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WEAK-INTERIOR IDEALS OF Γ-SEMIRINGS

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Abstract. In this paper, as a further generalization of ideals, we introduce the notion of weak -interior ideal of a $\Gamma-$semiring. The properties of weak-interior ideals are studied and various characterizations of them are given.

1. Introduction

Ideals play an important role in advance studies and uses of algebraic structures. Generalization of ideals in algebraic structures is necessary for further study of algebraic structures. Many mathematicians introduced various generalizations of the concept of ideals in algebraic structures, establishing important results and characterizations of algebraic structures. The notion of a semiring was introduced by Vandiver [**33**] in 1934, but semirings had appeared in earlier studies on the theory of ideals of rings. The concept of semiring generalizes the notion of ring as well as that of distributive lattice. Semirings are structurally similar to semigroups than to rings.

In 1995, M. M. K. Rao [**15, 16, 17, 18**] introduced the notion of Γ-semiring as a generalization of Γ-ring, ternary semiring and semiring. As a generalization of ring, the notion of Γ-ring was introduced by Nobusawa [**29**] in 1964. In 1981, Sen [**30**] introduced the notion of a Γ-semigroup as a generalization of semigroup. The notion of a ternary algebraic system was introduced by Lehmer [**13**] in 1932. M. M. K. Rao and Venkateswarlu [**26, 27, 28**] studied regular Γ-incline, field Γ-semiring and derivations. Lister [**14**] introduced ternary ring. Dutta and Sardar [**2**] introduced the notion of operator semirings of Γ-semiring.

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Key words and phrases. bi-quasi-interior ideal, quasi-interior ideal, bi-interior ideal, bi-quasi ideal, bi-ideal, quasi ideal, interior ideal, regular Γ-semiring, weak -interior ideal, tri-ideal, weak-interior simple Γ-semiring.

Henriksen [**4**] and Shabir et al. [**31**] studied ideals in semirings. We know that the notion of an one sided ideal of any algebraic structure is a generalization of notion of an ideal. The quasi ideals are generalization of left ideal and right ideal whereas the bi-ideals are generalization of quasi ideals.

In 1952, the concept of bi-ideals was introduced by Good and Hughes [**3**] for semigroups. The notion of bi-ideals in rings and semirings were introduced by Lajos and Szasz [**11, 12**]. Bi-ideal is a special case of $(m-n)$ ideal. Steinfeld [**32**] first introduced the notion of quasi ideals for semigroups and then for rings. Iseki [**7, 5, 6, 8**] introduced the concept of quasi ideal for a semiring. Quasi ideals, bi-ideals in $\Gamma-$semirings studied by Jagtap and Pawar [**9, 10**]. M. M. K. Rao [**19, 20, 21, 22, 24, 23, 25, 26, 27, 28**] introduced the notion of left (right) bi-quasi ideal, the notion of bi-interior ideal and the notion of bi weak-interior ideal of Γ-semiring as a generalization of ideal of Γ-semiring, studied their properties and characterized the left bi-quasi simple Γ-semiring and regular Γ-semiring.

In this paper, we introduce the notion of weak-interior ideals as a generalization of quasi ideal, interior ideal, left(right) ideal and ideal of Γ-semiring and study the properties of weak-interior ideals of Γ-semiring.

2. Preliminaries

In this section we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1. ([**1**]) A set S together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called semiring provided

(i) Addition is a commutative operation.

(ii) Multiplication distributes over addition both from the left and from the right.

(iii) There exists $0 \in S$ such that $x+0=x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$.

Definition 2.2. ([**15**]) Let $(M,+)$ and $(\Gamma,+)$ be commutative semigroups. Then we call M a Γ-semiring, if there exists a mapping $M \times \Gamma \times M \to M$, the image of (x,α,y) will be denoted by $x\alpha y$, $x,y \in M$, $\alpha \in \Gamma$ satisfying the following axioms for all $x,y,z \in M$ and $\alpha,\beta \in \Gamma$

(i) $x\alpha(y+z) = x\alpha y + x\alpha z$

(ii) $(x+y)\alpha z = x\alpha z + y\alpha z$

(iii) $x(\alpha+\beta)y = x\alpha y + x\beta y$

(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Every semiring R is a Γ-semiring with $\Gamma = R$ and ternary operation $x\gamma y$ defined as the usual semiring multiplication.

Example 2.1. Let S be a semiring and $M_{p,q}(S)$ denote the additive abelian semigroup of all $p \times q$ matrices with identity element whose entries are from S. Then $M_{p,q}(S)$ is a Γ-semiring with $\Gamma = M_{p,q}(S)$ ternary operation is defined by $x\alpha z = x(\alpha^t)z$ as the usual matrix multiplication, where α^t denote the transpose of the matrix α; for all x, y and $\alpha \in M_{p,q}(S)$.

Example 2.2. Let M be the set of all natural numbers. Then $(M,\ max,\ min)$ is a semiring. If $\Gamma = M$, then M is a $\Gamma-$semiring.

Example 2.3. Let M be the additive semigroup of all $m \times n$ matrices over the set of non negative rational numbers and Γ be the additive semigroup of all $n \times m$ matrices over the set of non negative integers. Then with respect to usual matrix multiplication M is a Γ-semiring.

Definition 2.3. A Γ-semiring M is said to be commutative Γ-semiring if $x\alpha y = y\alpha x$, for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 2.4. A Γ-semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha\ 0 = 0$, for all $x \in M, \alpha \in \Gamma$.

Definition 2.5. An element $a \in \Gamma$-semiring M is said to be idempotent if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$.

Definition 2.6. A non-empty subset A of a Γ-semiring M is called

(i) a Γ-subsemiring of M if $(A, +)$ is a subsemigroup of $(M, +)$ and $A\Gamma A \subseteq A$.
(ii) a quasi ideal of M if A is a Γ-subsemiring of M and $A\Gamma M \cap M\Gamma A \subseteq A$.
(iii) a bi-ideal of M if A is a Γ-subsemiring of M and $A\Gamma M\Gamma A \subseteq A$.
(iv) an interior ideal of M if A is a Γ-subsemiring of M and $M\Gamma A\Gamma M \subseteq A$.
(v) a left (right) ideal of M if A is a Γ-subsemiring of M and $M\Gamma A \subseteq A(A\Gamma M \subseteq A)$.
(vi) an ideal if A is a Γ-subsemiring of $M, A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$.
(vii) a $k-$ideal if A is a Γ-subsemiring of $M, A\Gamma M \subseteq A, M\Gamma A \subseteq A$ and $x \in M,\ x + y \in A, y \in A$ then $x \in A$.

Definition 2.7. ([**20**]) Let M be a Γ-semiring. A non-empty subset B of M is said to be bi-interior ideal of M if B is a Γ-subsemiring of M and $M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B$.

Definition 2.8. ([**22**]) Let M be a Γ-semiring. A non-empty subset L of M is said to be left bi-quasi ideal (right bi-quasi ideal) of M if L is a subsemigroup of $(M, +)$ and $M\Gamma L \cap L\Gamma M\Gamma L \subseteq L$ (res. $L\Gamma M \cap L\Gamma M\Gamma L \subseteq L$).

Definition 2.9. ([**23**]) Let M be a Γ-semiring. A non-empty subset B of M is said to be bi-quasi-interior ideal of M if B is a Γ-subsemiring of M and $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$.

Definition 2.10. ([**24**]) Let M be a Γ-semiring. L is said to be bi-quasi ideal of M if it is both a left bi-quasi ideal and a right bi-quasi ideal of M.

Definition 2.11. ([**24**]) A Γ-semiring M is called a left bi-quasi simple Γ-semiring if M has no left bi-quasi ideal other than M itself.

Remark 2.1. A bi-quasi-interior ideal of a Γ-semiring M need not be bi-ideal,quasi-ideal, interior ideal bi-interior ideal and bi-quasi ideals of Γ-semiring M.

DEFINITION 2.12. Let M be a Γ-semiring. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

DEFINITION 2.13. In a Γ-semiring M with unity 1, an element $a \in M$ is said to be left invertible (right invertible) if there exist $b \in M, \alpha \in \Gamma$ such that $b\alpha a = 1(a\alpha b = 1)$.

DEFINITION 2.14. In a Γ-semiring M with unity 1, an element $a \in M$ is said to be invertible if there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 1$.

DEFINITION 2.15. A Γ-semiring M is called a division Γ-semiring if for each non-zero element of M has multiplication inverse.

EXAMPLE 2.4. (i) Let Q be the set of all rational numbers,

$$M = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in Q \right\}$$

be the additive semigroup of M matrices and $\Gamma = M$. The ternary operation $A\alpha B$ is defined as usual matrix multiplication of A, α, B, for all $A, \alpha, B \in M$.Then M is a Γ-semiring

(a) If $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}$ then R is a quasi ideal of a Γ-semiring M and R is neither a left ideal nor a right ideal.

(b) If $S = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid 0 \neq a \in Q \right\}$ then S is a bi-ideal of Γ-semiring M.

(ii) If $M = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in Q \right\}$ and $\Gamma = M$ then M is a Γ-semiring with respect to usual addition of matrices and ternary operation is defined as usual matrix multiplication and $A = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}$. Then A is not a bi-ideal of Γ-semiring M.

EXAMPLE 2.5. Let N be a the set of all even natural numbers and $\Gamma = 4\mathcal{N}$ be additive abelian semigroups. Tennary operation is defined as $(x, \alpha, y) \to x + \alpha + y$, where $+$ is the usual addition of integers. Then N is a $\Gamma-$semiring. A subset $I = 4N$ of N is a bi-quasi-interior ideal of N but not bi-ideal,quasi-ideal, interior ideal, bi-interior ideal and bi-quasi ideal of Γ-semiring N.

3. Weak-interior ideals of Γ-semirings

In this section we introduce the notion of weak-interior ideal as a generalization of quasi-ideal and interior ideal of Γ-semiring and study the properties of weak-interior ideal of Γ-semiring. Throughout this paper$M$ is a Γ-semiring with unity element.

DEFINITION 3.1. A non-empty subset B of a Γ-semiring M is said to be left weak-interior ideal of M if B is a Γ-subsemiring of M and $M\Gamma B\Gamma B \subseteq B$.

DEFINITION 3.2. A non-empty subset B of a Γ-semiring M is said to be right weak-interior ideal of M if B is a Γ-subsemiring of M and $B\Gamma B\Gamma M \subseteq B$.

Definition 3.3. A non-empty subset B of a Γ-semiring M is said to be weak-interior ideal of M if B is a Γ-subsemiring of M and $B$is left and right weak-interior ideal of M.

Remark 3.1. A weak-interior ideal of a Γ-semiring M need not be quasi-ideal, interior ideal, bi-interior ideal. and bi-quasi ideal of Γ-semiring M.

Example 3.1. Let Q be the set of all rational numbers,

$$M = \left\{ \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \mid b, d \in Q \right\}$$

be the additive semigroup of M matrices and $\Gamma = M$. The ternary operation $A\alpha B$ is defined as usual matrix multiplication of A, α, B, for all $A, \alpha, B \in M$.Then M is a Γ-semiring If $R = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid 0 \neq b \in Q \right\}$ then R is a left weak interior ideal of the Γ-semiring M and R is neither a left ideal nor a right ideal , not a weak interior ideal and not a interior ideal of the Γ-semiring M .

In the following theorem, we mention some important properties and we omit the proofs since they are straight forward.

Theorem 3.1. *Let M be a Γ-semring. Then the following are hold.*

1. *Every left ideal is a left weak-interior ideal of M.*
2. *Every right ideal is a right weak-interior ideal of M.*
3. *Every quasi ideal is a weak-interior ideal of M.*
4. *Every ideal is a weak-interior ideal of M.*
5. *If B is a weak-interior ideal and T is a Γ-subsemiring of M then $B \cap T$ is a weak-interior ideal of ring M.*
6. *If L is a left ideal and R is a right ideal of a Γ-semiring M then $B = L\Gamma R$ is a weak-interior ideal of M.*
7. *Let M be a Γ-semiring and B be a Γ-subsemiring of M. If $M\Gamma M\Gamma B \subseteq B$ then B is a left weak-interior ideal of M.*
8. *Let M be a Γ-semiring and B be a Γ-subsemiring of M. If $M\Gamma M\Gamma B \subseteq B$ and $B\Gamma M\Gamma M \subseteq B$ then B is a weak-interior ideal of M.*

Theorem 3.2. *If B be an interior ideal of a Γ-semiring M, then B is a left weak-interior ideal of M.*

Proof. Suppose that B is an interior ideal of the Γ-semiring M. Then

$$M\Gamma B\Gamma B \subseteq M\Gamma B\Gamma M \subseteq B.$$

Hence B is a left weak-interior ideal of M. □

Corollary 3.1. *If B be an interior ideal of a Γ-semiring M, then B is a right weak-interior ideal of M.*

Corollary 3.2. *If B be an interior ideal of a Γ-semiring M, then B is a weak-interior ideal of M.*

THEOREM 3.3. *Let M be a Γ-semiring and B be a Γ-subsemiring of M. B is a left weak-interior ideal of M if and only if there exists left ideal L such that $L\Gamma L \subseteq B \subseteq L$.*

PROOF. Suppose B is a left weak-interior ideal of the Γ-semiring$M$. Then $M\Gamma B\Gamma B \subseteq B$. Let $L = M\Gamma B$. Then L is a left ideal of M. Therefore $L\Gamma L \subseteq B \subseteq L$.

Conversely suppose that there is a left ideal L of M such that $L\Gamma L \subseteq B \subseteq L$. Then

$$M\Gamma B\Gamma B \subseteq M\Gamma(L)\Gamma(L) \subseteq L\Gamma(L) \subseteq B.$$

Hence B is a left weak-interior ideal of M. □

COROLLARY 3.3. *Let M be a Γ-semiring and B be a Γ-subsemiring of M. B is a right weak-interior ideal of M if and only if there exist right ideal R such that $R\Gamma R \subseteq B \subseteq R$.*

COROLLARY 3.4. *Let M be a Γ-semiring and B be a Γ-subsemiring of M. B is a weak-interior ideal of M if and only if there exist ideal R such that $R\Gamma R \subseteq B \subseteq R$.*

THEOREM 3.4. *The intersection of a left weak-interior ideal B of a Γ-semiring M and a right ideal A of M is always a left weak-interior ideal of M.*

PROOF. Suppose $C = B \cap A$. Then $M\Gamma C\Gamma C \subseteq M\Gamma B\Gamma B \subseteq B$ and $M\Gamma C\Gamma C \subseteq M\Gamma A\Gamma A \subseteq A$ since A is a left ideal of M. Therefore $M\Gamma C\Gamma C \subseteq B \cap A = C$. Hence the intersection of a left weak-interior ideal B of a Γ-semiring M and a left ideal A of M is always a left weak-interior ideal of M. □

COROLLARY 3.5. *The intersection of a right weak-interior ideal B of a Γ-semiring M with a right ideal A of M is a right weak-interior ideal of M.*

COROLLARY 3.6. *The intersection of a weak-interior ideal B of a Γ-semiring M with an ideal A of M is a weak-interior ideal of M.*

THEOREM 3.5. *Let A and C be left weak-interior ideals of a Γ-semiring M such that $B = A\Gamma C$ is an additively subsemigroup of M. If $A\Gamma C = C\Gamma A$, then B is a left weak-interior ideal of M.*

PROOF. Let A and C be left weak-interior ideals of the Γ-semiring M and $B = A\Gamma C$. Then

$$B\Gamma B = A\Gamma C\Gamma A\Gamma C = A\Gamma C\Gamma C\Gamma C\Gamma A\Gamma C \subseteq A\Gamma C\Gamma M\Gamma C\Gamma M\Gamma C \subseteq A\Gamma C = B.$$

Therefore $B = A\Gamma C$ is a Γ-subsemiring of M. Moreover,

$$M\Gamma B\Gamma B = M\Gamma A\Gamma C\Gamma A\Gamma C \subseteq M\Gamma A\Gamma A\Gamma C \subseteq A\Gamma C = B.$$

Hence B is a left weak-interior ideal of M. □

COROLLARY 3.7. *Let A and C be weak-interior ideals of a Γ-semiring M such that $B = C\Gamma A$ is an additively subsemigroup of M. If $C\Gamma C = C$, then B is a weak-interior ideal of M.*

Theorem 3.6. *Let A and C be Γ-subsemirings of a Γ-semiring M such that $B = A\Gamma C$ is an additively subsemigroup of M. If A is the left ideal of M then B is a weak-interior ideal of M.*

Proof. Let A and C be Γ-subsemirings of M and $B = A\Gamma C$. Suppose A is the left ideal of M. Then $B\Gamma B = A\Gamma C\Gamma A\Gamma C \subseteq A\Gamma C = B$.

$$M\Gamma B\Gamma B = M\Gamma A\Gamma C\Gamma A\Gamma C \subseteq A\Gamma C = B.$$

Hence B is a left weak-interior ideal of M. □

Corollary 3.8. *Let A and C be Γ-subsemirings of a Γ-semiring M and $B = A\Gamma C$. If B is an additively subsemigroup of M and C is a right ideal, then B is a right weak-interior ideal of M.*

Theorem 3.7. *Let M be a Γ-semiring and T be a non-empty subset of M. If $B \subseteq T$ is a Γ- subsemiring of M containing $M\Gamma T\Gamma M\Gamma T$, then B is a left weak-interior ideal of Γ-semiring M.*

Proof. Let B be a Γ-subsemiring of M containing $M\Gamma T\Gamma M\Gamma T$. Then

$$M\Gamma B\Gamma M\Gamma B \subseteq M\Gamma T\Gamma M\Gamma T \subseteq B.$$

Therefore $B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B$. Hence B is a left weak-interior ideal of M. □

Theorem 3.8. *If B is a left weak-interior ideal of Γ-semiring M, $B\Gamma T$ is an additively subsemigroup of M and $T \subseteq B$ then $B\Gamma T$ is a left weak-interior ideal of M.*

Proof. Suppose B is a left weak-interior ideal of the Γ-semiring M, $B\Gamma T$ is an additively subsemigroup of M and $T \subseteq B$. Then $B\Gamma T\Gamma B\Gamma T \subseteq B\Gamma T$. Hence $B\Gamma T$ is a Γ-subsemiring of M. We have

$$M\Gamma B\Gamma T\Gamma B\Gamma T \subseteq M\Gamma B\Gamma B\Gamma T \subseteq B\Gamma T,$$

hence $B\Gamma T$ is a left weak-interior ideal of the Γ-semiring M. □

Theorem 3.9. *Let B be a left weak-interior ideal of a Γ-semiring M and I be a left weak interior ideal of M. Then $B \cap I$ is a left weak-interior ideal of M.*

Proof. Suppose B is a left weak-interior ideal of M and I is a left weak interior ideal of M. Obviously $B \cap I$ is a Γ-subsemiring of M. Then

$$M\Gamma(B\cap I)\Gamma(B\cap I) \subseteq M\Gamma B\Gamma B \subseteq B \quad \text{and} \quad M\Gamma(B\cap I)\Gamma(B\cap I) \subseteq M\Gamma I\Gamma I \subseteq I.$$

Therefore $M\Gamma(B\cap I)\Gamma(B\Gamma I) \subseteq B\cap I$. Hence $B \cap I$ is a left weak-interior ideal of M. □

Theorem 3.10. *Let M be a Γ-semiring and T be a Γ-subsemiring of M. Then every Γ-subsemiring of T containing $M\Gamma T\Gamma T$ is a left weak-interior ideal of M.*

Proof. Let C be a Γ-subsemiring of T containing $M\Gamma T\Gamma M\Gamma T$. Then

$$M\Gamma C\Gamma C \subseteq M\Gamma T\Gamma T \subseteq C.$$

Hence C is a left weak-interior ideal of M . □

Theorem 3.11. *The intersection of a family* $\{B_\lambda \mid \lambda \in A\}$ *of left weak-interior ideals of a* Γ*-semiring* M *is a left weak-interior ideal of* M.

Proof. Let $B = \bigcap\limits_{\lambda\in A} B_\lambda$. Then B is a Γ-subsemiring of M. Since B_λ is a left weak-interior ideal of M, for all $\lambda \in A$ we have $M\Gamma B_\lambda \Gamma B_\lambda \subseteq B_\lambda$, hence $M\Gamma \cap B_\lambda \cap B_\lambda \subseteq \cap B_\lambda$. It follows that, $M\Gamma B\Gamma B \subseteq B$. Hence B is a left weak-interior ideal of M. □

Corollary 3.9. *The intersection of a family* $\{B_\lambda \mid \lambda \in A\}$ *of right weak-interior ideals of a* Γ*-semiring* M *is a right weak-interior ideal of* M.

Corollary 3.10. *The intersection of a family* $\{B_\lambda \mid \lambda \in A\}$ *of weak-interior ideals of a* Γ*-semiring* M *is a weak-interior ideal of* M.

Theorem 3.12. *Let* B *be a left weak-interior ideal of a* Γ*-semiring* M *and* e *be a* $\beta-$*idempotent element of* B. *Then* $e\Gamma B$ *is a left weak-interior ideal of* M.

Proof. Let B be a left weak-interior ideal of the Γ-semiring M. Suppose $x \in B \cap e\Gamma M$. Then $x \in B$ and $x = e\alpha y$, $\alpha \in \Gamma, y \in M$.

$$x = e\alpha y = e\beta e\alpha y = e\beta(e\alpha y) = e\beta x \in e\Gamma B.$$

Therefore $B \cap e\Gamma M \subseteq e\Gamma B$ $e\Gamma B \subseteq B$ and $e\Gamma B \subseteq e\Gamma M$. It follows that $e\Gamma B \subseteq B \cap e\Gamma M$ and thereby, $e\Gamma B = B \cap e\Gamma M$. Hence $e\Gamma B$ is a left weak-interior ideal of M. □

Corollary 3.11. *Let* M *be a* Γ*-semiring* M *and* e *be* $\alpha-$*idempotent. Then* $e\Gamma M$ *and* $M\Gamma e$ *are left weak-interior ideal and right weak-interior ideal of* M *respectively.*

Theorem 3.13. *Let* M *be a* Γ*-semiring. If* $M = M\Gamma a$, *for all* $a \in M$, *then every left weak-interior ideal of* M *is a quasi ideal of* M.

Proof. Let B be a left weak-interior ideal of the Γ-semiring M and $a \in B$. Then we successively obtain $M\Gamma a \subseteq M\Gamma B$ and $M \subseteq M\Gamma B \subseteq M$, hence$M\Gamma B = M$. Thus, $M\Gamma B\Gamma B = M\Gamma B \subseteq BB$, consequently $B\Gamma M \cap M\Gamma B \subseteq B \cap B\Gamma M \subseteq B$. Therefore B is a left quasi ideal of M. □

Theorem 3.14. B *is a left weak-interior ideal of a* Γ*-semiring* M *if and only if* B *is a left ideal of some left ideal of* Γ*-semiring*M.

Proof. Suppose B is a left ideal of left ideal R of Γ-semiring$M$. Then $R\Gamma B \subseteq B$, $M\Gamma R \subseteq R$ and $M\Gamma B\Gamma B \subseteq M\Gamma R\Gamma B \subseteq R\Gamma B \subseteq B$. Therefore B is a left weak-interior ideal of a Γ-semiring M.

Conversely suppose that B is a left weak-interior ideal of a Γ-semiring M. Then $M\Gamma B\Gamma B \subseteq B$. Therefore B is a left ideal of left ideal of $M\Gamma B$ of the Γ-semiring M. □

Corollary 3.12. B *is a right weak-interior ideal of a* Γ*-semiring* M *if and only if* B *is a right ideal of some right ideal of* Γ*-semiring* M.

Corollary 3.13. B *is a weak-interior ideal of a* Γ*-semiring* M *if and only if* B *is an ideal of some ideal of* Γ*-semiring*M.

4. Left weak-interior simple Γ-semiring

In this section, we introduce the notion of left weak-interior simple Γ-semiring and these are then characterized using left weak-interior ideals of Γ-semirings.

Definition 4.1. A Γ-semiring M is a left (right) simple Γ-semiring if M has no proper left (right) ideals of M.

Definition 4.2. A Γ-semiring M is said to be simple Γ-semiring if M has no proper ideals of M.

Definition 4.3. A Γ-semiring M is said to be left (right) weak-interior simple Γ-semiring if M has no left (right) weak-interior ideal other than M itself.

Definition 4.4. A Γ-semiring M is said to be weak-interior simple Γ-semiring if M has no weak-interior ideal other than M itself.

Theorem 4.1. *If M is a division Γ-semiring then M is a left weak-interior simple Γ-semiring.*

Proof. Let B be a proper left weak-interior ideal of the division Γ-semiring M, $x \in M$ and $0 \neq a \in B$. Since M is a division Γ-semiring, there exist $b \in M$, $\alpha \in \Gamma$ such that $a\alpha b = 1$. Then there exist $\beta \in \Gamma$ such that $a\alpha b\beta x = x = x\beta a\alpha b$. Therefore $x \in B\Gamma M$ and $M \subseteq B\Gamma M$. We have $B\Gamma M \subseteq M$. Hence $M = B\Gamma M$. Similarly we can prove $M\Gamma B = M$. It follows that $M = M\Gamma B = M\Gamma B\Gamma B \subseteq B$. Therefore $M = B$. Hence division Γ-semiring M has no proper left-quasi-interior ideals. □

Corollary 4.1. *If M is a division Γ-semiring, then M is a right weak-interior simple Γ-semiring.*

Corollary 4.2. *If M is a division Γ-semiring, then M is a weak-interior simple Γ-semiring.*

Theorem 4.2. *Let M be a left simple Γ-semiring. Every left weak-interior ideal of M is a left ideal of M.*

Proof. Let M be a left simple Γ-semiring and B be a left weak-interior ideal of M. Then $M\Gamma B\Gamma B \subseteq B$ and $M\Gamma B$ is a left ideal of M. Since M is a left simple Γ-semiring, we have $M\Gamma B = M$. Therefore $M\Gamma B\Gamma B \subseteq B$, whence $M\Gamma B \subseteq B$. □

Corollary 4.3. *Let M be a right simple Γ-semiring. Every right weak-interior ideal is a right ideal of M.*

Corollary 4.4. *Let M be a simple Γ-semiring. Every weak-interior ideal is an ideal of M.*

Theorem 4.3. *Let M be a Γ-semiring. M is a left weak-interior simple Γ-semiring if and only if $\langle a\rangle = M$ for all $a \in M$, where $\langle a\rangle$ is the smallest left weak-interior ideal generated by a.*

Proof. Let M be a Γ-semiring. Suppose M is the left weak-interior simple Γ-semiring, $a \in M$ and $B = M\Gamma a$. Then B is a left ideal of M. By Theorem 3.5, B is a left weak-interior ideal of M. Therefore $B = M$. Hence $M\Gamma a = M$, for all $a \in M$. From $M = M\Gamma a \subseteq \langle a\rangle \subseteq M$ it follows that $M = \langle a\rangle$.

Conversely, suppose that $\langle a\rangle = M$. Let A be a left weak-interior ideal such that $a \in A$. Then $\langle a\rangle \subseteq A \subseteq M$, whence $M \subseteq A \subseteq M$. Therefore $A = M$. Hence M is a left weak-interior simple Γ-semiring. □

Theorem 4.4. *Let M be a Γ-semiring. Then M is a left weak-interior simple Γ-semrriring if and only if $M\Gamma a\Gamma a = M$, for all $a \in M$.*

Proof. Suppose M is a left-weak interior simple Γ-semiring and $a \in M$. Then $M\Gamma a\Gamma a$ is a left weak-interior ideal of M. Hence $M\Gamma a\Gamma a = M$, for all $a \in M$.

Conversely suppose that $M\Gamma a\Gamma a = M$, for all $a \in M$. Let B be a left weak-interior ideal of the Γ-semiring M and $a \in B$. Then, $M = M\Gamma a\Gamma a \subseteq M\Gamma B\Gamma B \subseteq B$, Therefore $M = B$. Hence M is a left weak-interior simple Γ-semiring. □

Corollary 4.5. *Let M be a Γ-semiring. Then M is a right weak-interior simple Γ-semrriring if and only if $a\Gamma a\Gamma M = M$, for all $a \in M$.*

Corollary 4.6. *Let M be a Γ-semiring. Then M is a weak-interior simple Γ-semrriring if and only if $a\Gamma a\Gamma M = M$ and $M\Gamma a\Gamma a = M$, for all $a \in M$.*

Theorem 4.5. *If a Γ-semiring M is a left simple Γ-semiring, then every left weak-interior ideal of M is a right ideal of M.*

Proof. Let B be a left weak-interior ideal of the left simple Γ-semiring$M$. Then $M\Gamma B$ is a left ideal of M and $M\Gamma B \subseteq M$. Therefore $M\Gamma B = M$. Then

$$B\Gamma M = B\Gamma M\Gamma B \subseteq M\Gamma B\Gamma M\Gamma B \subseteq B,$$

whence $B\Gamma M \subseteq B$. Hence every left weak-interior ideal is a right ideal of M. □

Corollary 4.7. *If a Γ-semiring M is right simple Γ-semiring then every right weak-interior ideal of M is a left ideal of M.*

Corollary 4.8. *Every weak-interior ideal of a left and right simple Γ-semiring M is an ideal of M.*

Theorem 4.6. *Let M be a Γ-semiring and B be a left weak-interior ideal of M. Then B is a minimal left weak-interior ideal of M if and only if B is a left weak-interior simple Γ-subsemiring of M.*

Proof. Let B be a minimal left weak-interior ideal of the Γ-semiring M and C be a left weak-interior ideal of B. Then $B\Gamma C\Gamma B\Gamma C \subseteq C$ and $B\Gamma C\Gamma B\Gamma C$ is a left weak-interior ideal of M. Since C is a weak-interior ideal of B, we have $B = B\Gamma C\Gamma C \subseteq C$, hence $B = C$.

Conversely suppose that B is the left weak-interior simple Γ-subsemiring of M. Let C be a left weak-interior ideal of M and $C \subseteq B$. Then,

$$B\Gamma C\Gamma C \subseteq M\Gamma C\Gamma C \subseteq M\Gamma B\Gamma B \subseteq B.$$

Therefore C is a left weak-interior of B. It follows that $B = C$. Hence B is a minimal left weak-interior ideal of M. □

Corollary 4.9. *Let M be a Γ-semiring and B be a right weak-interior ideal of M. Then B is a minimal right weak-interior ideal of M if and only if B is a right weak-interior simple Γ-subsemiring of M.*

Corollary 4.10. *Let M be a Γ-semiring and B be a weak-interior ideal of M. Then B is a minimal weak-interior ideal of M if and only if B is a weak-interior simple Γ-subsemiring of M.*

Theorem 4.7. *Let M be a Γ-semiring and $B = L\Gamma L$, where L is a minimal left ideal of M and $L\Gamma M = M\Gamma L$. Then B is a minimal left weak-interior ideal of M.*

Proof. Obviously $B = L\Gamma L$ is a left weak-interior ideal of M. Let A be a left weak-interior ideal of M such that $A \subseteq B$. Clearly, $M\Gamma A$ is a left ideal of M. Then

$$M\Gamma A \subseteq M\Gamma B = M\Gamma L\Gamma L \subseteq L$$

(since L is a left ideal of M). Therefore $M\Gamma A = L$ (since L is a minimal left ideal of M). Hence

$$B = M\Gamma A\Gamma M\Gamma A = M\Gamma M\Gamma A\Gamma A \subseteq M\Gamma A\Gamma A \subseteq A$$

. Therefore $A = B$. Hence B is a minimal left weak-interior ideal of M. □

Corollary 4.11. *Let M be a Γ-semiring and $B = R\Gamma R$, where R is a minimal right ideal of M. Then B is a minimal right weak-interior ideal of M.*

Theorem 4.8. *M is regular $\Gamma-$semiring if and only if $A\Gamma B = A \cap B$ for any right ideal A and left ideal B of $\Gamma-$semiring M.*

Theorem 4.9. *Let M be a Γ-semiring such that $A\Gamma M = M\Gamma A$ for all weak-interior ideals A of M. Then M is a regular Γ-semiring if and only if $B\Gamma B\Gamma M = B\Gamma B\Gamma M = B$ for all weak-interior ideals B of M.*

Proof. Suppose M is a regular semiring. Let B be a weak-interior ideal of M and $x \in B$. Then $M\Gamma B\Gamma B \subseteq B$ and $y \in M,\ \alpha, \beta \in \Gamma$ such that $x = x\alpha y\beta x \in M\Gamma B\Gamma B$. Therefore $x \in M\Gamma B\Gamma B$. Hence $M\Gamma B\Gamma B = B$. Similarly we can prove $B\Gamma B\Gamma M = B$.

Conversely suppose that $B\Gamma B\Gamma M = B\Gamma B\Gamma M = B$ for all weak-interior ideals B of M. Let $B = R \cap L$ where R, L are right and left ideals of M respectively. Then B is a weak-interior ideal of M. Therefore $(R \cap L)\Gamma(R \cap L)\Gamma M = R \cap L$,

$$R \cap L = (R \cap L)\Gamma(R \cap L)\Gamma M \subseteq R\Gamma R\Gamma M$$

and

$$R \cap L = (R \cap L)\Gamma(R \cap L)\Gamma M \subseteq R\Gamma R\Gamma M\Gamma M\Gamma L\Gamma L \subseteq R\Gamma L \subseteq R \cap L$$

since $R\Gamma L \subseteq L$ and $R\Gamma L \subseteq R$. Therefore $R \cap L = R\Gamma L$. Hence M is a regular Γ-semiring. □

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