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APPROXIMATE FIXED POINT PROPERTY FOR DIGITAL TREES AND PRODUCTS

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Abstract. We add to our knowledge of the approximate fixed point property (AFPP) in digital topology. We show that a digital image that is a tree has the AFPP. Given two digital images (X, κ) and (Y, λ) that have the approximate fixed point property, does their Cartesian product have the AFPP? We explore conditions that yield an affirmative answer. A general answer to this question is not known at the current writing.

1. Introduction

The study of fixed points of continuous functions $f : X \to X$ has long captured the attention of researchers in many areas of mathematics. It was introduced in digital topology by A. Rosenfeld [**12**]. Rosenfeld showed that even a digital image as simple as a digital interval need not have a fixed point property (FPP), but does have an "almost" or "approximate" fixed point property (AFPP) (precisely defined in [**6**]). It was shown in [**6**] that among digital images, only singletons have the FPP; perhaps as a consequence, attention shifted to the AFPP for digital images in such papers as [**3, 4, 5, 6, 9, 11**].

In this paper, we continue to study the AFPP for digital images; in particular, for trees and for Cartesian products.

2. Preliminaries

Much of this section is quoted or paraphrased from the references, especially [**4**].

We use $\mathbb{Z}$ to indicate the set of integers, $\mathbb{N}$ for the set of natural numbers, and $\mathbb{N}^*$ for the set of nonnegative integers.

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2.1. Adjacencies. The c_u-adjacencies are commonly used. Let $x, y \in \mathbb{Z}^n$, $x \neq y$, where we consider these points as n-tuples of integers:

$$x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n).$$

Let $u \in \mathbb{Z}$, $1 \leqslant u \leqslant n$. We say x and y are *c_u-adjacent* if

- there are at most u indices i for which $|x_i - y_i| = 1$, and
- for all indices j such that $|x_j - y_j| \neq 1$ we have $x_j = y_j$.

Often, a c_u-adjacency is denoted by the number of points adjacent to a given point in $\mathbb{Z}^n$ using this adjacency. E.g.,

- In $\mathbb{Z}^1$, c_1-adjacency is 2-adjacency.
- In $\mathbb{Z}^2$, c_1-adjacency is 4-adjacency and c_2-adjacency is 8-adjacency.
- In $\mathbb{Z}^3$, c_1-adjacency is 6-adjacency, c_2-adjacency is 18-adjacency, and c_3-adjacency is 26-adjacency.

For κ-adjacent x, y, we write $x \leftrightarrow_\kappa y$ or $x \leftrightarrow y$ when κ is understood. We write $x \leftrightarroweq_\kappa y$ or $x \leftrightarroweq y$ to mean that either $x \leftrightarrow_\kappa y$ or $x = y$.

We say $\{x_n\}_{n=0}^k \subset (X, \kappa)$ is a *κ-path* (or a *path* if κ is understood) from x_0 to x_k if $x_i \leftrightarrow_\kappa x_{i+1}$ for $i \in \{0, \ldots, k-1\}$, and k is the *length* of the path.

A subset Y of a digital image (X, κ) is *κ-connected* [**12**], or *connected* when κ is understood, if for every pair of points $a, b \in Y$ there exists a κ-path in Y from a to b.

We define

$$N(X, \kappa, x) = \{y \in X \mid x \leftrightarrow_\kappa y\},$$

$$N^*(X, \kappa, x) = \{y \in X \mid x \leftrightarroweq_\kappa y\} = N(X, \kappa, x) \cup \{x\}.$$

Definition 2.1. ([**1**]) Given digital images (X, κ) and (Y, λ), the *normal product adjacency* $NP(\kappa, \lambda)$ for the Cartesian product $X \times Y$ is as follows. For $x, x' \in X$, $y, y' \in Y$, we have $(x, y) \leftrightarrow_{NP(\kappa,\lambda)} (x', y')$ if

- $x \leftrightarrow_\kappa x'$ and $y = y'$, or
- $x = x'$ and $y \leftrightarrow_\lambda y'$, or
- $x \leftrightarrow_\kappa x'$ and $y \leftrightarrow_\lambda y'$. $\square$

2.2. Digitally continuous functions. The following generalizes a definition of [**12**].

Definition 2.2. ([**2**]) Let (X, κ) and (Y, λ) be digital images. A function $f : X \to Y$ is *(κ, λ)-continuous* if for every κ-connected $A \subset X$ we have that $f(A)$ is a λ-connected subset of Y. If $(X, \kappa) = (Y, \lambda)$, we say such a function is *κ-continuous*, denoted $f \in C(X, \kappa)$. $\square$

When the adjacency relations are understood, we will simply say that f is *continuous*. Continuity can be expressed in terms of adjacency of points:

Theorem 2.1 ([**12, 2**]). *A single-valued function $f : X \to Y$ is continuous if and only if $x \leftrightarrow x'$ in X implies $f(x) \leftrightarroweq f(x')$.* $\square$

Similar notions are referred to as *immersions*, *gradually varied operators*, and *gradually varied mappings* in [**7, 8**].

Composition and restriction preserve continuity, in the sense of the following assertions.

THEOREM 2.2 ([**2**]). *Let (X, κ), (Y, λ), and (Z, μ) be digital images. Let $f : X \to Y$ be (κ, λ)-continuous and let $g : Y \to Z$ be (λ, μ)-continuous. Then $g \circ f : X \to Z$ is (κ, μ)-continuous.* □

THEOREM 2.3 ([**9**]). *Let (X, κ) and (Y, λ) be digital images. Let $f : X \to Y$ be (κ, λ)-continuous.*

- *Let $A \subset X$. Then $f|_A : A \to Y$ is (κ, λ)-continuous.*
- *$f : X \to f(X)$ is (κ, λ)-continuous.* □

Given $X = \Pi_{i=1}^{v} X_i$, we denote throughout this paper the projection onto the i^{th} factor by p_i; i.e., $p_i : X \to X_i$ is defined by $p_i(x_1, \ldots, x_v) = x_i$, where $x_j \in X_j$.

THEOREM 2.4 ([**10**]). *Given digital images (X, κ) and (Y, λ), the projection maps p_1 and p_2 are $(NP(\kappa, \lambda), \kappa)$-continuous and $(NP(\kappa, \lambda), \lambda)$-continuous, respectively.* □

2.3. Approximate fixed points. Let $f \in C(X, \kappa)$ and let $x \in X$. We say

- x is a *fixed point* of f if $f(x) = x$;
- If $f(x) \leftrightarrows_\kappa x$, then x is an *almost fixed point* [**12, 13**] or *approximate fixed point* [**5**] of (f, κ).
- A digital image (X, κ) has the *approximate fixed point property* (AFPP) [**5**] if for every $g \in C(X, \kappa)$ there is an approximate fixed point of g.

REMARK 2.1. What we call the AFPP was denoted in [**4**] as the $AFPP_S$ in order to distinguish it from its more general version for multivalued continuous functions, denoted $AFPP_M$. In this paper, we discuss only single-valued continuous functions, so we use the simpler notation.

THEOREM 2.5 ([**5**]). *Let X and Y be digital images such that (X, κ) and (Y, λ) are isomorphic. If (X, κ) has the AFPP, then (Y, λ) has the AFPP.* □

THEOREM 2.6 ([**5**]). *Let X and Y be digital images such that Y is a κ-retract of X. If (X, κ) has the AFPP, then (Y, κ) has the AFPP.* □

3. Trees

A *tree* is a triple $T = (X, \kappa, s)$, where $s \in X$ and (X, κ) is a connected graph that is acyclic, i.e., lacking any subgraph isomorphic to a cycle of more than 2 points. The vertex s is the *root*. Given $x \leftrightarrow_\kappa y$ in X, we say x is the *parent of* y, and y is a *child of* x, if $x \leftrightarrow_\kappa y$ and the unique shortest path from y to the root contains x. Every vertex of the tree, except the root, has a unique parent vertex. A vertex, in general, may have multiple children. We define, recursively, a *descendant* of x in a tree $T = (X, \kappa, r)$ as follows: $y \in X$ is a descendant of $x \in X$ if y is a child of x or y is a descendant of a child of x.

We will use the following.

Proposition 3.1. *Let (X, κ) have the AFPP. Let $X' = X \cup \{x_0\}$, where $x_0 \notin X$, and let there be a κ-retraction $r : X' \to X$ such that $N^*(X', \kappa, x_0) \subset N^*(X', \kappa, r(x_0))$. Then (X', κ) has the AFPP.*

Proof. Let $f \in C(X', \kappa)$. Then $g = r \circ f|_X \in C(X, \kappa)$. Therefore, there is an approximate fixed point $y \in X$ of g.

- If $f(y) \in X$, then $f(y) = g(y) \leftrightarrows_\kappa y$, as desired.
- Otherwise, $f(y) = x_0$ and $y \leftrightarrows_\kappa g(y) = r(x_0)$. The continuity of f implies $f(g(y)) \leftrightarrows_\kappa f(y) = x_0$, hence

  $$f(g(y)) \in N^*(X', \kappa, x_0) \subset N^*(X', \kappa, r(x_0)) = N^*(X', \kappa, g(y)).$$

  So $g(y)$ is an approximate fixed point of f.

In either case, f has an approximate fixed point. Since f was taken as an arbitrary member of $C(X', \kappa)$, the assertion follows. $\square$

Theorem 3.1. *A digital image (T, κ) that is a tree has the AFPP.*

Proof. We argue by induction on $\#T$, the number of vertices in T. The assertion is trivial for $\#T = 1$.

Suppose $k \in \mathbb{N}$ such that the assertion is correct for all digital trees T satisfying $\#T \leqslant k$. Now let (T, κ) be a digital tree with $\#T = k + 1$. Let $v_0 \in T$ be a leaf of T, with $v_1 \in T$ as the parent of v_0. Then $(T \setminus \{v_0\}, \kappa)$ is a digital tree of k points. The function $r : T \to T \setminus \{v_0\}$ defined by $r(v_0) = v_1$, $r(x) = x$ for $x \neq v_0$, is clearly a κ-retraction, and $N^*(T, \kappa, v_0) = \{v_0, v_1\} \subset N^*(T, \kappa, r(v_0))$. It follows from the inductive hypothesis and Proposition 3.1 that (T, κ) has the $AFPP$. This completes the induction. $\square$

4. Cartesian products

In this section, we demonstrate an affirmative response to the following question.

Question 4.1. ([**4**]) Let $X = \Pi_{i=1}^{v}[a_i, b_i]_\mathbb{Z}$, where for at least 2 indices i we have $b_i > a_i$. Does (X, c_v) have the AFPP?

Several authors have written that this question was answered by Theorem 4.1 of [**12**]. However, it wasn't, as observed in [**4**]:

> A. Rosenfeld's paper [**12**] states the following as its Theorem 4.1 (quoted verbatim).
>
> > Let I be a digital picture, and let f be a continuous function from I into I; then there exists a point $P \in I$ such that $f(P) = P$ or is a neighbor or diagonal neighbor of P.
>
> Several subsequent papers have incorrectly concluded that this result implies that I with some c_u adjacency has the $AFPP$. By *digital picture* Rosenfeld means a digital cube, $I = [0, n]_\mathbb{Z}^v$.

By a "continuous function" he means a (c_1, c_1)-continuous function; by "a neighbor or diagonal neighbor of P" he means a c_v-adjacent point.

A partial solution to this problem is given in the following (restated here in our terminology), which is Theorem 1 of [**11**]. The "proof" in [**11**] has multiple errors; a correct proof is given in [**4**].

Theorem 4.1. *Let $X = [-1, 1]_{\mathbb{Z}}^v$ and $1 \leqslant u \leqslant v$. Then (X, c_u) has the AFPP if and only if $u = v$.* $\square$

We make use of the following.

Theorem 4.2 ([**6**]). *For $X \subset \mathbb{Z}^m$ and $Y \subset Z^n$, $NP(c_m, c_n) = c_{m+n}$, i.e., given $x, x' \in X$, $y, y' \in Y$,*

$$(x, y) \leftrightarrow_{NP(c_m,c_n)} (x', y') \text{ if and only if } (x, y) \leftrightarrow_{c_{m+n}} (x', y'). \quad \square$$

Remark 4.1. It is shown in [**6**] that for $m \leqslant M$, $n \leqslant N$, $m + n < M + N$, if $X \subset \mathbb{Z}^M$ and $Y \subset Z^N$, then we can have $NP(c_m, c_n) \neq c_{m+n}$.

Theorem 4.3. *Let (X, κ) be a digital image with the AFPP. Then the image $(X \times [0, n]_{\mathbb{Z}}, NP(\kappa, c_1))$ has the AFPP.*

Proof. We argue by induction on n.

For $n = 0$ we argue as follows. Since

$$(X \times [0, 0]_{\mathbb{Z}}, NP(\kappa, c_1)) = (X \times \{0\}, NP(\kappa, c_1))$$

is isomorphic to (X, κ), it follows from Theorem 2.5 that $(X \times [0, 0]_{\mathbb{Z}}, NP(\kappa, c_1))$ has the AFPP.

Now suppose $k \in \mathbb{N}^*$ and $(X \times [0, k]_{\mathbb{Z}}, NP(\kappa, c_1))$ has the AFPP. To complete the induction, we must show that $(X \times [0, k + 1]_{\mathbb{Z}}, NP(\kappa, c_1))$ has the AFPP. Let $r : X \times [0, k + 1]_{\mathbb{Z}} \to X \times [0, k]_{\mathbb{Z}}$ be defined by

$$r(x, t) = \begin{cases} (x, t) & \text{if } 0 \leqslant t \leqslant k; \\ (x, k) & \text{if } t = k + 1. \end{cases}$$

Clearly, r is $NP(\kappa, c_1)$-continuous and is a retraction.

Let $f \in C(X \times [0, k + 1]_{\mathbb{Z}}, NP(\kappa, c_1))$. Let $g : X \times [0, k]_{\mathbb{Z}} \to X \times [0, k]_{\mathbb{Z}}$ be defined by $g(x, t) = r \circ f \circ I(x, t)$, where $I : X \times [0, k]_{\mathbb{Z}} \to X \times [0, k + 1]_{\mathbb{Z}}$ is the inclusion function. By the inductive hypothesis, g has an approximate fixed point; i.e., there exists $p = (x_0, t_0) \in X \times [0, k]_{\mathbb{Z}}$ such that

$$p \leftrightarroweq_{NP(\kappa,c_1)} g(p). \tag{4.1}$$

- If $f(p) \in X \times [0, k]_{\mathbb{Z}}$ then

  $$p \leftrightarroweq_{NP(\kappa,c_1)} g(p) = f(p),$$

  so p is an approximate fixed point of f.

- Otherwise, we have that for some $x_1 \in X$, $f(p) = (x_1, k+1)$ and $g(p) = (x_1, k)$. Let $p_1 : X \times [0, k+1]_{\mathbb{Z}} \to X$ and $p_2 : X \times [0, k+1]_{\mathbb{Z}} \to [0, k+1]_{\mathbb{Z}}$ be the projections defined for $x \in X$, $t \in [0, k+1]_{\mathbb{Z}}$ by

  $$p_1(x,t) = x, \;\; p_2(x,t) = t.$$

  By Theorems 2.2 and 2.4, the functions $f \circ g$, $p_1 \circ f$, $p_1 \circ f \circ g$, $p_2 \circ f$, $p_2 \circ g$, and $p_2 \circ f \circ g$ are all continuous. By continuity of f and (4.1), $f(g(p)) \leftrightarroweq f(p)$, so

  $$p_1(f(g(p))) \leftrightarroweq_{\kappa} p_1(f(p)) = x_1 = p_1(g(p)) \tag{4.2}$$

  and $p_2(f(g(p))) \leftrightarroweq_{c_1} p_2(f(p)) = k+1$, so $p_2(f(g(p))) \in \{k, k+1\}$, hence

  $$p_2(f(g(p))) \leftrightarroweq_{c_1} p_2(g(p)). \tag{4.3}$$

  By (4.2) and (4.3), $g(p)$ is an approximate fixed point of f.

In either case, f has an approximate fixed point. This completes the induction argument. □

LEMMA 4.1. *Let (X, κ) be a digital image. Consider (Y, c_v), where $Y = [0, n]_{\mathbb{Z}}^v$. For $X \times Y \times [0, n]_{\mathbb{Z}}$, $NP(NP(\kappa, c_k)), c_1) = NP(\kappa, c_{k+1})$.*

PROOF. Let $x, x' \in X$, $y, y' \in Y$, $t, t' \in [0, n]_{\mathbb{Z}}$, where

$$y = (y_1, \ldots, y_v), \;\; y' = (y_1', \ldots, y_v'),$$

$y_i, y_i' \in [0, n]_{\mathbb{Z}}$ for $i = 1, \ldots, v$, such that $(x, y, t) \neq (x', y', t')$. Then

$$(x, y, t) \leftrightarrow_{NP(NP(\kappa, c_v), c_1)} (x', y', t') \;\text{ if and only if}$$

$$(x, y) \leftrightarroweq_{NP(\kappa, c_v)} (x', y') \text{ and } t \leftrightarroweq_{c_1} t' \;\text{ if and only if}$$

$$x \leftrightarroweq_{\kappa} x' \text{ and } y \leftrightarroweq_{c_v} y' \text{ and } t \leftrightarroweq_{c_1} t' \;\text{ if and only if}$$

$$x \leftrightarroweq_{\kappa} x' \text{ and } (y, t) \leftrightarroweq_{NP(c_v, c_1)} (y', t') \;\text{ if and only if}$$

(by Theorem 4.2)

$$x \leftrightarroweq_{\kappa} x' \text{ and } (y, t) \leftrightarroweq_{c_{v+1}} (y', t') \;\text{ if and only if}$$

$(x, y, t) \leftrightarroweq_{NP(\kappa, c_{v+1})} (x', y', t')$. The assertion is established. □

THEOREM 4.4. *Let (X, κ) be a digital image with the AFPP. Let $Y = [0, n]_{\mathbb{Z}}^v$. Then the image $(X \times Y, NP(\kappa, c_v))$ has the AFPP.*

PROOF. We argue by induction on v. For $v = 1$, the assertion is correct by Theorem 4.3.

Suppose, for some $k \in \mathbb{N}^*$, for $Y = \Pi_{i=1}^k [0, n]_{\mathbb{Z}}$, $(X \times Y, NP(\kappa, c_k))$ has the AFPP. Then by Theorem 4.3, $(X \times Y \times [0, n]_{\mathbb{Z}}, NP(NP(\kappa, c_k)), c_1)$ has the AFPP. Note that $X \times Y \times [0, n]_{\mathbb{Z}} = X \times [0, n]_{\mathbb{Z}}^{k+1}$, and, by Lemma 4.1, that $NP(NP(\kappa, c_k), c_1) = NP(\kappa, c_{k+1})$. This completes our induction. □

THEOREM 4.5. *Let (X, κ) be a digital image with the AFPP. Then the image $(X \times \Pi_{i=1}^v [a_i, b_i]_{\mathbb{Z}}, NP(\kappa, c_v))$ has the AFPP.*

PROOF. This follows from Theorems 4.4, 2.5, and 2.6, as the image

$$(X \times \Pi_{i=1}^{v}[a_i, b_i]_{\mathbb{Z}}, NP(\kappa, c_v))$$

is clearly isomorphic to an $NP(\kappa, c_v)$-retract of $X \times [0, n]^v$ for some n. □

THEOREM 4.6 ([**12**]). *The digital image* $([a, b]_{\mathbb{Z}}, c_1)$ *has the AFPP.*

THEOREM 4.7 ([**4**]). *Let* $X \subset \mathbb{Z}^v$ *be such that* X *has a subset* $Y = \Pi_{i=1}^{v}[a_i, b_i]_{\mathbb{Z}}$*, where* $v > 1$*; for all indices* i*,* $b_i \in \{a_i, a_i + 1\}$*; and, for at least 2 indices* i*,* $b_i = a_i + 1$*. Then* (X, c_u) *fails to have the AFPP for* $1 \leqslant u < v$*.* □

As noted in [**4**], Theorem 4.7 states a severe limitation on the AFPP for digital images $X \subset \mathbb{Z}^v$ and the c_u adjacency, where $1 \leqslant u < v$. We have the following.

THEOREM 4.8. *For* $1 \leqslant u \leqslant v$*,* $(\Pi_{i=1}^{v}[a_i, b_i]_{\mathbb{Z}}, c_u)$ *has the AFPP if and only if* $u = v$*.*

PROOF. For $u < v$, the assertion comes from Theorem 4.7. Now consider the case $u = v$. For $v = 1$, the assertion follows from Theorem 4.6. For $v > 1$, Theorem 4.2 lets us conclude that

$$(\Pi_{i=1}^{v}[a_i, b_i]_{\mathbb{Z}}, c_v) = ([a_1, b_1]_{\mathbb{Z}} \times \Pi_{i=2}^{v}[a_i, b_i]_{\mathbb{Z}}, NP(c_1, c_{v-1})).$$

The assertion follows from Theorem 4.5. □

5. Further remarks

We have shown that a digital image that is a tree has the AFPP.

A general answer to the question posed in the abstract is not known at this writing. We have shown that given a digital image (X, κ) with the AFPP, then $(X \times \Pi_{i=1}^{v}[a_i, b_i]_{\mathbb{Z}}, NP(\kappa, c_v))$ has the AFPP. It follows that $(\Pi_{i=1}^{v}[a_i, b_i]_{\mathbb{Z}}, c_v)$ has the AFPP.

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