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ON N-POWER HYPONORMAL OPERATORS IN INDEFINITE INNER PRODUCT SPACE

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Abstract. In this paper, we extend the concept of n-power hyponormal operators with reference to indefinite inner product, which is weaker than the case of normal operators. Furthermore, we give some basic properties of these operators.

1. Introduction

An indefinite inner product is a conjugate symmetric sesquilinear form $[x, y] = \langle x, Jy \rangle$, where $\langle \cdot, \cdot \rangle$ denote the Euclidean inner product. The indefinite product of matrices and applications to indefinite inner product space and extended some formulae from Euclidean space to an indefinite inner product space were investigated by Ramanathan et al. [**11**] in 2004. Kamaraj et al. [**7**] was introduced the concept of Moore-Penrose inverse in indefinite inner product space in 2005. The concept of hyponormal operators were introduced by Stampfli [**12**] in 1962. In 1990, Aluthge [**1**] extended the concept of p-hyponormal operators. Alzuraigi et al. [**2**] studied the n-normal operators in 2010. Guesba et al. [**5**] developed the concept of n-power-hyponormal operators in 2016. For B(H) and HN denotes to the set of all bounded linear and hyponormal operators with reference to indefinite inner product. For T is called n-EP if $T^nT^{[\dagger]} = T^{[\dagger]}T^n$, normal if $TT^{[*]} = T^{[*]}T$, skew-EP if $T^2 = -{T^{[\dagger]}}^2$ and projection if $T^2 = T = T^{[*]}$. T is called unitary if

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$TT^{[*]} = T^{[*]}T = I$. For T is called hypo-EP if $TT^{[\dagger]} \leqslant T^{[\dagger]}T$. We refer various properties and advantages of this product in ([**4**], [**6**], [**8**], [**9**], [**10**]).

2. n-power hyponormal operators

Definition 2.1. For an operator $T \in B(H)$, if $T^nT^{[*]} \leqslant T^{[*]}T^n$ then T is called n-power hyponormal (HN) operator.

Proposition 2.1. *If $S, T \in B(H)$ are unitarily equivalent and if T is n-power HN operators then so is S.*

Proof. Let T be an n-power HN operator and S be unitary equivalent of T. Then there exists unitary operator U such that $S = UTU^{[*]}$ so $S^n = UT^nU^{[*]}$.

We have,
$$\begin{aligned} S^nS^{[*]} &= UT^nU^{[*]}(UTU^{[*]})^{[*]} \\ &= UT^nU^{[*]}UT^{[*]}U^{[*]} \\ &= UT^nT^{[*]}U^{[*]} \\ &\leqslant UT^{[*]}T^nU^{[*]} \text{ (Since T is n-power HN)} \\ &= S^{[*]}S^n. \end{aligned}$$

Thus, $S^nS^{[*]} \leqslant S^{[*]}S^n$. Therefore S is a n-power HN operator. □

Proposition 2.2. *Let $T \in B(H)$ be an n-power HN operator. Then $T^{[*]}$ is n-power HN operator.*

Proof. Since, T is n-power-HN operator. We have
$$\begin{aligned} T^nT^{[*]} \leqslant T^{[*]}T^n &\Rightarrow (T^nT^{[*]})^{[*]} \leqslant (T^{[*]}T^n)^{[*]} \\ &\Rightarrow (T^{[*]})^{[*]}(T^n)^{[*]} \leqslant (T^n)^{[*]}(T^{[*]})^{[*]} \\ &\Rightarrow T(T^{[*]})^n \leqslant (T^{[*]})^nT \\ &\Rightarrow (T^{[*]})^nT \geqslant T(T^{[*]})^n. \end{aligned}$$

Thus $T^{[*]}$ is n-power HN operator. □

Corollary 2.1. *If T and $T^{[*]}$ are two n-power HN operators, then T is n-normal operator.*

Theorem 2.1. *If S and T are commuting n-power-HN operators and $ST^{[*]} = T^{[*]}S$, then ST is an n-power HN operator.*

Proof. Since $ST = TS$, so $S^nT^n = (ST)^n$ and $ST^{[*]} = T^{[*]}S$, so $S^nT^{[*]} = T^{[*]}S^n$. Now,
$$\begin{aligned} &ST^{[*]} = T^{[*]}S \\ &\Rightarrow TS^{[*]} = S^{[*]}T \\ &\Rightarrow T^n\, S^{[*]} = S^{[*]}T^n. \end{aligned}$$

We have,
$$\begin{aligned} (ST)^n(ST)^{[*]} &= S^nT^n\, T^{[*]}S^{[*]} \\ &\leqslant S^nT^{[*]}T^nS^{[*]} \text{ (since T is n-power hyponormal)} \\ &= T^{[*]}S^nS^{[*]}T^n \end{aligned}$$

$$\leqslant T^{[*]}S^{[*]}S^nT^n \text{ (since } S \text{ is n-power hyponormal).}$$

Hence, $(ST)^n(ST)^{[*]} \leqslant (ST)^{[*]}(ST)^n$. Therefore ST is an n-power HN operator. □

PROPOSITION 2.3. *Let S and T be commuting n-power HN operators, such that $TS^{[*]} = S^{[*]}T$ and $(S+T)^{[*]}$ is commutes with $\sum_{k=1}^{n-1} C_n^k\ S^{n-k}T^k$. Then $(S+T)$ is an n-power HN operator.*

PROOF.

$$(S+T)^n\ (S+T)^{[*]} = \left(\sum_{k=0}^{n-1} C_n^k\ S^{n-k}T^k\right)\left(S^{[*]}T^{[*]}\right)$$

$$= S^nT^{[*]} + \sum_{k=0}^{n-1} C_n^kS^{n-k}T^k(S+T)^{[*]} + T^nS^{[*]} + S^nT^{[*]} + T^nT^{[*]}$$

and since $TS^{[*]}= S^{[*]}T$ it folows $T^nS^{[*]} + S^{[*]}T^n$. Now, $TS^{[*]} = S^{[*]}T$. Then $ST^{[*]} = T^{[*]}S$ and $S^nT^{[*]} = T^{[*]}S^n$.

Since, $(S+T)^{[*]}$ is commute with $\sum_{k=1}^{n-1} C_n^k\ S^{n-k}T^k$, we have

$$(S+T)^n\,(S+T)^{[*]}$$

$$= S^nS^{[*]} + (S+T)^{[*]}\sum_{k=1}^{n-1} C_n^kS^{n-k}T^k + S^{[*]}T^n + T^{[*]}S^n + T^nT^{[*]}$$

$$\leqslant\ S^{[*]}S^n + (S+T)^{[*]}\sum_{k=1}^{n-1} C_n^k\ S^{n-k}T^k\ +\ S^{[*]}T^n + T^{[*]}T^n$$

$$= (S+T)^{[*]}\left(\sum_{k=0}^{n-1} C_n^k\ S^{n-k}T^k\right)$$

$$= (S+T)^{[*]}\,(S+T)^n. \qquad \square$$

PROPOSITION 2.4. *If $S, T \in B(H)$ are 2-power-HN operators such that $TS^{[*]} = S^{[*]}T$ and $ST + TS = 0$, then $S+T$ and ST are 2-power-HN operators.*

PROOF. Since $ST + TS = 0$, hence $S^2T^2 = T^2S^2$. So, $(S+T)^2 = S^2 + T^2$. Now

$$(S+T)^2(S+T)^{[*]} = \left(S^2+T^2\right)\left(S^{[*]}+T^{[*]}\right)$$

$$= S^2S^{[*]} + S^2T^{[*]} + T^2S^{[*]} + T^2T^{[*]}$$

$$= S^2S^{[*]} + T^{[*]}S^2 + S^{[*]}T^2 + T^2T^{[*]} \text{ since } TS^{[*]} = S^{[*]}T$$

$$\leqslant S^{[*]}S^2 + T^{[*]}S^2 + S^{[*]}T^2 + T^{[*]}T^2 = (S+T)^{[*]}\,(S+T)^2.$$

Now, $(\mathrm{ST})^2\,(\mathrm{ST})^{[*]} = S^2T^2T^{[*]}S^{[*]}$

$$\leqslant\ S^2T^{[*]}T^2S^{[*]} = T^{[*]}S^2S^{[*]}T^2$$

$$\leqslant T^{[*]}S^{[*]}S^2T^{[*]} = (ST)^{[*]}(ST)^2. \qquad \square$$

THEOREM 2.2. *Let $T_1, T_2, \cdots, T_m$ be n-power HN operators in B(H). Then $(T_1 \oplus T_2 \oplus ... \oplus T_n)$ and $(T_1 \otimes T_2 \otimes \cdots T_n)$ are the n-power HN operators.*

PROOF. Since we have

$$
\begin{aligned}
&(T_1 \oplus T_2 \oplus \cdots \oplus T_n)^n (T_1 \oplus T_2 \oplus \cdots \oplus T_n)^{[*]} \\
&\qquad = (T_1^n \oplus T_2^n \oplus \cdots \oplus T_n^n)(T_1^{[*]} \oplus T_2^{[*]} \oplus \cdots \oplus T_n^{[*]}) \\
&\qquad = T_1^n T_1^{[*]} \oplus T_2^n T_2^{[*]} \oplus \cdots \oplus T_m^n T_m^{[*]} \\
&\qquad \leqslant T_1^{[*]} T_1^n \oplus T_2^{[*]} T_2^n \oplus \cdots \oplus T_m^{[*]} T_m^n \\
&\qquad = (T_1^{[*]} \oplus T_2^{[*]} \oplus \ldots \oplus T_m^{[*]})(T_1^n \oplus T_2^n \oplus \cdots \oplus T_m^n) \\
&\qquad = (T_1 \oplus T_2 \oplus \cdots \oplus T_m)^{[*]} (T_1 \oplus T_2 \oplus \cdots \oplus T_m)^n,
\end{aligned}
$$

then $(T_1 \oplus T_2 \oplus \cdots \oplus T_m)$ is an n-power HN operator.

Now, for $x_1, \ldots, x_m \in H$

$$
\begin{aligned}
&(T_1 \otimes T_2 \otimes \cdots \otimes T_m)^n (T_1 \otimes T_2 \otimes \cdots \otimes T_m)^{[*]} (x_1 \otimes \cdots \otimes x_m) \\
&\quad = (T_1^n \otimes T_2^n \otimes \cdots \otimes T_m^n)(T_1^{[*]} \otimes T_2^{[*]} \otimes \cdots \otimes T_m^{[*]})\ (x_1 \otimes \cdots \otimes x_m) \\
&\quad = T_1^n T_1^{[*]} x_1 \otimes \cdots \otimes T_m^n T_m^{[*]} x_m \\
&\quad \leqslant T_1^{[*]} T_1^n x_1 \otimes \cdots \otimes T_m^{[*]} T_m^n x_m \text{ (since } T \text{ is an } n\text{-power HN operator)} \\
&\quad = (T_1^{[*]} \otimes T_2^{[*]} \otimes \ldots \otimes T_m^{[*]})(T_1^n \otimes T_2^n \otimes \cdots \otimes T_m^n)(x_1 \otimes \ldots, \otimes x_m) \\
&\quad = (T_1 \otimes T_2 \otimes \cdots \otimes T_m)^{[*]} (T_1 \otimes T_2 \otimes \cdots \otimes T_m)^n (x_1 \otimes \cdots \otimes x_m)
\end{aligned}
$$

So $(T_1 \otimes T_2 \otimes \cdots \otimes T_m)^n (T_1 \otimes T_2 \otimes \cdots \otimes T_m)^{[*]}$

$$\leqslant (T_1 \otimes T_2 \otimes \cdots \otimes T_m)^{[*]} (T_1 \otimes T_2 \otimes \cdots \otimes T_m)^n.$$

Hence $(T_1 \otimes T_2 \otimes \cdots \otimes T_m)$ is an n-power HN operator. □

PROPOSITION 2.5. *If T is 3-power-HN and $T^2 = -T^{[*]^2}$, then T is 3-normal operator.*

PROOF. Since $T^3 T^{[*]} = T T^2 T^{[*]} = -T T^{[*]^3}$ and $T^{[*]} T^3 = T^{[*]} T^2 T = -T^{[*]^3} T$, we have T is 3-power-HN. Then $T^3 T^{[*]} \leqslant T^{[*]} T^3$ and $-T T^{[*]^3} \leqslant -T^{[*]^3} T$. Thus $T T^{[*]^3} \geqslant T^{[*]^3} T$ and $\left(T T^{[*]^3}\right)^{[*]} \geqslant (T^{[\dagger]^3} T)^3$. Hence $T^3 T^{[*]} \geqslant T^{[*]} T^3$. So, $T^3 T^{[*]} \geqslant T^{[*]} T^3$. □

PROPOSITION 2.6. *If T is 4-power HN and T is skew-normal operator, then T is 4-normal operator.*

PROOF. If T is a skew-normal operator, then $T^2 = -T^{[*]^2}$.

Since $T^4 T^{[*]} = T^2 T^2 T^{[*]} = T^{[*]^5}$ and $T^{[*]} T^4 = T^{[*]} T^2 T^2 = T^{[\dagger]^5}$, thus $T^4 T^{[*]} = T^{[*]} T^4$. □

PROPOSITION 2.7. *If T is a 2-power HN operator and T is idempotent, then T is a HN operator.*

Proof. Since T is 2-power HN operator, then $T^2T^{[*]} \leqslant T^{[*]}T^2$. Since T is idempotent, isince $T^2 = T$ holds, it implies $TT^{[*]} \leqslant T^{[*]}T$. Hence, T is a HN operator. □

Proposition 2.8. *If T is a 3-power HN operator and T is idempotent, then T is a 2-power HN operator.*

Proof. Since T is a 3-power HN operator, then $T^3T^{[*]} \leqslant T^{[*]}T^3$. Since T is idempotent, it implies $T^2T^{[*]} \leqslant T^{[*]}T^2$. Hence T is a 2-power HN operator. □

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