E(G) = \sum_{i=1}^{n} \sqrt{\gamma_i^+}.$$

In [**8**], Cheng and Liu referred to this quantity as "*normalized incidence energy*" and obtained some upper and lower bounds for it as well as its Coulson integral formula. For more details on Randić (normalized) incidence energy, see [**4, 8, 15, 33**].

For a real number $\alpha \neq 0$, the sum of the αth powers of the non-zero normalized Laplacian eigenvalues of a connected graph G was defined by [**3**]

$$s_\alpha = s_\alpha(G) = \sum_{i=1}^{n-1} \gamma_i^\alpha.$$

This sum is closely associated with various graph invariants. For $\alpha = 1$ and $\alpha = 2$, $s_1 = n$ and $s_2 = n + 2R_{-1}$[**38**], where $R_{-1} = R_{-1}(G) = \sum_{i\sim j} \frac{1}{d_i d_j}$ is the general Randić index [**5**]. Notice that $2ms_{-1}$ is equal to the degree Kirchoff index [**6**] and $s_{1/2} = LIE$. For more information on s_α, see [**1, 3, 10, 24**].

Short time ago, for a real number $\alpha \neq 0$, we introduced the sum of the αth powers of the normalized signless Laplacian eigenvalues of a connected graph G as [**4**]

$$\sigma_\alpha = \sigma_\alpha(G) = \sum_{i=1}^{n} \left(\gamma_i^+\right)^\alpha.$$

The cases $\alpha = 1$ and $\alpha = 2$ are (same with s_α) equal to

$$\sigma_1 = \sum_{i=1}^{n} \gamma_i^+ = n \quad \text{and } \sigma_2 = \sum_{i=1}^{n} \left(\gamma_i^+\right)^2 = n + 2R_{-1}, \tag{1.4}$$

see [**8**]. Moreover for $\alpha = 1/2$, $\sigma_{1/2} = I_R E$. Note that the normalized Laplacian and normalized signless Laplacian eigenvalues of bipartite graphs coincide [**4**]. Then, for bipartite graphs, s_α is equal to σ_α and so LIE is equal to Randić (normalized) incidence energy $I_R E$ [**4**].

In this paper, we obtain some bounds on $\sigma_\alpha(G)$. As the special case of some of these bounds, we present new and better results for $I_R E$.

2. Preliminaries

We now state some preliminarily results that will be used in the subsequent section.

Lemma 2.1 ([**4**]). *If G is a bipartite graph, then the eigenvalues of $\mathcal{L}(G)$ and $\mathcal{L}^+(G)$ coincide.*

Lemma 2.2 ([**4**]). *If G is a bipartite graph, then σ_α coincide with s_α. Especially, for bipartite graphs, $\sigma_{1/2} = I_R E = LIE = s_{1/2}$.*

Lemma 2.3 ([**15**]). *For any connected graph G, the largest normalized signless Laplacian eigenvalue $\gamma_1^+ = 2$.*

Lemma 2.4 ([**15**]). *Let G be a graph of order $n \geqslant 2$ with no isolated vertices. Then $\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_n^+ = \frac{n-2}{n-1}$ if and only if $G \cong K_n$.*

Let $q_1 \geqslant q_2 \geqslant \cdots \geqslant q_n$ denote the eigenvalues of the signless Laplacian matrix $Q(G)$. It is a well known fact that for any connected non-bipartite graph G of order n, $q_i > 0$, for $i = 1, 2, \ldots, n$ [**12**]. Considering this with the definition of normalized signless Laplacian matrix given by Eq. (1.2), one can arrive at:

Remark 2.1. If G is a connected non-bipartite graph of order n, then $\gamma_i^+ > 0$, for $i = 1, 2, \dots, n$.

Lemma 2.5 ([**10, 29**]). *Let G be a connected graph of order n. Then*

$$\gamma_1 \geqslant 1 + \frac{2R_{-1}}{n}.$$

Equality holds if and only if $G \cong K_n$.

Lemma 2.6 ([**9, 24, 37**]). *Let G be a connected graph of order n with maximum vertex degree Δ. Then*

$$\gamma_1 \geqslant 1 + \frac{1}{\Delta} \geqslant \frac{n}{n-1}.$$

Each of the equalities hold if and only if $G \cong K_n$.

Lemma 2.7 ([**34**]). *Let G be a connected graph of order n with maximum vertex degree Δ and minimum vertex degree δ. Then*

$$\frac{n}{2\Delta} \leqslant R_{-1} \leqslant \frac{n}{2\delta}.$$

Equality holds in both sides if and only if G is a regular graph.

Remark 2.2. By Lemmas 2.5, 2.6 and 2.7, it is easy to see that

$$\gamma_1 \geqslant 1 + \frac{2R_{-1}}{n} \geqslant 1 + \frac{1}{\Delta} \geqslant \frac{n}{n-1}.$$

Each of the equalities hold if and only if $G \cong K_n$. Then, combining this fact with Eq. (1.3), we have that

$$\gamma_n^+ \leqslant 1 - \frac{2R_{-1}}{n} \leqslant 1 - \frac{1}{\Delta} \leqslant \frac{n-2}{n-1}.$$

Each of the equalities hold if and only if $G \cong K_n$.

Let $K_{p,q} - e$ denote the graph obtained by deleting any edge e from the complete bipartite graph $K_{p,q}$. In [**36**], the first two smallest values on γ_2 among all connected bipartite graphs with the fixed size of bipartition were determined as the following:

Lemma 2.8 ([**23, 36**]). *Let G ($\ncong K_{p,q}$) be a connected bipartite graph with bipartition $V = X \cup Y$ and $p = |X| > 1$, $q = |Y| > 1$. Then*

$$\gamma_2(G) \geqslant 1 + \frac{1}{\sqrt{pq}} > \gamma_2(K_{p,q}) = 1.$$

The first equality holds if and only if $G \cong K_{p,q} - e$ (e is any edge in $K_{p,q}$).

Lemma 2.9 ([**9**]). *Let G be a bipartite graph of order n. Then, $\gamma_i = 2 - \gamma_{n-i+1}$, for $i = 1, 2, \dots, n$.*

3. Main Results

In this section, we present the main results of this paper on σ_α and $I_R E$.

THEOREM 3.1. *Let G be a connected non-bipartite graph with $n \geqslant 3$ vertices.*
(i) *If $0 \leqslant \alpha \leqslant 1$, then*

$$\sigma_\alpha(G) \leqslant 2^\alpha + \left(1 - \frac{2R_{-1}}{n}\right)^\alpha + \frac{\left(n - 3 + \frac{2R_{-1}}{n}\right)^\alpha}{(n-2)^{\alpha-1}}. \tag{3.1}$$

(ii) *If $\alpha \leqslant 0$ or $\alpha \geqslant 1$, then*

$$\sigma_\alpha(G) \geqslant 2^\alpha + \left(1 - \frac{2R_{-1}}{n}\right)^\alpha + \frac{\left(n - 3 + \frac{2R_{-1}}{n}\right)^\alpha}{(n-2)^{\alpha-1}}. \tag{3.2}$$

Equality in both (3.1) and (3.2) occurs if and only if either $\alpha = 0$ or $\alpha = 1$ or $G \cong K_n$.

PROOF. In order to prove this theorem, we utilize the similar proof technique applied in Theorem 4.1 of [**7**]. We first start with the case $0 < \alpha < 1$. Recall that x^α is concave for $x > 0$ and $0 < \alpha < 1$. Then

$$\left(\sum_{i=2}^{n-1} \frac{\gamma_i^+}{n-2}\right)^\alpha \geqslant \frac{1}{n-2} \sum_{i=2}^{n-1} \left(\gamma_i^+\right)^\alpha,$$

that is equivalent to

$$\sum_{i=2}^{n-1} \left(\gamma_i^+\right)^\alpha \leqslant \frac{1}{(n-2)^{\alpha-1}} \left(\sum_{i=2}^{n-1} \gamma_i^+\right)^\alpha.$$

with equality holding if and only if $\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_{n-1}^+$. Thus, it follows from Eq. (1.4) and Lemma 2.3 that

$$\begin{aligned} \sigma_\alpha(G) &\leqslant \left(\gamma_1^+\right)^\alpha + \left(\gamma_n^+\right)^\alpha + \frac{\left(\sum\limits_{i=2}^{n-1} \gamma_i^+\right)^\alpha}{(n-2)^{\alpha-1}} \\ &= \left(\gamma_1^+\right)^\alpha + \left(\gamma_n^+\right)^\alpha + \frac{\left(n - \gamma_1^+ - \gamma_n^+\right)^\alpha}{(n-2)^{\alpha-1}} \\ &= 2^\alpha + \left(\gamma_n^+\right)^\alpha + \frac{\left(n - 2 - \gamma_n^+\right)^\alpha}{(n-2)^{\alpha-1}}. \end{aligned}$$

For $0 < x \leqslant \frac{n-2}{n-1}$, let us consider the following function

$$f(x) = x^\alpha + \frac{(n-2-x)^\alpha}{(n-2)^{\alpha-1}}.$$

It can be easily seen that f is increasing for $x \leqslant \frac{n-2}{n-1}$. By Remark 2.1 and Remark 2.2, we have

$$0 < \gamma_n^+ \leqslant 1 - \frac{2R_{-1}}{n} \leqslant \frac{n-2}{n-1}.$$

Therefore

$$\sigma_\alpha(G) \leqslant 2^\alpha + \left(1 - \frac{2R_{-1}}{n}\right)^\alpha + \frac{\left(n - 3 + \frac{2R_{-1}}{n}\right)^\alpha}{(n-2)^{\alpha-1}}.$$

Hence we get the inequality (3.1). We now assume that the equality holds in (3.1). Then all inequalities used in the above arguments must be equalities. That is,

$$\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_{n-1}^+ \text{ and } \gamma_n^+ = 1 - \frac{2R_{-1}}{n}.$$

Furthermore, from Eq. (1.4) and Lemma 2.3, $\sum_{i=2}^{n} \gamma_i^+ = n - 2$. These together with Remark 2.2 and Lemma 2.4 imply that $G \cong K_n$.

Conversely one can easily check that the equality holds in (3.1) for $G \cong K_n$. We now consider the case $\alpha < 0$ or $\alpha > 1$. Notice that x^α is convex for $x > 0$ and $\alpha < 0$ or $\alpha > 1$. Then

$$\left(\sum_{i=2}^{n-1} \frac{\gamma_i^+}{n-2}\right)^\alpha \leqslant \frac{1}{n-2} \sum_{i=2}^{n-1} \left(\gamma_i^+\right)^\alpha$$

with equality holding if and only if $\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_{n-1}^+$. Furthermore, f is decreasing for $x \leqslant \frac{n-2}{n-1}$. Then, similarly to the above arguments, we obtain the second part of the theorem. □

By taking $\alpha = 1/2$ in Theorem 3.1, we have:

Corollary 3.1. *Let G be a connected non-bipartite graph with $n \geqslant 3$ vertices. Then*

$$I_R E(G) \leqslant \sqrt{2} + \sqrt{1 - \frac{2R_{-1}}{n}} + \sqrt{(n-2)\left(n - 3 + \frac{2R_{-1}}{n}\right)} \tag{3.3}$$

with equality holding if and only if $G \cong K_n$.

Considering Remark 2.2 with the proof of Theorem 3.1, we also have the following result.

Theorem 3.2. *Let G be a connected non-bipartite graph with $n \geqslant 3$ vertices and maximum vertex degree Δ.*

(i) *If $0 \leqslant \alpha \leqslant 1$, then*

$$\sigma_\alpha(G) \leqslant 2^\alpha + \left(1 - \frac{1}{\Delta}\right)^\alpha + \frac{\left(n - 3 + \frac{1}{\Delta}\right)^\alpha}{(n-2)^{\alpha-1}}. \tag{3.4}$$

(ii) *If $\alpha \leqslant 0$ or $\alpha \geqslant 1$, then*

$$\sigma_\alpha(G) \geqslant 2^\alpha + \left(1 - \frac{1}{\Delta}\right)^\alpha + \frac{\left(n - 3 + \frac{1}{\Delta}\right)^\alpha}{(n-2)^{\alpha-1}}. \tag{3.5}$$

Equality in both (3.4) and (3.5) occurs if and only if either $\alpha = 0$ or $\alpha = 1$ or $G \cong K_n$.

As the special case of Theorem 3.2, we obtain:

COROLLARY 3.2. *Let G be a connected non-bipartite graph with $n \geqslant 3$ vertices and maximum vertex degree Δ. Then*

$$I_R E(G) \leqslant \sqrt{2} + \sqrt{1 - \frac{1}{\Delta}} + \sqrt{(n-2)\left(n - 3 + \frac{1}{\Delta}\right)} \tag{3.6}$$

with equality holding if and only if $G \cong K_n$.

REMARK 3.1. For a (connected) graph G of order $n \geqslant 2$, in [**8, 15**] the authors obtained that

$$I_R E(G) \leqslant \sqrt{2} + \sqrt{(n-1)(n-2)} \tag{3.7}$$

with equality holding if and only if $G \cong K_n$. Considering Remark 2.2 with the proof of Theorem 3.1, we conclude that the upper bounds (3.3) and (3.6) are better than the upper bound (3.7) for connected non-bipartite graphs. Moreover, (3.3) is the best for $I_R E$ among the mentioned upper bounds.

Let $RS(G) = (\rho_1, \rho_2, \ldots, \rho_n)$ and $NSLS(G) = \left(\gamma_1^+, \gamma_2^+, \ldots, \gamma_n^+\right)$ denote the spectrum of $R(G)$ and $\mathcal{L}^+(G)$, respectively. In [**13**], Das et al. found that

$$RS(K_{p,q} - e) = \left(1, \frac{1}{\sqrt{pq}}, \underbrace{0, \ldots, 0}_{n-4}, -\frac{1}{\sqrt{pq}}, -1\right) \tag{3.8}$$

where $K_{p,q} - e$ is the graph obtained by deleting any edge e from $K_{p,q}$. Then, by Eqs. (1.3) and (3.8), one may get that

$$NSLS(K_{p,q} - e) = \left(2, 1 + \frac{1}{\sqrt{pq}}, \underbrace{1, \ldots, 1}_{n-4}, 1 - \frac{1}{\sqrt{pq}}, 0\right). \tag{3.9}$$

Considering the similar proof technique applied in Theorem 4.1 of [**7**], we now give the following result on σ_α of connected bipartite graphs.

THEOREM 3.3. *Let G be a connected bipartite graph with bipartition $V = X \cup Y$ and $p = |X| > 1$, $q = |Y| > 1$. If $G \cong K_{p,q}$, then $s_\alpha(G) = \sigma_\alpha(G) = 2^\alpha + n - 2$* [**3**]. *Otherwise,*

(i) *If $0 \leqslant \alpha \leqslant 1$, then*

$$s_\alpha(G) = \sigma_\alpha(G) \leqslant 2^\alpha + \left(1 + \frac{1}{\sqrt{pq}}\right)^\alpha + \left(1 - \frac{1}{\sqrt{pq}}\right)^\alpha + n - 4. \tag{3.10}$$

(ii) *If $\alpha \leqslant 0$ or $\alpha \geqslant 1$, then*

$$s_\alpha(G) = \sigma_\alpha(G) \geqslant 2^\alpha + \left(1 + \frac{1}{\sqrt{pq}}\right)^\alpha + \left(1 - \frac{1}{\sqrt{pq}}\right)^\alpha + n - 4. \tag{3.11}$$

Equality in both (3.10) and (3.11) occurs if and only if either $\alpha = 0$ or $\alpha = 1$ or $G \cong K_{p,q} - e$ (e is any edge in $K_{p,q}$).

Proof. If $G \cong K_{p,q}$, then by Theorem 3.7 of [**3**] and Lemma 2.2, we have that $s_\alpha(G) = \sigma_\alpha(G) = 2^\alpha + n - 2$. Otherwise, G is not complete bipartite graph. We now consider the case $0 < \alpha < 1$. Note that x^α is concave for $x > 0$ and $0 < \alpha < 1$. Thus,

$$\left(\sum_{i=3}^{n-2} \frac{\gamma_i^+}{n-4}\right)^\alpha \geqslant \frac{1}{n-4}\sum_{i=3}^{n-2}\left(\gamma_i^+\right)^\alpha$$

i.e.,

$$\sum_{i=3}^{n-2}\left(\gamma_i^+\right)^\alpha \leqslant \frac{1}{(n-4)^{\alpha-1}}\left(\sum_{i=3}^{n-2}\gamma_i^+\right)^\alpha .$$

with equality holding if and only if $\gamma_3^+ = \gamma_4^+ = \cdots = \gamma_{n-2}^+$. Note that $\gamma_n^+ = 0$, by Lemma 2.1 and the fact that $\gamma_n = 0$ [**9**]. Bearing in mind these with Eq. (1.4) and Lemmas 2.1, 2.3 and 2.9, we get

$$\begin{aligned}
\sigma_\alpha(G) &\leqslant \left(\gamma_1^+\right)^\alpha + \left(\gamma_2^+\right)^\alpha + \left(\gamma_{n-1}^+\right)^\alpha + \frac{\left(\sum\limits_{i=3}^{n-2}\gamma_i^+\right)^\alpha}{(n-4)^{\alpha-1}} \\
&= \left(\gamma_1^+\right)^\alpha + \left(\gamma_2^+\right)^\alpha + \left(\gamma_{n-1}^+\right)^\alpha + \frac{\left(n-\gamma_1^+-\gamma_2^+-\gamma_{n-1}^+\right)^\alpha}{(n-4)^{\alpha-1}} \\
&= 2^\alpha + \left(2-\gamma_{n-1}^+\right)^\alpha + \left(\gamma_{n-1}^+\right)^\alpha + n - 4.
\end{aligned}$$

For $0 < x < 1$, let

$$g(x) = (2-x)^\alpha + x^\alpha.$$

It is elementary to see that g is increasing for $x < 1$. Since G is connected bipartite, $\gamma_{n-1}^+ > 0$, by Lemma 2.1 and the fact that $\gamma_{n-1} > 0$ [**9**]. Taking into account this with Lemmas 2.1, 2.8 and 2.9, we have

$$0 < \gamma_{n-1}^+ \leqslant 1 - \frac{1}{\sqrt{pq}} < 1.$$

Therefore

$$\sigma_\alpha(G) \leqslant 2^\alpha + \left(1+\frac{1}{\sqrt{pq}}\right)^\alpha + \left(1-\frac{1}{\sqrt{pq}}\right)^\alpha + n - 4.$$

Hence the inequality (3.10) holds. We now suppose that the equality holds in (3.10). Then

$$\gamma_3^+ = \gamma_4^+ = \cdots = \gamma_{n-2}^+ \text{ and } \gamma_{n-1}^+ = 1 - \frac{1}{\sqrt{pq}}.$$

Since G is bipartite, by Lemmas 2.1 and 2.9, $\gamma_{n-1}^+ = 1 - \frac{1}{\sqrt{pq}}$ implies that $\gamma_2^+ = 1 + \frac{1}{\sqrt{pq}}$. Then, by Lemma 2.8, $G \cong K_{p,q} - e$. By Eq. (1.4) and Lemmas 2.1, 2.3 and 2.9, we have that $\sum_{i=3}^{n-2}\gamma_i^+ = n-4$. Thus, $\gamma_3^+ = \gamma_4^+ = \cdots = \gamma_{n-2}^+ = 1$. This together with Eq. (3.9) confirm that $G \cong K_{p,q} - e$.

Conversely, it is easy to check that the equality holds in (3.10) for $G \cong K_{p,q} - e$, by Eq. (3.9). The proof of the case $\alpha < 0$ or $\alpha > 1$ is obtained similarly to the above considering x^α is convex for $x > 0$ and g is decreasing for $x < 1$. This completes the proof of theorem. □

By setting $\alpha = 1/2$ in Theorem 3.3, we have the following result.

COROLLARY 3.3. *Let G be a connected bipartite graph with bipartition $V = X \cup Y$ and $p = |X| > 1$, $q = |Y| > 1$. If $G \cong K_{p,q}$, then $LIE\,(G) = I_R E\,(G) = \sqrt{2} + n - 2$* [**15**]. *Otherwise,*

$$LIE\,(G) = I_R E\,(G) \leqslant \sqrt{2} + \sqrt{1 + \frac{1}{\sqrt{pq}}} + \sqrt{1 - \frac{1}{\sqrt{pq}}} + n - 4. \tag{3.12}$$

with equality holding if and only if $G \cong K_{p,q} - e$ (e is any edge in $K_{p,q}$).

REMARK 3.2. For a bipartite graph G of order n with no isolated vertices, it was derived that [**15**]

$$I_R E\,(G) \leqslant \sqrt{2} + n - 2 \tag{3.13}$$

with equality holding if and only if G is a complete bipartite graph. From the proof of Theorem 3.3 and Corollary 3.3, one can easily see that the upper bound (3.12) is better than the upper bound (3.13) for any connected bipartite graph G ($\ncong K_{p,q}$) with bipartition $V = X \cup Y$ and $p = |X| > 1$, $q = |Y| > 1$.

REMARK 3.3. For any n vertex tree T, it was determined that [**15**],

$$I_R E\,(T) \leqslant I_R E\,(S_n) \tag{3.14}$$

where S_n denote the star graph with n vertices. In other words, among all trees with n vertices, the star graph S_n is the unique tree with maximum Randić (normalized) incidence energy [**15**]. Because trees are bipartite graphs, from Lemma 2.2, the result in Eq. (3.14) can be re-stated as: For any n vertex tree T

$$LIE\,(T) \leqslant LIE\,(S_n)\,. \tag{3.15}$$

Furthermore, by Lemma 2.8, the proof of Theorem 3.3 and Corollary 3.3, we deduce that among all connected bipartite graphs except complete biparite graph, $K_{p,q} - e$ has the maximum Randić (normalized) incidence energy or Laplacian incidence energy.

For a connected bipartite graph G, Li et al. [**24**] obtained the result in (3.16) on $s_\alpha\,(G)$. By Lemma 2.2, one can re-state their result as:

THEOREM 3.4. [**24**] *Let G be a connected bipartite graph with $n \geqslant 3$ vertices. If $\alpha < 0$ or $0 < \alpha \leqslant 1$ or $\alpha \geqslant 2$, then*

$$s_\alpha\,(G) = \sigma_\alpha\,(G) \geqslant 2^\alpha + \frac{(n-2)^{2-\alpha}}{(n + 2R_{-1} - 4)^{1-\alpha}}. \tag{3.16}$$

Equality holds in (3.16) if and only if either $\alpha = 1$ or $\alpha = 2$ or $G \cong K_{p,q}$ ($p+q = n$). If $1 \leqslant \alpha \leqslant 2$, then the inequality (3.16) is reversed.

The above result on σ_α can be extended to all connected graphs as follows:

THEOREM 3.5. *Let G be a connected graph with $n \geqslant 3$ vertices. If $\alpha < 0$ or $0 < \alpha \leqslant 1$ or $\alpha \geqslant 2$, then*

$$\sigma_\alpha(G) \geqslant 2^\alpha + \frac{(n-2)^{2-\alpha}}{(n+2R_{-1}-4)^{1-\alpha}}. \tag{3.17}$$

Equality holds in (3.17) if and only if either $\alpha = 1$ or $\alpha = 2$ or $G \cong K_n$. If $1 \leqslant \alpha \leqslant 2$, then the inequality (3.17) is reversed.

PROOF. By considering Hölder's inequality [**31**], Eq. (1.4) and Lemmas 2.3 and 2.4, the proof can be easily obtained in a similar way to the proofs of Theorems 3.14 and 3.16 in [**24**]. □

From Theorem 3.5, we get the following result.

COROLLARY 3.4. *Let G be a connected graph with $n \geqslant 3$ vertices. Then*

$$I_R E(G) \geqslant \sqrt{2} + (n-2)\sqrt{\frac{n-2}{n+2R_{-1}-4}} \tag{3.18}$$

with equality holding if and only if $G \cong K_n$.

REMARK 3.4. For a graph G of order n with no isolated vertices, in [**8**] Cheng and Liu established a lower bound for Randić (normalized) incidence energy as:

$$I_R E(G) \geqslant \sqrt{\frac{n^3}{n+2R_{-1}}} \tag{3.19}$$

with equality holding if and only if n is even and G is disjoint union of $\frac{n}{2}$ paths of length 1. It can be easily seen that (3.18) is better than (3.19) on many example. For instance, for a graph G with vertex set $V = \{v_1, v_2, v_3, v_4, v_5\}$ and edge set $E = \{v_1v_2, v_1v_4, v_2v_3, v_2v_5, v_4v_5\}$, $I_R E(G) \approx 4.370$. For this graph, at rounded three decimal places, the lower bound (3.18) gives $I_R E(G) \geqslant 4.260$ whereas the lower bound (3.19) gives $I_R E(G) \geqslant 4.128$.

From Theorem 3.5 and Lemma 2.7, we have:

COROLLARY 3.5. *Let G be a connected graph with $n \geqslant 3$ vertices, maximum vertex degree Δ and minimum vertex degree δ.*

(i) *If $\alpha < 0$ or $0 < \alpha \leqslant 1$ (resp., $1 \leqslant \alpha < 2$), then*

$$\sigma_\alpha(G) \geqslant (resp., \leqslant)\ 2^\alpha + \frac{(n-2)^{2-\alpha}}{\left(n\left(1+\frac{1}{\delta}\right)-4\right)^{1-\alpha}}. \tag{3.20}$$

Equality holds in (3.20) if and only if $\alpha = 1$ or $G \cong K_n$.

(ii) *If $\alpha > 2$, then*

$$\sigma_\alpha(G) \geqslant 2^\alpha + \frac{(n-2)^{2-\alpha}}{\left(n\left(1+\frac{1}{\Delta}\right)-4\right)^{1-\alpha}}. \tag{3.21}$$

Equality holds in (3.21) if and only if $G \cong K_n$.

From Corollary 3.5, we obtain:

COROLLARY 3.6. *Let G be a connected graph with $n \geqslant 3$ vertices and minimum vertex degree δ. Then*

$$I_R E\left(G\right) \geqslant \sqrt{2} + (n-2)\sqrt{\frac{n-2}{n\left(1+\frac{1}{\delta}\right)-4}} \tag{3.22}$$

with equality holding if and only if $G \cong K_n$.

References

[1] M. Bianchi, A. Cornaro, J. L. Palacios and A. Torriero. Bounding the sum of powers of normalized Laplacian eigenvalues of graphs through majorization methods. *MATCH Commun. Math. Comput. Chem.*, **70**(2)(2013), 707–716.

[2] Ş. B. Bozkurt, A. D. Gungor, I. Gutman and A. S. Cevik. Randić matrix and Randić energy. *MATCH Commun. Math. Comput. Chem.*, **64**(1)(2010), 239–250.

[3] Ş. B. Bozkurt and D. Bozkurt. On the sum of powers of normalized Laplacian eigenvalues of graphs. *MATCH Commun. Math. Comput. Chem.*, **68**(3)(2012), 917–930.

[4] Ş. B. Bozkurt Altındağ. Note on the sum of powers of normalized signless Laplacian eigenvalues of graphs. *Math. Interdisc. Res.*, **4**(2)(2019), 171–182.

[5] M. Cavers, S. Fallat and S. Kirkland. On the normalized Laplacian energy and the general Randić index R_{-1} of graphs. *Lin. Algebra Appl.*, **433**(1)(2010), 172–190.

[6] H. Chen and F. Zhang. Resistance distance and the normalized Laplacian spectrum. *Discr. Appl. Math.*, **155**(5)(2007), 654–661.

[7] X. Chen and K. C. Das. Characterization of extremal graphs from Laplacian eigenvalues and the sum of powers of the Laplacian eigenvalues of graphs. *Discr. Math.*, **338**(7)(2015), 1252–1263 .

[8] B. Cheng and B. Liu. The normalized incidence energy of a graph. *Lin. Algebra Appl.*, **438**(11)(2013), 4510–4519. .

[9] F. R. K. Chung. *Spectral Graph Theory.* Amer. Math. Soc., Providence, 1997.

[10] G. P. Clemente and A. Cornaro. New bounds for the sum of powers of normalized Laplacian eigenvalues of graphs. *Ars Math. Contemp.*, **11**(2)(2016), 403–413.

[11] D. Cvetković, M. Doob and H. Sachs. *Spectra of Graphs.* Academic Press, New York, 1980.

[12] D. Cvetković, P. Rowlinson and S. Simić. Signless Laplacian of finite graphs. *Lin. Algebra Appl.*, **423**(1)(2007), 155–171.

[13] K. C. Das, S. Sun and I. Gutman. Normalized Laplacian eigenvalues and Randić energy of graphs. *MATCH Commun. Math. Comput. Chem.*, **77**(1)(2017), 45–59.

[14] K. C. Das and S. Elumalai. On Energy of graphs. *MATCH Commun. Math. Comput. Chem.*, **77**(1)(2017), 3–8.

[15] R. Gu, F. Huang and X. Li. Randić incidence energy of graphs. *Trans. Comb.*, **3**(4)(2014), 1–9.

[16] I. Gutman. The energy of a graph. *Ber. Math. Statist. Sekt. Forschungsz. Graz.*, **103**(100-105)(1978), 1–22.

[17] I. Gutman. Topology and stability of conjugated hydrocarbons. The dependence of total π-electron energy on molecular topology. *J. Serb. Chem. Soc.*, **70**(3)(2005), 441–456.

[18] I. Gutman. Oboudi-type bounds for graph energy. *Math. Interdisc. Res.*, **4**(2)(2019), 151–155.

[19] I. Gutman, D. Kiani and M. Mirzakhah. On incidence energy of graphs. *MATCH Commun. Math. Comput. Chem.*, **62**(3)(2009), 573–580.

[20] I. Gutman, D. Kiani, M. Mirzakhah and B. Zhou. On incidence energy of a graph. *Lin. Algebra Appl.*, **431**(8)(2009), 1223–1233.

[21] M. Jooyandeh, D. Kiani and M. Mirzakhah. Incidence energy of a graph. *MATCH Commun. Math. Comput. Chem.*, **62**(3)(2009), 561–572.

[22] X. Li, Y. Shi and I. Gutman. *Graph Energy.* Springer, New York, 2012.

[23] J. Li, J. M. Guo and W. C. Shiu. Bounds on normalized Laplacian eigenvalues of graphs. *J. Inequal. Appl.*, **316**(2014), 1–8.

[24] J. Li, J. M. Guo, W. C. Shiu, Ş. B. Bozkurt Altındağ and D. Bozkurt. Bounding the sum of powers of normalized Laplacian eigenvalues of a graph. *Appl. Math. Comput.*, **324**(2018), 82–92.

[25] B. Liu, Y. Huang and J. Feng. A note on the Randić spectral radius. *MATCH Commun. Math. Comput. Chem.*, **68**(3)(2012), 913–916.

[26] J. Liu and B. Liu. A Laplacian-energy-like invariant of a graph. *MATCH Commun. Math. Comput. Chem.*, **59**(2)(2008), 355–372.

[27] B. J. McClelland. Properties of the latent roots of a matrix: The estimation of π-electron energies. *J. Chem. Phys.*, **54**(2)(1971), 640–643.

[28] R. Merris. Laplacian matrices of graphs A survey. *Lin. Algebra Appl.*, **197-198**(1994), 143–176.

[29] E.I. Milovanović, M.M. Matejić and I. Ž. Milovanović. On the normalized Laplacian spectral radius, Laplacian incidence energy and Kemeny's constant. *Linear Algebra Appl.*, **582**(2019), 181–196.

[30] I. Ž. Milovanović, E.I. Milovanović, M.R. Popović and R.M. Stanković. Remark on the Laplacian energy like and Laplacian incidence energy invariants of graphs. *Creat. Math. Inform.*, **24**(2)(2015), 183–187.

[31] D. S. Mitrinović and P. M. Vasić. *Analytic Inequalities* (pp.74–94). Springer-Verlag, New York, 1970.

[32] V. Nikiforov. The energy of graphs and matrices. *J. Math. Anal. Appl.*, **326**(2)(2007), 1472–1475.

[33] L. Qiao, S. Zhang, B. Ning and J. Li. Coulson-type integral formulas for the general Laplacian energy-like invariant of graphs I. *J. Math. Anal. Appl.*, **435**(2)(2016), 1249–1261.

[34] L. Shi. Bounds on Randić indices. *Discrete Math.*, **309**(16)(2009), 5238–5241.

[35] L. Shi and H. Wang. The Laplacian incidence energy of graphs. *Linear Algebra Appl.*, **439**(12)(2013), 4056–4062.

[36] S. Sun and K. Ch. Das. On the second largest normalized Laplacian eigenvalue of graphs. *Appl. Math. Comput.*, **348**(2019), 531–541.

[37] X. Zhang. On the Laplacian spectra of graphs. *Ars Comb.*, **72**(2004), 191–198.

[38] P. Zumstein. *Comparison of spectral methods through the adjacency matrix and the Laplacian of a graph.* Diploma Thesis, ETH Zurich, 2005.

Received by editors 17.07.2020; Revised version 27.07.2020 and 27.08.2020; Available online 07.09.2020.

Konya, Turkey
E-mail address: `srf_burcu_bozkurt@hotmail.com`