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TRI-QUASI IDEALS OF Γ-SEMIGROUPS

Murali Krishna Rao Marapureddy

Abstract. In this paper, as a further generalization of ideals, we introduce the notion of tri-quasi ideals as a generalization of ideal,left ideal,right ideal, tri-ideal, bi-ideal, quasi ideal, interior ideal, bi-interior ideal, bi-quasi-interior ideal and bi-quasi ideal of Γ-semigroup and study the properties of tri-quasi ideals of Γ-semigroup.

1. Introduction

As a generalization of a ring, the notion of a $\Gamma-$ring was introduced by Nobusawa [**6**] in 1964. In 1981. Sen [**15**] introduced the notion of a Γ-semigroup as a generalization of semigroup. The notion of a ternary algebraic system was introduced by Lehmer [**5**] in 1932. Ideals play an important role in advance studies and uses of algebraic structures. Generalization of ideals in algebraic structures is necessary for further study of algebraic structures. Many mathematicians proved important results and characterization of algebraic structures by using the concept and the properties of generalization of ideals in algebraic structures. We know that the notion of a one sided ideal of any algebraic structure is a generalization of an ideal. The quasi ideals are generalization of left ideal and right ideal whereas the bi-ideals are generalization of quasi ideals. In 1952, the concept of bi-ideals was introduced by Good and Hughes [**1**] for semigroups. The notion of bi-ideals in rings and semigroups were introduced by Lajos and Szasz [**3, 4**]. Bi-ideal is a special case of $(m-n)$ ideal. Steinfeld [**14**] first introduced the notion of quasi ideals for semigroups and then for rings. Murali Krishna Rao [**7, 8, 9, 10, 11, 12, 13**] introduced the notion of left (right) bi-quasi ideal, bi-interior ideal, quasi-interior ideal and bi-quasi-interior ideal of semiring, Γ-semiring,Γ-semigroup and studied

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their properties. In this paper,as a further generalization of ideals, we introduce the notion of tri-quasi ideal as a generalization of bi-ideal, quasi ideal, interior ideal, bi-interior ideal, tri-ideal and bi-quasi ideal of Γ-semigroup and study some of the properties of tri-quasi ideals of Γ-semiring. Some charecterizations of tri-quasi ideals of Γ -semigroup, regular Γ -semigroup and simple Γ -semigroup are given.

2. Preliminaries

In this section, we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. Let M and Γ be non-empty subsets. Then we call M a $\Gamma-$semigroup, if there exists a mapping $M \times \Gamma \times M \to M$ (images of (x, α, y) will be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) such that it satisfies $x\alpha(y\beta z) = (x\alpha y)\beta z$ for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.

Definition 2.2. Let M be a $\Gamma-$semigroup. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 2.3. A $\Gamma-$semigroup M is said to be commutative if $a\alpha b = b\alpha a$, for all $a, b \in M, \alpha \in \Gamma$.

Definition 2.4. Let M be a $\Gamma-$semigroup. An element $a \in M$ is said to be an idempotent of M if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$ and a is also said to be α- idempotent.

Definition 2.5. Let M be a $\Gamma-$semigroup. If every element of M is an idempotent of M then $\Gamma-$semigroup M is said to be band.

Definition 2.6. Let M be a $\Gamma-$semigroup. An element $a \in M$ is said to be regular element of M if there exist $x \in M, \alpha, \beta \in \Gamma$ such that $a = a\alpha x\beta a$.

Definition 2.7. Let M be a $\Gamma-$semigroup. Every element of M is a regular element of M then M is said to be a regular $\Gamma-$semigroup M.

Definition 2.8. A non-empty subset A of a $\Gamma-$ semigroup$M$ is called

(i) a $\Gamma-$subsemigroup of M if A is a subsemigroup of M and $A\Gamma A \subseteq A$.
(ii) a quasi ideal of M if A is a $\Gamma-$subsemigroup of M and $A\Gamma M \cap M\Gamma A \subseteq A$.
(iii) a bi-ideal of M if A is a $\Gamma-$subsemigroup of M and $A\Gamma M\Gamma A \subseteq A$.
(iv) an interior ideal of M if A is a $\Gamma-$subsemigroup of M and $M\Gamma A\Gamma M \subseteq A$.
(v) a left (right) ideal of M if A is a $\Gamma-$subsemigroup of M and
$$M\Gamma A \subseteq A \ \ (A\Gamma M \subseteq A).$$
(vi) an ideal if A is a $\Gamma-$subsemigroup of $M, A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$.
(vii) a $k-$ideal if A is a $\Gamma-$subsemigroup of $M, A\Gamma M \subseteq A, M\Gamma A \subseteq A$ and $x \in M,\ x + y \in A, y \in A$ then $x \in A$.
(viii) a bi-interior ideal of M if A is a $\Gamma-$subsemigroup of M and
$$M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B.$$

(ix) a left bi-quasi ideal (right bi-quasi ideal) of M if A is a $\Gamma-$subsemigroup of M and $M\Gamma A \cap A\Gamma M\Gamma A \subseteq A$ (res. $A\Gamma M \cap A\Gamma M\Gamma A \subseteq A$).

(x) a left quasi-interior ideal (right quasi-interior ideal) of M if A is a $\Gamma-$subsemigroup of M and $M\Gamma A\Gamma M\Gamma A \subseteq A$ (res. $A\Gamma M\Gamma A\Gamma M \subseteq A$).

(xi) a bi-quasi-interior ideal of M if A is a $\Gamma-$subsemigroup of M and

$$B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B.$$

(xii) a left tri-ideal (right tri-ideal) of M if A is a $\Gamma-$subsemigroup of M and $A\Gamma M\Gamma A\Gamma A \subseteq A$ (res. $A\Gamma A\Gamma M\Gamma A \subseteq A$).

(xiii) a tri-ideal of M if A is a $\Gamma-$subsemigroup of M and $A\Gamma M\Gamma A\Gamma A \subseteq A$ and $A\Gamma A\Gamma M\Gamma A \subseteq A$.

(xiv) a left (right) weak-interior ideal of M if B is a $\Gamma-$subsemigroup of M and $M\Gamma B\Gamma B \subseteq B$ (res. $B\Gamma B\Gamma M \subseteq B$).

(xv) a weak-interior ideal of M if B is a $\Gamma-$subsemigroup of M and B is a left weak-interior ideal and a right weak-interior ideal of M.

3. Tri-quasi ideals of Γ-semigroups

In this section, we introduce the notion of tri-quasi ideal as a generalization of bi-ideal,quasi-ideal and interior ideal of Γ-semigroups and study the properties of tri-quasi ideal of Γ-semigroups.

Definition 3.1. A non-empty subset B of a Γ-semigroup M is said to be tri-quasi ideal of M if B is a Γ-subsemiring of M and $B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B$.

Every tri-quasi ideal of a Γ-semigroup M need not be bi-ideal,quasi-ideal, interior ideal bi-interior ideal. and bi-quasi ideals of a Γ-semigroup M.

Example 3.1. Let Q be the set of all rational numbers. If

$$M = \left\{ \begin{pmatrix} a & c \\ b & 0 \end{pmatrix} \mid a, b, c \in Q \right\}$$

and $\Gamma = M$ and ternary operation is defined as usual matrix multiplication and

$$A = \left\{ \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}.$$

Then A is tri-quasi ideal of the Γ-semigroup M.

In the following theorem, we mention some important properties and we omit the proofs since proofs are straightforward.

Theorem 3.1. *Let M be a Γ-semigroup. Then the following are hold.*

(1) *Every left ideal is a tri-quasi ideal of M.*
(2) *Every right ideal is a tri-quasi ideal of M.*
(3) *Every quasi ideal is tri-quasi ideal of M.*
(4) *Every ideal is a tri-quasi ideal of M.*
(5) *Intersection of a right ideal and a left ideal of M is a tri-quasi ideal of M.*
(6) *If L is a left ideal and R is a right ideal of a Γ-semigroup M then $B = R\Gamma L$ is a tri-quasi ideal of M.*

(7) *Every bi-ideal of a* Γ*-semigroup* M *is a tri-quasi ideal of a* Γ*-semigroup* M.
(8) *Every interior ideal of* Γ*-semigroup* M *is a tri-quasi ideal of* M.
(9) *Let* B *be bi-ideal of a* Γ*-semigroup* M *and* I *be interior ideal of* M. *Then* $B \cap I$ *is a tri-quasi ideal of* M.

THEOREM 3.2. *Let* M *be a* Γ*-semigroup.* B *is a tri-quasi ideal of* M *and* $B\Gamma B = B$ *if and only if there exist a left ideal* L *and a right ideal* R *such that* $R\Gamma L \subseteq B \subseteq R \cap L$.

PROOF. Suppose B is a tri-quasi ideal of the Γ-semigroup$M$. Then

$$B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B.$$

Let $R = B\Gamma M$ and $L = M\Gamma B$. Then L and R are left and right ideals of M respectively. Therefore $R\Gamma L \subseteq B \subseteq R \cap L$.

Conversely suppose that there exist L and R are left and right ideals of M respectively such that $R\Gamma L \subseteq B \subseteq R \cap L$. Then

$$\begin{aligned} B\Gamma B\Gamma M\Gamma B\Gamma B &\subseteq (R\cap L)\Gamma(R\cap L)\Gamma M\Gamma(R\cap L)\Gamma(R\cap L) \\ &\subseteq (R)\Gamma R\Gamma M\Gamma L\Gamma(L) \subseteq R\Gamma L \subseteq B\ . \end{aligned}$$

Hence B is a tri-quasi ideal of the Γ-semigroup M. □

THEOREM 3.3. *The intersection of a tri-quasi ideal* B *of* Γ*-semigroup* M *and a right ideal* A *of* M *is always tri-quasi ideal of* M.

PROOF. Suppose $C = B \cap A$.

$$C\Gamma C\Gamma M\Gamma C\Gamma C \subseteq B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B$$
$$C\Gamma C\Gamma M\Gamma C\Gamma C \subseteq A\Gamma M\Gamma A\Gamma M\Gamma A \subseteq A$$

since A is a left ideal of M. Therefore $C\Gamma A\Gamma C\Gamma M\Gamma C\Gamma M \subseteq B \cap A = C$.

Hence the intersection of a tri-quasi ideal B of the Γ-semigroup M and a right ideal A of M is always a tri-quasi ideal of M. □

COROLLARY 3.1. *The intersection of a tri-quasi ideal* B *of* Γ*-semigroup* M *and a left ideal* A *of* M *is always tri-quasi ideal of* M.

COROLLARY 3.2. *The intersection of a tri-quasi ideal* B *of* Γ*-semigroup* M *and an ideal* A *of* M *is always tri-quasi ideal of* M.

THEOREM 3.4. *Let* A *and* C *be* Γ*-subsemigroups of* M *and* $B = A\Gamma C$. *If* A *is the left ideal then* B *is a tri-quasi-interior ideal of* M.

PROOF. Let A and C be Γ-subsemigroups of M and $B = A\Gamma C$. Suppose A is the left ideal of M. Then $B\Gamma B = A\Gamma C\Gamma A\Gamma C \subseteq A\Gamma C = B$ and

$$B\Gamma B\Gamma M\Gamma B\Gamma B = A\Gamma C\Gamma A\Gamma C\Gamma M\Gamma A\Gamma C\Gamma A\Gamma C \subseteq A\Gamma C = B.$$

Hence B is a tri-quasi ideal of M. □

COROLLARY 3.3. *Let* A *and* C *be* Γ*-subsemigroups of a* Γ*-semigroup* M *and* $B = A\Gamma C$. *If* C *is a right ideal then* B *is a tri-quasi ideal of* M.

Theorem 3.5. *Let M be a Γ-semigroup and T be a non-empty subset of M. Then every subsemiring of T containing $T\Gamma T\Gamma M\Gamma T\Gamma T$ is a tri-quasi ideal of Γ-semigroup M.*

Proof. Let B be a subsemiring of T containing $T\Gamma T\Gamma M\Gamma T\Gamma T$. Then

$$B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq T\Gamma T\Gamma M\Gamma T\Gamma T \subseteq B.$$

Therefore $B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B$. Hence B is a tri-quasi ideal of M. □

Theorem 3.6. *Let M be a Γ-semigroup.Then B is a tri-quasi ideal of a Γ-semigroup M if and only if B is a left ideal of some right ideal of a Γ-semigroup$M$.*

Proof. Let B be a tri-quasi ideal of the Γ-semigroup M. Then

$$B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B.$$

Therefore $B\Gamma B$ is a left ideal of right ideal $B\Gamma B\Gamma M$ of M.

Conversely suppose that B is a left ideal of some right ideal R of the Γ-semigroup$M$. Then $R\Gamma B \subseteq B, R\Gamma M \subseteq R$. Hence $B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B\Gamma M\Gamma B \subseteq R\Gamma M\Gamma B \subseteq R\Gamma B \subseteq B$. Therefore B is a tri-quasi ideal of the Γ-semigroup M. □

Corollary 3.4. *B is a tri-quasi ideal of a Γ-semigroup M if and only if B is a right ideal of some left ideal of a Γ-semigroup M.*

Theorem 3.7. *If B is a tri-quasi ideal of a Γ-semigroup M, T is a Γ-subsemigroup of M and $T \subseteq B$ then $B\Gamma T$ is a tri-quasi ideal of M.*

Proof. Obviously, $B\Gamma T$ is a Γ-subsemigroup of M. $B\Gamma T\Gamma B\Gamma T \subseteq B\Gamma T$. Hence $B\Gamma T$ is a Γ-subsemiring of M. From here it follows

$$B\Gamma T\Gamma B\Gamma T\Gamma M\Gamma B\Gamma T\Gamma B\Gamma T \subseteq B\Gamma B\Gamma M\Gamma B\Gamma B\Gamma T \subseteq B\Gamma T.$$

Hence $B\Gamma T$ is a tri-quasi ideal of the Γ-semigroup M. □

Theorem 3.8. *Let M be a Γ-semigroup. If $M = M\Gamma < a >$, for all $a \in M$, where $< a >$ is the smallest tri-quasi ideal generated by a. Then every tri-quasi ideal of M is a quasi ideal of M.*

Proof. Let B be a tri-quasi ideal of a Γ-semigroup M and $a \in B$. Then $B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B$ and $M\Gamma < a > \subseteq M\Gamma B$, ($B\Gamma M = M$). Thus $M \subseteq M\Gamma B \subseteq M$ and $M\Gamma B = M$. From here it follows $B\Gamma M = B\Gamma M\Gamma B \subseteq B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B$ and $M\Gamma B \cap B\Gamma M \subseteq M\Gamma M \cap B\Gamma M \subseteq B$. Therefore B is a quasi ideal of M. Hence the theorem. □

Theorem 3.9. *The intersection of $\{B_\lambda \mid \lambda \in A\}$ tri-quasi ideals of a Γ-semigroup M is a tri-quasi-interior ideal of M.*

Proof. Let $B = \bigcap\limits_{\lambda \in A} B_\lambda$. Then B is a Γ-subsemiring of M. Since B_λ is a tri-quasi ideal of M, we have $B_\lambda\Gamma B_\lambda\Gamma M\Gamma B_\lambda\Gamma B_\lambda \subseteq B_\lambda$, for all $\lambda \in A$ and $B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B$. Hence B is a tri-quasi ideal of M. □

Theorem 3.10. *If B be a left bi-quasi ideal of a Γ-semigroup M, then B is a tri-quasi ideal of M.*

Proof. Suppose B is a left bi-quasi ideal of the Γ-semigroup M. Then

$$B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq M\Gamma B \text{ and } B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B\Gamma M\Gamma B.$$

Therefore $B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq M\Gamma B \cap B\Gamma M\Gamma B \subseteq B$. Hence B is a tri-quasi ideal of M. □

Corollary 3.5. *If B be a right bi-quasi ideal of a Γ-semigroup M, then B is a tri-quasi ideal of M.*

Corollary 3.6. *If B be a bi-quasi ideal of a Γ-semigroup M, then B is a tri-quasi ideal of M.*

Theorem 3.11. *If B be a bi–interior ideal of a Γ- semiring M, then B is a tri-quasi ideal of M.*

Proof. Suppose B is a bi–interior ideal of the Γ-semigroup M. Then

$$B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq M\Gamma B\Gamma M \text{ and } B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B\Gamma M\Gamma B.$$

Therefore $B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$. Hence B is a tri-quasi ideal of M. □

4. Tri-quasi simple Γ-semigroup and Regular Γ-semigroup

In this section,we introduce the notion of a tri-quasi simple Γ-semigroup and characterize the tri-quasi simple Γ-semigroup using tri-quasi ideals of Γ-semigroup and study the properties of minimal tri-quasi ideals of Γ-semigroup and we characterize regular Γ-semigroup using tri-quasi ideals of Γ-semigroup.

Definition 4.1. A Γ-semigroup M is a left (right) simple Γ-semigroup if M has no proper left (right) ideals of M

Definition 4.2. A Γ-semigroup M is said to be simple Γ-semigroup if M has no proper ideals of M

Definition 4.3. A Γ-semigroup M is said to be tri-quasi simple Γ-semigroup M if M has no tri-quasi ideals other than M itself.

Theorem 4.1. *If M is a $\Gamma-$group then M is tri-quasi simple Γ-group.*

Proof. Let B be a proper tri-quasi ideal of the $\Gamma-$group M and $0 \neq a \in B$. Since M is a $\Gamma-$group, there exist $b \in M$, $\alpha \in \Gamma$ such that $a\alpha b = 1$. Then there exist $\beta \in \Gamma$, $x \in M$ such that $a\alpha b\beta x = x = x\beta a\alpha b$. Then $x \in B\Gamma M$. Therefore $M \subseteq B\Gamma M$. We have $B\Gamma M \subseteq M$. Hence $M = B\Gamma M$. Similarly we can prove $M\Gamma B = M$. Further on, we have

$$M = M\Gamma B = B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B.$$

So, $M \subseteq B$. Therefore $M = B$. Hence $\Gamma-$group M has no proper tri-quasi-interior ideals. □

Theorem 4.2. *Let M be a left and right simple Γ-semigroup.Then M is a tri-quasi simple Γ-semigroup.*

PROOF. Let M be a simple Γ-semigroup and B be a tri-quasi ideal of M. Then $B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B$ and $M\Gamma B$ and $B\Gamma M$ are left and right ideals of M. Since M is a left and right simple Γ-semigroup, we have $M\Gamma B = M$. $B\Gamma M = M$. Hence $B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B$ and $B\Gamma M\Gamma B \subseteq B$. So, $M \subseteq B$. □

THEOREM 4.3. *Let M be a Γ-semigroup. Then M is a tri-quasi simple Γ-semigroup if and only if $(a)_{tqi} = M$, for all $a \in M$, where $(a)_{tqi}$ is the tri-quasi ideal generated by a.*

PROOF. Let M be a Γ-semigroup. Suppose that $(a)_{tqi}$ is a tri-quasi ideal generated by a and M is a tri-quasi -interior simple Γ-semigroup. Then $(a)_{tqi} = M$, for all $a \in M$.

Conversely suppose that B is a tri-quasi ideal of Γ-semigroup M and $(a)_{tqi} = M$, for all $a \in M$. Let $b \in B$. Then $(b)_{tqi} \subseteq B \;\Rightarrow\; M = (b)_{tqi} \subseteq B \subseteq M$. Therefore M is a tri-quasi simple Γ-semigroup. □

THEOREM 4.4. *Let M be a Γ-semigroup. M is a tri-quasi simple Γ-semigroup if and only if $< a >= M$, for all $a \in M$ and where $< a >$ is the smallest tri-quasi ideal generated by a.*

PROOF. Let M be a Γ-semigroup. Suppose M is a tri-quasi simple Γ-semigroup, $a \in M$ and $B = M\Gamma a$. Then B is a left ideal of M. Therefore, by Theorem 3.1, B is a tri-quasi ideal of M. So, $B = M$. Hence $M\Gamma a = M$, for all $a \in M$. Further on, we have $M\Gamma a \subseteq < a > \subseteq M$ and $M \subseteq < a > \subseteq M$. Thus $M =< a >$.

Conversely suppose that $< a >$ is the smallest tri-quasi ideal of M generated by a and $< a >= M$. Let A be the tri-quasi ideal and $a \in A$. Then $< a > \subseteq A \subseteq M$ and $M \subseteq A \subseteq M$. Therefore $A = M$. Hence M is a tri-quasi simple Γ-semigroup. □

THEOREM 4.5. *Let M be a Γ-semigroup. Then M is a tri-quasi simple Γ-semrring if and only if $a\Gamma a\Gamma M\Gamma a\Gamma a = M$, for all $a \in M$.*

PROOF. Suppose M is left tri-quasi simple Γ-semigroup and $a \in M$. Therefore $a\Gamma a\Gamma M\Gamma a\Gamma a$ is a tri-quasi ideal of M. Hence $a\Gamma M\Gamma a\Gamma M\Gamma a = M$, for all $a \in M$.

Conversely suppose that $a\Gamma a\Gamma M\Gamma a\Gamma a = M$, for all $a \in M$. Let B be a tri-quasi ideal of the Γ-semigroup M and $a \in B$. Then $M = a\Gamma a\Gamma M\Gamma a\Gamma a$ and $M = a\Gamma a\Gamma M\Gamma a\Gamma a \subseteq B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B$ Therefore $M = B$. Hence M is a tri-quasi simple Γ-semigroup. □

THEOREM 4.6. *If B is a minimal tri-quasi ideal of a Γ-semigroup M then any two non-zero elements of B generated the same right ideal of M.*

PROOF. Let B be a minimal tri-quasi ideal of M and $x \in B$. Then $(x)_R \cap B$ is a tri-quasi ideal of M. Therefore $(x)_R \cap B \subseteq B$. Since B is a minimal tri-quasi ideal of M, we have $(x)_R \cap B = B \Rightarrow B \subseteq (x)_R$. Suppose $y \in B$. Then $y \in (x)_R$, $(y)_R \subseteq (x)_R$. Therefore $(x)_R = (y)_R$. Hence the theorem. □

COROLLARY 4.1. *If B is a minimal tri-quasi ideal of a Γ-semigroup M then any two non-zero elements of B generates the same left ideal of M.*

Theorem 4.7. *Let M be a Γ-semigroup and $B = R\Gamma L$, where L and R are minimal left and right ideals of M repectively. Then B is a minimal tri-quasi ideal of M.*

Proof. Obviously $B = R\Gamma L$ is a tri-quasi ideal of M. Let A be a tri-quasi ideal of M such that $A \subseteq B$. Then $M\Gamma A\Gamma A$ is a left ideal of M. Thus

$$M\Gamma A\Gamma A \subseteq M\Gamma B\Gamma B = M\Gamma R\Gamma L\Gamma R\Gamma L \subseteq L,$$

since L is a left ideal of M. Similarly, we can prove $A\Gamma A\Gamma M \subseteq R$. Therefore $M\Gamma A\Gamma A = L$ and $A\Gamma A\Gamma M = R$. Hence

$$B = A\Gamma A\Gamma M\Gamma M\Gamma A\Gamma A \subseteq A\Gamma A\Gamma M\Gamma A\Gamma A \subseteq A.$$

Therefore $A = B$. Hence B is a minimal tri-quasi ideal of M. □

Theorem 4.8. *M is a regular $\Gamma-$semigroup if and only if $A\Gamma B = A \cap B$ for any right ideal A and left ideal B of $\Gamma-$semigroup M.*

Theorem 4.9. *Let M be a regular idempotent Γ-semigroup. Then B is a tri-quasi ideal of M if and only if $B\Gamma B\Gamma M\Gamma B\Gamma B = B$, for all tri-quasi ideals B of M.*

Proof. Suppose M is a regular Γ-semigroup, B is a tri-quasi ideal of M and $x \in B$. Then $B\Gamma B\Gamma M\Gamma B\Gamma B \subseteq B$ and there exist $y \in M$, $\alpha, \beta, \delta \in \Gamma$,such that $x = x\delta x\alpha y\beta x\delta x \in B\Gamma B\Gamma M\Gamma B\Gamma B$. Therefore $x \in B\Gamma B\Gamma M\Gamma B\Gamma B$. Hence $B\Gamma B\Gamma M\Gamma B\Gamma B = B$.

Conversely suppose that $B\Gamma B\Gamma M\Gamma B\Gamma B = B$, for all tri-quasi ideals B of M. Let $B = R \cap L$, where R is a right ideal and L is a left ideal of M. Then B is a tri-quasi ideal of M. Therefore $(R \cap L)\Gamma M\Gamma(R \cap L)\Gamma M\Gamma(R \cap L) = R \cap L$

$$\begin{aligned} R \cap L &= (R \cap L)\Gamma(R \cap L)M\Gamma M\Gamma(R \cap L)\Gamma(R \cap L) \\ &\subseteq R\Gamma M\Gamma L\Gamma M\Gamma L \\ &\subseteq R\Gamma L \\ &\subseteq R \cap L \text{ (since } R\Gamma L \subseteq L \text{ and } R\Gamma L \subseteq R). \end{aligned}$$

Therefore $R \cap L = R\Gamma L$. Hence M is a regular Γ-semigroup. □

Theorem 4.10. *Let B be Γ-subsemigroup of a regular idempotent Γ-semigroup M. B can be represented as $B = R\Gamma L$, where R is a right ideal and L is a left ideal of M if and only if B is a tri-quasi ideal of M.*

Proof. Suppose $B = R\Gamma L$, where R is right ideal of M and L is a left ideal of M.

$$B\Gamma B\Gamma M\Gamma B\Gamma B = R\Gamma L\Gamma R\Gamma L\Gamma M\Gamma R\Gamma L\Gamma R\Gamma L \subseteq R\Gamma L = B.$$

Hence B is a tri-quasi ideal of the Γ-semigroup M.

Conversely suppose that B is a tri-quasi ideal of the regular idempotent Γ-semigroup M.Then $B\Gamma B\Gamma M\Gamma B\Gamma B = B$. Let $R = B\Gamma M$ and $L = M\Gamma B$. Then $R = B\Gamma M$ is a right ideal of M and $L = M\Gamma B$ is a left ideal of M. Thus $B\Gamma M \cap M\Gamma B \subseteq B\Gamma B\Gamma M\Gamma B\Gamma B = B$ and $B\Gamma M \cap M\Gamma B \subseteq B$. From here it follows

$R \cap L \subseteq B$. We have $B \subseteq B\Gamma M = R$ and $B \subseteq M\Gamma B = L$. So, $B \subseteq R \cap L$ and $B = R \cap L = R\Gamma L$, since M is a regular Γ-semigroup. Hence B can be represented as $R\Gamma L$, where R is the right ideal and L is the left ideal of M. □

The following theorem is a necessary and sufficient condition for Γ-semigroup M to be regular using tri-quasi ideal.

THEOREM 4.11. *M is a regular Γ-semigroup if and only if $B \cap I \cap L \subseteq B\Gamma I\Gamma L$, for any tri-quasi ideal B, ideal I and left ideal L of M.*

PROOF. Suppose M be a regular Γ-semigroup, B, I and L are tri-quasi ideal, ideal and left ideal of M respectively. Let $a \in B \cap I \cap L$. Then $a \in a\Gamma M\Gamma a$, since M is regular. Thus

$$a \in a\Gamma M\Gamma a \subseteq a\Gamma M\Gamma a\Gamma M\Gamma a \subseteq B\Gamma I\Gamma L.$$

Hence $B \cap I \cap L \subseteq B\Gamma I\Gamma L$.

Conversely suppose that $B \cap I \cap L \subseteq B\Gamma I\Gamma L$, for any tri-quasi ideal B, ideal I and left ideal L of M. Let R be a right ideal and L be a left ideal of M. Then by assumption, $R \cap L = R \cap M \cap L \subseteq R\Gamma M\Gamma L \subseteq R\Gamma L$. We have $R\Gamma L \subseteq R$, $R\Gamma L \subseteq L$. Therefore $R\Gamma L \subseteq R \cap L$. Hence $R \cap L = R\Gamma L$. Thus M is a regular Γ-semigroup. □

COROLLARY 4.2. *M is a regular Γ-semigroup if and only if $B \cap I \cap R \subseteq R\Gamma I\Gamma B$, for any tri-quasi ideal B, ideal I and right ideal R of M.*

THEOREM 4.12. *Let M be a regular Γ-semigroup. If B is a tri-quasi idealof M and B is itself a regular $\Gamma-subsemigroup$ of M then any tri-quasi ideal of B is a tri-quasi ideal of M.*

PROOF. Suppose A is a tri-quasi ideal of B and B is a tri-quasi ideal of M. Then by Theorem 4.9, we have $B\Gamma B\Gamma M\Gamma B\Gamma B = B$ and $A\Gamma A\Gamma B\Gamma A\Gamma A = A$. Thus $A\Gamma A\Gamma M\Gamma A\Gamma A \subseteq A\Gamma A\Gamma B\Gamma B\Gamma M\Gamma B\Gamma B\Gamma A\Gamma A$, since $A \subset A\Gamma B$ and $A \subset B\Gamma A$. So, $EB\Gamma B\Gamma M\Gamma B\Gamma B \subseteq A\Gamma A\Gamma B\Gamma A\Gamma A = A$. Hence $A\Gamma A\Gamma M\Gamma A\Gamma A \subseteq A$. Therefore A is a tri-quasi ideal of M. □

5. Conclusion

As a further generalization of ideals, we introduced the notion of tri-quasi ideal of Γ-semigroup as a generalization of ideal ,left ideal, right ideal, bi-ideal,quasi ideal and interior ideal of Γ-semigroup and studied some of their properties. We introduced the notion of tri-quasi simple Γ-semigroup and characterized the tri-quasi simple Γ-semigroup, regular Γ-semigroup using tri-quasi ideals of Γ-semigroup.In continuity of this paper, we study prime tri-quasi ideals, maximal and minimal tri-quasi ideals of Γ-semigroup.

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Department of Mathematics, GITAM University, Visakhapatnam- 530 045, A.P., India.

E-mail address: `mmarapureddy@gmail.com`