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ON CUBIC h-IDEALS OF Γ-HEMIRING

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ABSTRACT. The concepts of cubic h-ideal, cubic h-bi-ideal and cubic h-quasi-ideal of a Γ-hemiring are introduced and some related properties are investigated. Relation between cubic h-ideals, cubic h-bi-ideals and cubic h-quasi-ideals are discussed and some characterizations of cubic h-ideals are also considered.

1. Introduction

Semiring is a well known universal algebra generalizing of associative ring $(R, +, .)$. It has been found very useful for solving problems in different areas of applied mathematics and information sciences, since the structure of a semiring provides an algebraic framework for modelling and studying the key factors in these applied areas. Ideals of semiring play a central role in the structure theory and useful for many purposes. However they do not in general coincide with the usual ring ideals and for this reason, their use is somewhat limited in trying to obtain analogues of ring theorems for semiring. To ammend this gap, Henriksen [**3**] defined a more restricted class of ideals, which are called k-ideals. A still more restricted class of ideals in hemirings are given by Iizuka [**4**], which are called h-ideals. La Torre [**9**], investigated h-ideals and k-ideals in hemirings in an effort to obtain analogues of ring theorems for hemiring and to amend the gap between ring ideals and semiring ideals.

The theory of fuzzy sets, proposed by Zadeh [**13**], has provided a useful mathematical tool for describing the behavior of the systems that are too complex or undefined to admit precise mathematical analysis by classical methods and tools. The study of fuzzy algebraic structure has started by Rosenfeld [**12**]. Since then many researchers developed this ideas. The concept of fuzzy h-ideals in hemiring

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was introduced by Jun et al [**5**]. Since then many researchers enriched this idea. Sardar et al [**11**], Ma et al [**10**] extended some of these results in Γ-hemiring, a more general setting of hemiring. The concept of cubic subgroups and cubic sets were initiated by Jun et al [**6, 7**]. Khan et al [**8**] applied this in case of cubic h-ideals of hemirings. Chinnadurai [**1, 2**] used this notion to study cubic bi-ideals and cubic lateral ideals in near-ring and ternary near-ring respectively.

The main aim of this paper is to introduce and study cubic h-ideal in Γ-hemiring. At first we define some basic operation such as intersection, cartesian product etc on them and use these to obtain some of its basic properties. After that we define composition of cubic h-ideals and use it to define cubic h-bi-ideals and cubic h-quasi-ideals and obtain some characterizations.

2. Preliminaries

We recall the following preliminaries for subsequent use.

Definition 2.1. Let S and Γ be two additive commutative semigroups with zero. Then S is called a $\Gamma-$hemiring if there exists a mapping $\times\Gamma\times \ni (a,\alpha,b) \longmapsto a\alpha b \in S$ satisfying the following conditions:
(i) $(a+b)\alpha c = a\alpha c + b\alpha c$,
(ii) $a\alpha(b+c) = a\alpha b + a\alpha c$,
(iii) $a(\alpha+\beta)b = a\alpha b + a\beta b$,
(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$.
(v) $0_S\alpha a = 0_S = a\alpha 0_S$,
(vi) $a0_\Gamma b = 0_S = b0_\Gamma a$
for all $a, b, c \in S$ and for all $\alpha, \beta \in \Gamma$.

For simplification we write 0 instead of 0_S and 0_Γ.

Example 2.1. Let S be the set of all $m \times n$ matrices over $\mathbf{Z_0^-}$ (the set of all non-positive integers) and Γ be the set of all $n \times m$ matrices over $\mathbf{Z_0^-}$, then S forms a Γ-hemiring with usual addition and multiplication of matrices.

A subset A of a Γ-hemiring S is called a left (resp. right) ideal of S if A is closed under addition and $S\Gamma A \subseteq A$ (resp. $A\Gamma S \subseteq A$). A subset A of a hemiring S is called an ideal if it is both left and right ideal of S. A subset A of a Γ-hemiring S is called a quasi-ideal of S if A is closed under addition and $S\Gamma A \cap A\Gamma S \subseteq A$. A subset A of a Γ-hemiring S is called a bi-ideal if A is closed under addition and $A\Gamma S\Gamma A \subseteq A$. A left ideal A of S is called a left h-ideal if $x, z \in S$, $a, b \in A$ and $x + a + z = b + z$ implies $x \in A$. A right h-ideal is defined analoguesly. The h-closure $\overline{A}$ of A in S is defined as $\overline{A} = \{x \in S \mid\ x + a + z = b + z,$ for some $a, b \in A$ and $z \in S\}$. Now if A is a left (right) ideal of S, then $\overline{A}$ is the smallest left (right) h-ideal containing A. A quasi-ideal (bi-ideal) A of S is called an h-quasi-ideal(resp. h-bi-ideal) of S if $\overline{S\Gamma A} \cap \overline{A\Gamma S}$ (resp. $\overline{A\Gamma S\Gamma A}$) and $x + a + z = b + z$ implies $x \in A$ for all $x, z \in S$ and $a, b \in A$.

Definition 2.2. A fuzzy subset of a nonempty set X is defined as a function $\mu : X \to [0, 1]$.

DEFINITION 2.3. Let X be a non-empty set. A cubic set A in X is a structure $A = \{< x, \widetilde{\mu}, f >: x \in X\}$ which briefly denoted as $A =< \widetilde{\mu}, f >$ where $\widetilde{\mu} = [\mu^-, \mu^+]$ is an interval valued fuzzy set (briefly, IVF) in X and f is a fuzzy set in X.

DEFINITION 2.4. For any non-empty set G of a set X, the characteristic cubic set of G is defined to be the structure $\chi_G(x) =< x, \widetilde{\zeta}_{\chi_G}(x), \eta_{\chi_G}(x) : x \in X >$ where

$$\widetilde{\zeta}_{\chi_G}(x) \begin{array}{l} = [1,1] \approx \widetilde{1}\ , \text{ if } x \in G \\ = [0,0] \approx \widetilde{0}\ , \text{ otherwise.} \end{array} \quad \text{and} \quad \eta_{\chi_G}(x) \begin{array}{l} = 0\ , \text{ if } x \in G \\ = 1\ , \text{ otherwise.} \end{array}$$

Throughout this paper unless otherwise mentioned S denotes the Γ-hemiring and for any two set P and Q, we use the following notation:

$$\cap\{P, Q\} = P \cap Q \text{ and } \cup \{P, Q\} = P \cup Q.$$

3. Basic definitions and results of cubic h-ideals

In this section, the notions of cubic h-ideals in Γ-hemiring are introduced and some of their basic properties are investigated.

DEFINITION 3.1. Let $< \widetilde{\mu}, f >$ be a non-empty cubic subset of a Γ-hemiring S. Then $< \widetilde{\mu}, f >$ is called a cubic left ideal [cubic right ideal] of S if

(i) $\widetilde{\mu}(x + y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$, $f(x + y) \leqslant \max\{f(x), f(y)\}$ and

(ii) $\widetilde{\mu}(x\gamma y) \supseteq \widetilde{\mu}(y)$, $f(x\gamma y) \leqslant f(y)$
[respectively $\widetilde{\mu}(x\gamma y) \supseteq \widetilde{\mu}(x), f(x\gamma y) \leqslant f(x)$].

for all $x, y \in S$, $\gamma \in \Gamma$.

A cubic ideal of a Γ-hemiring S is a non-empty cubic subset of S which is a cubic left ideal as well as a cubic right ideal of S.

Note that if $< \widetilde{\mu}, f >$ is a cubic left or right ideal of a Γ-hemiring S, then $\widetilde{\mu}(0) \supseteq \widetilde{\mu}(x)$ and $f(0) \leqslant f(x)$ for all $x \in S$.

DEFINITION 3.2. A cubic left ideal $< \widetilde{\mu}, f >$ of a Γ-hemiring S is called a cubic left h-ideal if for all $a, b, x, z \in S$, $x + a + z = b + z \Rightarrow \widetilde{\mu}(x) \supseteq \cap\{\widetilde{\mu}(a), \widetilde{\mu}(b)\}$ and $f(x) \leqslant \max\{f(a), f(b)\}$.

A cubic right h-ideal is defined similarly. By a cubic h-ideal $< \widetilde{\mu}, f >$, we mean that $< \widetilde{\mu}, f >$ is both cubic left and cubic right h-ideal.

EXAMPLE 3.1. Let S be the additive commutative semigroup of all non-positive integers and Γ be the additive commutative semigroup of all non-positive even integers. Then S is a Γ-hemiring if $a\gamma b$ denotes the usual multiplication of integers a, γ, b where $a, b \in S$ and $\gamma \in \Gamma$. Let $< \widetilde{\mu}, f >$ be a cubic subset of S, defined as follows

$$\widetilde{\mu}(x) \begin{array}{l} = [0, 1] \text{ if } x = 0 \\ = [0.1, 0.9] \text{ if } x \text{ is even} \\ = [0.2, 0.3] \text{ if } x \text{ is odd} \end{array} \quad \text{and} \quad f(x) \begin{array}{l} = 0 \text{ if } x = 0 \\ = 0.5 \text{ if } x \text{ is even} \\ = 0.8 \text{ if } x \text{ is odd} \end{array}$$

The cubic subset $< \widetilde{\mu}, f >$ of S is both a cubic ideal and cubic h-ideal of S.

Throughout this section, we prove results only for cubic left h-ideals. Similar results can be obtained for cubic right h-ideals and cubic ideals.

THEOREM 3.1. *A cubic set $C =< \widetilde{\mu}, f >$ of S is a cubic left h-ideal of S if and only if any level subset $C_t =< \widetilde{\mu}_t, f_t >:= \{x \in S : \widetilde{\mu}(x) \supseteq [t,t]$ and $f(x) \leqslant t, t \in [0,1]\}$ is a left h-ideal of S, provided it is non-empty.*

PROOF. Let $< \widetilde{\mu}, f >$ be a cubic left h-ideal of S and assume that $< \widetilde{\mu}_t, f_t > \neq \phi$ for $t \in$ [0,1]. Let $x, z \in S$ and $a, b \in < \widetilde{\mu}_t, f_t >$. Then $\widetilde{\mu}(a+b) \supseteq \cap\{\widetilde{\mu}(a), \widetilde{\mu}(b)\} \supseteq [t,t]$ and $f(a+b) \leqslant \max\{f(a), f(b)\} \leqslant t$; implies that $a + b \in < \widetilde{\mu}_t, f_t >$. Also, in addition for $\gamma \in \Gamma$, $\widetilde{\mu}(x\gamma a) \supseteq \widetilde{\mu}(a) \supseteq [t,t]$ and $f(x\gamma a) \leqslant f(a) \leqslant t$ which implies $x\gamma a \in < \widetilde{\mu}_t, f_t >$. So, $< \widetilde{\mu}_t, f_t >$ is a left ideal of S. Now for h-ideal, suppose $x + a + z = b + z$ with $a, b \in \widetilde{\mu}_t$. Then $\widetilde{\mu}(x) \supseteq \cap\{\widetilde{\mu}(a), \widetilde{\mu}(b)\} \supseteq [t,t]$ and $f(x) \leqslant \max\{f(a), f(b)\} = t \Rightarrow x \in < \widetilde{\mu}_t, f_t > \Rightarrow C_t$ is a left h-ideal of S.

Conversely, suppose $< \widetilde{\mu}_t, f_t >$ is a left h-ideal of S. If possible, suppose $< \widetilde{\mu}, f >$ is not a cubic left h-ideal of S. Then there exist $x_0, z_0, a_0, b_0 \in S$ such that $x_0 + a_0 + z_0 = b_0 + z_0$ and $\widetilde{\mu}(x_0) \subset \cap\{\widetilde{\mu}(a_0), \widetilde{\mu}(b_0)\}$ or $f(x_0) > \max\{f(a_0), f(b_0)\}$. Taking $t_0 = \frac{1}{2}[f(x_0) + \max\{f(a_0), f(b_0)\}]$, we see that $t_0 \in$ [0,1] and $f(x_0) > t_0 > \max\{f(a_0), f(b_0)\}$ whence $a_0, b_0 \in f_{t_0}$ but $x_0 \notin f_{t_0}-$ which is a contradiction. Therefore $\widetilde{\mu}(x) \supseteq \cap\{\widetilde{\mu}(a), \widetilde{\mu}(b)\}$ and $f(x) \leqslant \max\{f(a), f(b)\}$for all $a, b, x, z \in S$ with $x + a + z = b + z$. Consequently, $< \widetilde{\mu}, f >$ is a cubic left h-ideal of S. □

THEOREM 3.2. *Let A be a non-empty subset of a Γ-hemiring S. Then A is a left h-ideal of S if and only if the characteristic function $\chi_A =< \widetilde{\mu}_{\chi_A}, f_{\chi_A} >$ is a cubic left h-ideal of S.*

PROOF. Assume that A is a left h-ideal of S and $x, y \in S$, $\gamma \in \Gamma$. Suppose $\widetilde{\mu}_{\chi_A}(x+y) \subset \cap\{\widetilde{\mu}_{\chi_A}(x), \widetilde{\mu}_{\chi_A}(y)\}$ and $f_{\chi_A}(x+y) > \max\{f_{\chi_A}(x), f_{\chi_A}(y)\}$. It follows that $\widetilde{\mu}_{\chi_A}(x+y) = \widetilde{0}$, $\cap\{\widetilde{\mu}_{\chi_A}(x), \widetilde{\mu}_{\chi_A}(y)\} = \widetilde{1}$ and $f_{\chi_A}(x+y) = 1$, $\max\{f_{\chi_A}(x), f_{\chi_A}(y)\} = 0$. This imply that $x, y \in A$ but $x + y \notin A -$ a contradiction. So, $\widetilde{\mu}_{\chi_A}(x+y) \supseteq \cap\{\widetilde{\mu}_{\chi_A}(x), \widetilde{\mu}_{\chi_A}(y)\}$ and $f_{\chi_A}(x+y) \leqslant \max\{f_{\chi_A}(x), f_{\chi_A}(y)\}$. Similarly we can show that $\widetilde{\mu}(x\gamma y) \supseteq \widetilde{\mu}(y)$, $f(x\gamma y) \leqslant f(y)$ and for all $a, b, x, z \in S$ with $x + a + z = b + z$, $\widetilde{\mu}(x) \supseteq \cap\{\widetilde{\mu}(a), \widetilde{\mu}(b)\}$, $f(x) \leqslant \max\{f(a), f(b)\}$. Therefore $\chi_A =< \widetilde{\mu}_{\chi_A}, f_{\chi_A} >$ is a cubic left h-ideal of S.

Conversely, assume that for any subset A of S, $\chi_A =< \widetilde{\mu}_{\chi_A}, f_{\chi_A} >$ is a cubic left h-ideal of S. Let $x, y \in A$, $a, b \in S$ and $\gamma \in \Gamma$. Then $\widetilde{\mu}_{\chi_A}(x) = \widetilde{\mu}_{\chi_A}(y) = \widetilde{1}$ and $f_{\chi_A}(x) = f_{\chi_A}(y) = 0$. Now $\widetilde{\mu}(x+y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\} = \widetilde{1}$, $f(x+y) \leqslant \max\{f(x), f(y)\} = 0$ and $\widetilde{\mu}(x\gamma y) \supseteq \widetilde{\mu}(y) = \widetilde{1}$, $f(x\gamma y) \leqslant f(y) = 0$. This implies $x + y, x\gamma y \in A$. Also if, $a + x + b = y + b$, then $\widetilde{\mu}(a) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$ and $f(a) \leqslant \max\{f(x), f(y)\}$ which implies $a \in A$. Hence A is a left h-ideal of S. □

DEFINITION 3.3. Let $A =< \widetilde{\mu}, f >$ and $B =< \widetilde{\theta}, g >$ be two cubic sets of a $\Gamma-$hemiring S. Define intersection of A and B by

$$A \cap B =< \widetilde{\mu}, f > \cap < \widetilde{\theta}, g >=< \widetilde{\mu} \cap \widetilde{\theta}, f \cup g > .$$

PROPOSITION 3.1. *Intersection of a non-empty collection of cubic left h-ideals is a cubic left h-ideal of S.*

Proof. Let $A_i = \{< \widetilde{\mu}_i, f_i >: i \in I\}$ be a non-empty family of h-ideals of S. Let $a, b, x, y, z \in S$ and $\gamma \in \Gamma$. Then

$$\begin{aligned}(\bigcap_{i\in I}\widetilde{\mu}_i)(x+y) &= \bigcap_{i\in I}\{\mu_i(x+y)\} \supseteq \bigcap_{i\in I}\{\cap\{\widetilde{\mu}_i(x), \widetilde{\mu}_i(y)\}\}\\ &= \cap\{\bigcap_{i\in I}\widetilde{\mu}_i(x), \bigcap_{i\in I}\widetilde{\mu}_i(y)\} = \cap\{(\bigcap_{i\in I}\widetilde{\mu}_i)(x), (\bigcap_{i\in I}\widetilde{\mu}_i)(y)\}.\end{aligned}$$

$$\begin{aligned}(\bigcup_{i\in I}f_i)(x+y) &= \sup_{i\in I}\{f(x+y)\} \leqslant \sup_{i\in I}\{\max\{f_i(x), f_i(y)\}\}\\ &= \max\{\sup_{i\in I}f_i(x), \sup_{i\in I}f_i(y)\} = \max\{(\bigcup_{i\in I}f_i)(x), (\bigcup_{i\in I}f_i)(y)\}.\end{aligned}$$

Again

$$(\bigcap_{i\in I}\widetilde{\mu}_i)(x\gamma y) = \bigcap_{i\in I}\{\widetilde{\mu}_i(x\gamma y)\} \supseteq \bigcap_{i\in I}\{\widetilde{\mu}_i(y)\} = (\bigcap_{i\in I}\widetilde{\mu}_i)(y).$$

$$(\bigcup_{i\in I}f_i)(x\gamma y) = \sup_{i\in I}\{f_i(x\gamma y)\} \leqslant \sup_{i\in I}\{f_i(y)\} = (\bigcap_{i\in I}f_i)(y).$$

Suppose $x \in S$ be such that $x + a + z = b + z$, for $z, a, b \in S$. Then

$$\begin{aligned}(\bigcap_{i\in I}\widetilde{\mu}_i)(x) &= \bigcap_{x\in I}\{\mu_i(x)\} \supseteq \bigcap_{i\in I}\{\cap\{\widetilde{\mu}_i(a), \widetilde{\mu}_i(b)\}\}\\ &= \cap\{\bigcap_{i\in I}\widetilde{\mu}_i(a), \bigcap_{i\in I}\widetilde{\mu}_i(b)\} = \cap\{(\bigcap_{i\in I}\widetilde{\mu}_i)(a), (\bigcap_{i\in I}\widetilde{\mu}_i)(b)\}.\end{aligned}$$

$$\begin{aligned}(\bigcup_{i\in I}f_i)(x) &= \sup_{x\in I}\{f_i(x)\} \leqslant \sup_{i\in I}\{\max\{f_i(a), f_i(b)\}\}\\ &= \max\{\sup_{i\in I}f_i(a), \sup_{i\in I}f_i(b)\} = \max\{(\bigcup_{i\in I}f_i)(a), (\bigcup_{i\in I}f_i)(b)\}.\end{aligned}$$

Hence $\bigcap_{i\in I}A_i = \{< \bigcap_{i\in I}\widetilde{\mu}_i, \bigcup_{i\in I}f_i >: i \in I\}$ is a cubic left h-ideal of S. □

Proposition 3.2. *Let $f : R \to S$ be a morphism of $\Gamma-$hemirings and $A =< \widetilde{\phi}, g >$ be is a cubic left h-ideal of S, then $f^{-1}(A)$ is a cubic left h-ideal of R where $f^{-1}(A)(x) =< f^{-1}(\widetilde{\phi})(x), f^{-1}(g)(x) >=< \widetilde{\phi}(f(x)), g(f(x)) >$*

Proof. Let $f : R \to S$ be a morphism of Γ-hemirings. Suppose $A =< \widetilde{\phi}, g >$ and $r, s \in R$, $\gamma \in \Gamma$. Then

$$\begin{aligned}f^{-1}(\widetilde{\phi})(r+s) &= \widetilde{\phi}(f(r+s)) = \widetilde{\phi}(f(r)+f(s))\\ &\supseteq \cap\{\widetilde{\phi}(f(r)), \widetilde{\phi}(f(s))\} = \cap\{(f^{-1}(\widetilde{\phi}))(r), (f^{-1}(\widetilde{\phi}))(s)\}\end{aligned}$$

$$\begin{aligned}f^{-1}(g)(r+s) &= g(f(r+s)) = g(f(r)+f(s))\\ &\leqslant \max\{g(f(r)), g(f(s))\} = \max\{(f^{-1}(g))(r), (f^{-1}(g))(s)\}\end{aligned}$$

Again

$$(f^{-1}(\widetilde{\phi}))(r\gamma s) = \widetilde{\phi}(f(r\gamma s)) = \widetilde{\phi}(f(r)\gamma f(s)) \supseteq \widetilde{\phi}(f(s)) = (f^{-1}(\widetilde{\phi}))(s)$$

and

$$(f^{-1}(g))(r\gamma s) = g(f(r\gamma s)) = g(f(r)\gamma f(s)) \leqslant g(f(s)) = (f^{-1}(g))(s).$$

Thus $< f^{-1}(\widetilde{\phi})(x), f^{-1}(g)(x) >$ is a cubic left ideal of R. Suppose $x, a, b, z \in R$ be such that $x + a + z = b + z$. Then $f(x) + f(a) + f(z) = f(b) + f(z)$. Now

$$(f^{-1}(\widetilde{\phi}))(x) = \widetilde{\phi}(f(x)) \supseteq \cap\{\widetilde{\phi}(f(a)), \widetilde{\phi}(f(b))\} = \cap\{f^{-1}(\widetilde{\phi})(a), f^{-1}(\widetilde{\phi})(b)\}$$

and

$$(f^{-1}(g))(x) = g(f(x)) \leqslant \max\{g(f(a)), g(f(b))\} = \max\{f^{-1}(g)(a), f^{-1}(g)(b)\}.$$

Therefore $< f^{-1}(\widetilde{\phi})(x), f^{-1}(g)(x) >$ is a cubic left h-ideal of R. □

DEFINITION 3.4. A cubic left h-ideal $< \widetilde{\mu}, f >$ of a Γ-hemiring S, is said to be normal cubic left h-ideal if $\widetilde{\mu}(0) = \widetilde{1},\ f(0) = 0$.

PROPOSITION 3.3. *Given a cubic left h-ideal $< \widetilde{\mu}, f >$ of a Γ-hemiring S, let $< \widetilde{\mu}_+, f_+ >$ be a cubic set in S obtained by $\widetilde{\mu}_+(x) = \widetilde{\mu}(x) + \widetilde{1} - \widetilde{\mu}(0),\ f_+(x) = f(x) - f(0)$ for all $x \in S$. Then $< \widetilde{\mu}_+, f_+ >$ is a normal cubic left h-ideal of S, which contains $< \widetilde{\mu}, f >$.*

PROOF. For all $x, y \in S$ and $\gamma \in \Gamma$, we have $\widetilde{\mu}_+(0) = \widetilde{\mu}(0) + \widetilde{1} - \widetilde{\mu}(0) = \widetilde{1}$, $f_+(0) = f(0) - f(0) = 0$ Now,

$$\begin{aligned} \widetilde{\mu}_+(x+y) &= \widetilde{\mu}(x+y) + \widetilde{1} - \widetilde{\mu}(0) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\} + \widetilde{1} - \widetilde{\mu}(0) \\ &= \cap\{\{\widetilde{\mu}(x) + \widetilde{1} - \widetilde{\mu}(0)\}, \{\widetilde{\mu}(y) + \widetilde{1} - \widetilde{\mu}(0)\}\} = \cap\{\widetilde{\mu}_+(x), \widetilde{\mu}_+(y)\} \end{aligned}$$

$$\begin{aligned} f_+(x+y) &= f(x+y) - f(0) \leqslant \max\{f(x), f(y)\} - f(0) \\ &= \max\{\{f(x) - f(0)\}, \{f(y) - f(0)\}\} = \max\{f_+(x), f_+(y)\} \end{aligned}$$

and

$$\widetilde{\mu}_+(x\gamma y) = \widetilde{\mu}(x\gamma y) + \widetilde{1} - \widetilde{\mu}(0) \supseteq \widetilde{\mu}(y) + \widetilde{1} - \widetilde{\mu}(0) = \widetilde{\mu}_+(y).$$

$$f_+(x\gamma y) = f(x\gamma y) - f(0) \leqslant f(y) - f(0) = f_+(y).$$

Hence $< \widetilde{\mu}_+, f_+ >$ is a cubic left ideal of S.

Now, let $a, b, x, z \in S$ be such that $x + a + z = b + z$. Then

$$\begin{aligned} \widetilde{\mu}_+(x) &= \widetilde{\mu}(x) + \widetilde{1} - \widetilde{\mu}(0) \supseteq \cap\{\widetilde{\mu}(a), \widetilde{\mu}(b)\} + \widetilde{1} - \widetilde{\mu}(0) \\ &= \cap\{\{\widetilde{\mu}(a) + \widetilde{1} - \widetilde{\mu}(0)\}, \{\widetilde{\mu}(b) + \widetilde{1} - \widetilde{\mu}(0)\}\} = \cap\{\widetilde{\mu}_+(a), \widetilde{\mu}_+(b)\}. \end{aligned}$$

$$\begin{aligned} f_+(x) &= f(x) - f(0) \leqslant \max\{f(a), f(b)\} - f(0) \\ &= \max\{\{f(a) - f(0)\}, \{f(b) - f(0)\}\} = \max\{f_+(a), f_+(b)\}. \end{aligned}$$

Therefore, $< \widetilde{\mu}_+, f_+ >$ is a normal cubic left h-ideal of S and from definition of μ_+, $\mu \subseteq \mu_+$. □

DEFINITION 3.5. Let $A =< \widetilde{\mu}, f >$ and $B =< \widetilde{\nu}, g >$ be cubic subsets of X. The cartesian product of A and B is defined by $(A \times B)(x, y) = (< \widetilde{\mu}, f > \times < \widetilde{\nu}, g >)(x, y) = (< \widetilde{\mu} \times \widetilde{\nu}, f \times g >)(x, y) = [\cap\{\widetilde{\mu}(x), \widetilde{\nu}(y)\}, \max\{f(x), g(y)\}]$ for all $x, y \in X$.

THEOREM 3.3. *Let $A =< \widetilde{\mu}, f >$ and $B =< \widetilde{\nu}, g >$ be cubic left h-ideals of a Γ-hemiring S. Then $A \times B$ is a cubic left h-ideal of the Γ-hemiring $S \times S$.*

PROOF. Let $(x_1, x_2), (y_1, y_2) \in S \times S$ and $\gamma \in \Gamma$. Then

$$\begin{aligned} (\widetilde{\mu} \times \widetilde{\nu})((x_1, x_2) + (y_1, y_2)) &= (\widetilde{\mu} \times \widetilde{\nu})(x_1 + y_1, x_2 + y_2) \\ &= \cap\{\widetilde{\mu}(x_1 + y_1), \widetilde{\nu}(x_2 + y_2)\} \\ &\supseteq \cap\{\cap\{\widetilde{\mu}(x_1), \widetilde{\mu}(y_1)\}, \cap\{\widetilde{\nu}(x_2), \widetilde{\nu}(y_2)\}\} \\ &= \cap\{\cap\{\widetilde{\mu}(x_1), \widetilde{\nu}(x_2)\}, \cap\{\widetilde{\mu}(y_1), \widetilde{\nu}(y_2)\}\} \\ &= \cap\{(\widetilde{\mu} \times \widetilde{\nu})(x_1, x_2), (\widetilde{\mu} \times \widetilde{\nu})(y_1, y_2)\} \end{aligned}$$

$$\begin{aligned}(f\times g)((x_1,x_2)+(y_1,y_2)) &= (f\times g)(x_1+y_1,x_2+y_2)\\ &= \max\{f(x_1+y_1),g(x_2+y_2)\}\\ &\leqslant \max\{\max\{f(x_1),f(y_1)\},\max\{g(x_2),g(y_2)\}\}\\ &= \max\{\max\{f(x_1),g(x_2)\},\max\{f(y_1),f(y_2)\}\}\\ &= \max\{(f\times g)(x_1,x_2),(f\times g)(y_1,y_2)\}\end{aligned}$$

and

$$\begin{aligned}(\widetilde{\mu}\times\widetilde{\nu})((x_1,x_2)\gamma(y_1,y_2)) &= (\widetilde{\mu}\times\widetilde{\nu})(x_1\gamma y_1,x_2\gamma y_2) = \cap\{\widetilde{\mu}(x_1\gamma y_1),\widetilde{\nu}(x_2\gamma y_2)\}\\ &\supseteq \cap\{\widetilde{\mu}(y_1),\widetilde{\nu}(y_2)\} = (\widetilde{\mu}\times\widetilde{\nu})(y_1,y_2).\end{aligned}$$

$$\begin{aligned}(f\times g)((x_1,x_2)\gamma(y_1,y_2)) &= (f\times g)(x_1\gamma y_1,x_2\gamma y_2) = \max\{f(x_1\gamma y_1),g(x_2\gamma y_2)\}\\ &\leqslant \max\{f(y_1),g(y_2)\} = (f\times g)(y_1,y_2).\end{aligned}$$

Hence $A\times B$ is a cubic left ideal of $S\times S$. Now, let $(a_1,a_2),(b_1,b_2),(x_1,x_2),(z_1,z_2)\in S\times S$ be such that $(x_1,x_2)+(a_1,a_2)+(z_1,z_2)=(b_1,b_2)+(z_1,z_2)$ i.e., $(x_1+a_1+z_1,x_2+a_2+z_2)=(b_1+z_1,b_2+z_2)$. Then $x_1+a_1+z_1=b_1+z_1$ and $x_2+a_2+z_2=b_2+z_2$ so that

$$\begin{aligned}(\widetilde{\mu}\times\widetilde{\nu})(x_1,x_2) &= \cap\{\widetilde{\mu}(x_1),\widetilde{\nu}(x_2)\}\\ &\supseteq \cap\{\cap\{\widetilde{\mu}(a_1),\widetilde{\mu}(b_1)\},\cap\{\widetilde{\nu}(a_2),\widetilde{\nu}(b_2)\}\}\\ &= \cap\{\cap\{\widetilde{\mu}(a_1),\widetilde{\nu}(a_2)\},\cap\{\widetilde{\mu}(b_1),\widetilde{\nu}(b_2)\}\}\\ &= \cap\{(\widetilde{\mu}\times\widetilde{\nu})(a_1,a_2),(\widetilde{\mu}\times\widetilde{\nu})(b_1,b_2)\}.\end{aligned}$$

$$\begin{aligned}(f\times g)(x_1,x_2) &= \max\{f(x_1),g(x_2)\}\\ &\leqslant \max\{\max\{f(a_1),f(b_1)\},\max\{g(a_2),g(b_2)\}\}\\ &= \max\{\max\{f(a_1),g(a_2)\},\max\{f(b_1),g(b_2)\}\}\\ &= \max\{(f\times g)(a_1,a_2),(f\times g)(b_1,b_2)\}.\end{aligned}$$

Consequently, $A\times B$ is a cubic left h-ideal of $S\times S$. □

4. Cubic h-bi-ideals and Cubic h-quasi-ideals

DEFINITION 4.1. Let $A=<\widetilde{\mu},f>$ and $B=<\widetilde{\theta},g>$ be two cubic sets of a $\Gamma-$hemiring S. Define composition of A and B by

$$A\Gamma_{ch}B=<\widetilde{\mu},f>\Gamma_{ch}<\widetilde{\theta},g>=<\widetilde{\mu}\Gamma_{ch}\widetilde{\theta},f\Gamma_{ch}g>$$

where

$$\begin{aligned}\widetilde{\mu}\Gamma_{ch}\widetilde{\theta}(x) &= \bigcup_{x+a_1\gamma b_1+z=a_2\delta b_2+z}[\cap\{\widetilde{\mu}(a_1),\widetilde{\mu}(a_2),\widetilde{\theta}(b_1),\widetilde{\theta}(b_2)\}]\\ &= \widetilde{0},\ \text{if } x \text{ cannot be expressed as } x+a_1\gamma b_1+z=a_2\delta b_2+z.\end{aligned}$$

and

$$\begin{aligned}f\Gamma_{ch}g(x) &= \inf_{x+a_1\gamma b_1+z=a_2\delta b_2+z}\{\max\{f(a_1),f(c_1),g(b_1),g(d_1)\}\}\\ &= 1,\ \text{if } x \text{ cannot be expressed as above}\end{aligned}$$

for x,z,$a_1,a_2,b_1,b_2\in$ S and $\gamma,\delta\in\Gamma$.

DEFINITION 4.2. Let $A=<\widetilde{\mu},f>$ and $B=<\widetilde{\theta},g>$ be two cubic sets of a $\Gamma-$hemiring S. Define generalized composition of A and B by

$$A\circ_{ch}B=<\widetilde{\mu},f>\circ_{ch}<\widetilde{\theta},g>=<\widetilde{\mu}\circ_{ch}\widetilde{\theta},f\circ_{ch}g>$$

where

$$\widetilde{\mu}\circ_{ch}\widetilde{\theta}(x) = \cup[\bigcap_i\{\cap\{\widetilde{\mu}(a_i),\widetilde{\mu}(c_i),\widetilde{\theta}(b_i),\widetilde{\theta}(d_i)\ \}\}]$$
$$x+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z$$
$$=\widetilde{0},\text{ if } x \text{ cannot be expressed as above}$$

and

$$f\circ_{ch}g(x) = \inf[\max_i\{\max\{f(a_i),f(c_i),g(b_i),g(d_i)\}\}]$$
$$x+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z$$
$$=1,\text{ if } x \text{ cannot be expressed as above}$$

where x,z,a_i,b_i,c_i,d_i ∈S and $\gamma_i,\delta_i\in\Gamma$, for i=1,...,n.

Proposition 4.1. *Let $A=<\widetilde{\mu}_1,f>$, $B=<\widetilde{\mu}_2,g>$ be two cubic h-ideals of a $\Gamma-$hemiring S. Then $A\Gamma_{ch}B\subseteq Ao_{ch}B\subseteq A\cap B\subseteq A,B$ where $A\Gamma_{ch}B=<\widetilde{\mu}_1\Gamma_{ch}\widetilde{\mu}_1,f\Gamma_{ch}g>$ and $Ao_{ch}B=<\widetilde{\mu}_1o_{ch}\widetilde{\mu}_1,fo_{ch}g>$.*

Proof. Suppose $A=<\widetilde{\mu}_1,f>$, $B=<\widetilde{\mu}_2,g>$ be two cubic h-ideal of a Γ-hemiring S. Then

$$(\widetilde{\mu}_1\circ_{ch}\widetilde{\mu}_2)(x) = \cup\{\bigcap_i\{\cap\{\widetilde{\mu}_1(a_i),\widetilde{\mu}_1(c_i),\widetilde{\mu}_2(b_i),\ \widetilde{\mu}_2(d_i)\}\}\}$$
$$x+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z$$

where $x,z,a_i,b_i,c_i,d_i\in S,\ \ \gamma_i,\delta_i\in\Gamma,$ for $i=1,...,n$ and

$$\supseteq \bigcup_{x+a_1\gamma b_1+z=a_2\delta b_2+z}\{\cap\{\widetilde{\mu}_1(a_1),\ \widetilde{\mu}_1(a_2),\widetilde{\mu}_2(b_1),\widetilde{\mu}_2(b_2)\}\}=(\widetilde{\mu}_1\Gamma_{ch}\widetilde{\mu}_2)(x)$$

where $x,z,a_1,a_2,b_1,b_2\in S,\ \ \gamma,\delta\in\Gamma$.

$$(f\circ_{ch}g)(x) = \inf\{\max_i\{\max\{f(a_i),f(c_i),g(b_i),g(d_i)\}\}\}$$
$$x+\sum_{i=1}^{n}a_i\gamma_i b_i+z=\sum_{i=1}^{n}c_i\delta_i d_i+z$$

where $x,z,a_i,b_i,c_i,d_i\in S$ and $\gamma_i,\delta_i\in\Gamma,$ for $i=1,...,n$.

$$\leqslant \inf_{x+a_1\gamma b_1+z=a_2\delta b_2+z}\{\max\{f(a_1),\ f(a_2),g(b_1),g(b_2)\}\}$$

where $x,z,a_1,a_2,b_1,b_2\in S$ and $\gamma,\delta\in\Gamma$

$$=(f\Gamma_{ch}g)(x)$$

Therefore

$$A\Gamma_{ch}B\subseteq A\circ_{ch}B................(1)$$

$$
\begin{aligned}
(\widetilde{\mu}_1 \circ_{ch} \widetilde{\mu}_2)(x) \quad &= \cup\{\underset{i}{\cap}\{\cap\{\widetilde{\mu}_1(a_i), \widetilde{\mu}_1(c_i), \widetilde{\mu}_2(b_i), \widetilde{\mu}_2(d_i)\}\}\} \\
&\quad {}_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z} \\
&\quad \text{where } x, z, a_i, b_i, c_i, d_i \in S \text{ and } \gamma_i, \delta_i \in \Gamma, \text{ for } i = 1, ..., n. \\
&\subseteq \cup\{\underset{i}{\cap}\{\cap\{\widetilde{\mu}_1(a_i), \widetilde{\mu}_1(c_i)\}\}\} \\
&= \cup\{\cap\{\underset{i}{\cap}\widetilde{\mu}_1(a_i), \underset{i}{\cap}\widetilde{\mu}_1(c_i)\}\} \\
&\subseteq \cup\{\cap\{\widetilde{\mu}_1(\sum_{i=1}^{n} a_i\gamma_i b_i), \widetilde{\mu}_1(\sum_{i=1}^{n} c_i\delta_i d_i)\}\} \\
&\quad {}_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z} \\
&\subseteq \widetilde{\mu}_1(x) ....(2)
\end{aligned}
$$

$$
\begin{aligned}
(f \circ_{ch} g)(x) \quad &= \inf\{\underset{i}{\max}\{\max\{f(a_i), f(c_i), g(b_i), g(d_i)\}\}\} \\
&\quad {}_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z} \\
&\quad \text{where } x, z, a_i, b_i, c_i, d_i \in S \text{ and } \gamma_i, \delta_i \in \Gamma, \text{ for } i = 1, ..., n. \\
&\geqslant \inf\{\underset{i}{\max}\{\max\{f(a_i), f(c_i)\}\}\} \\
&= \inf\{\max\{\underset{i}{\max} f(a_i), \underset{i}{\max} f(c_i)\}\} \\
&\geqslant \inf\{\max\{f(\sum_{i=1}^{n} a_i\gamma_i b_i), f(\sum_{i=1}^{n} c_i\delta_i d_i)\}\} \\
&\quad {}_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z} \\
&\geqslant f(x) ....(2??????)
\end{aligned}
$$

Since this is true for every representation of x, $A \circ_{ch} B \subseteq A$.

Similarly we can prove that $A \circ_{ch} B \subseteq B$. Therefore $A \circ_{ch} B \subseteq A \cap B$. □

Definition 4.3. A cubic subset $< \widetilde{\mu}, f >$ of a Γ-hemiring S is called cubic h-bi-ideal if for all $x, y, z, a, b \in S$ and $\alpha, \beta \in \Gamma$ we have
(i) $\widetilde{\mu}(x + y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$, $f(x + y) \leqslant \max\{f(x), f(y)\}$
(ii) $\widetilde{\mu}(x\alpha y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$, $f(x\alpha y) \leqslant \max\{f(x), f(y)\}$
(iii) $\widetilde{\mu}(x\alpha y\beta z) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(z)\}$, $f(x\alpha y\beta z) \leqslant \max\{f(x), f(z)\}$
(iv) $x + a + z = b + z \Rightarrow \widetilde{\mu}(x) \supseteq \cap\{\widetilde{\mu}(a), \widetilde{\mu}(b)\}$, $f(x) \leqslant \max\{f(a), f(b)\}$

Definition 4.4. A cubic subset $< \widetilde{\mu}, f >$ of a Γ-hemiring S is called cubic h-quasi-ideal if for all $x, y, z, a, b \in S$ we have
(i) $\widetilde{\mu}(x + y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}$, $f(x + y) \leqslant \max\{f(x), f(y)\}$
(ii) $(\widetilde{\mu} \circ_{ch} \widetilde{\zeta}_{\chi_S}) \cap (\widetilde{\zeta}_{\chi_S} \circ_{ch} \widetilde{\mu}) \subseteq \widetilde{\mu}$, $(f \circ_{ch} \eta_{\chi_S}) \cup (\eta_{\chi_S} \circ_{ch} f) \supseteq f$
(iii) $x + a + z = b + z \Rightarrow \widetilde{\mu}(x) \supseteq \cap\{\widetilde{\mu}(a), \widetilde{\mu}(b)\}$, $f(x) \leqslant \max\{f(a), f(b)\}$.

Proposition 4.2. *Any cubic h-quasi-ideal of S is a cubic h-bi-ideal of S.*

PROOF. Let $<\widetilde{\mu}, f>$ be any cubic h-quasi-ideal of S. It is sufficient to show that $\widetilde{\mu}(x\alpha y\beta z) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(z)\},\ f(x\alpha y\beta z) \leqslant \max\{f(x), f(z)\}$ and $\widetilde{\mu}(x\alpha y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\},\ f(x\alpha y) \leqslant \max\{f(x), f(y)\}$ for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$. In fact, by the assumption,we have

$$\begin{aligned}\widetilde{\mu}(x\alpha y\beta z) &\supseteq ((\widetilde{\mu} o_{ch} \widetilde{\zeta}_{\chi_S}) \cap (\widetilde{\zeta}_{\chi_S} o_{ch} \widetilde{\mu}))(x\alpha y\beta z) \\ &= \cap\{(\widetilde{\mu} o_{ch} \widetilde{\zeta}_{\chi_S})(x\alpha y\beta z), (\widetilde{\zeta}_{\chi_S} o_{ch} \widetilde{\mu})(x\alpha y\beta z)\} \\ &= \cap\{\underset{x\alpha y\beta z+\sum_{i=1}^{n} a_i\gamma_i b_i+z=\sum_{i=1}^{n} c_i\delta_i d_i+z}{\cup(\cap(\widetilde{\mu}(a_i), \widetilde{\mu}(c_i))),\ \cup(\cap(\widetilde{\mu}(b_i), \widetilde{\mu}(d_i)))}\} \\ &\supseteq \cap\{\cap(\widetilde{\mu}(0), \widetilde{\mu}(x)), \cap(\widetilde{\mu}(0), \widetilde{\mu}(z))\} \\ &\quad (\text{since } x\alpha y\beta z + 0\gamma 0 + 0 = x\alpha y\beta z + 0) \\ &= \cap\{\widetilde{\mu}(x), \widetilde{\mu}(z)\}.\end{aligned}$$

$$\begin{aligned}f(x\alpha y\beta z) &\subseteq (f o_{ch} \eta_{\chi_S}) \cup (\eta_{\chi_S} o_{ch} f))(x\alpha y\beta z) \\ &= \max\{(f o_{ch} \eta_{\chi_S})(x\alpha y\beta z), (\eta_{\chi_S} o_{ch} f)(x\alpha y\beta z)\} \\ &= \max\{\underset{x\alpha y\beta z+\sum_{i=1}^{n} a_i\gamma_i b_i+z=\sum_{i=1}^{n} c_i\delta_i d_i+z}{\inf(\max(f(a_i), f(c_i))),\ \inf(\max(f(b_i), f(d_i)))}\} \\ &\leqslant \max\{\max(f(0), f(x)), \max(f(0), f(z))\} \\ &\quad (\text{since } x\alpha y\beta z + 0\gamma 0 + 0 = x\alpha y\beta z + 0) \\ &= \max\{f(x), f(z)\}\end{aligned}$$

Similarly, we can show that $\widetilde{\mu}(x\alpha y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\},\ f(x\alpha y) \leqslant \max\{f(x), f(y)\}$ for all $x, y \in S$ and for all $\alpha \in \Gamma$. ☐

THEOREM 4.1. *A cubic subset $<\widetilde{\mu}, f>$ of a Γ-hemiring S is a cubic left h-ideal of S if and only if for all $x, y, z, a, b \in S$, we have*

(i) $\widetilde{\mu}(x + y) \supseteq \cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\},\ f(x + y) \leqslant \max\{f(x), f(y)\}$

(ii) $\widetilde{\zeta}_{\chi_S} o_{ch} \widetilde{\mu} \subseteq \widetilde{\mu},\ \eta_{\chi_S} o_{ch} f \supseteq f$

(iii) $x + a + z = b + z \Rightarrow \widetilde{\mu}(x) \supseteq \cap\{\widetilde{\mu}(a), \widetilde{\mu}(b)\},\ f(x) \leqslant \max\{f(a), f(b)\}$.

PROOF. Assume that $<\widetilde{\mu}, f>$ is a cubic left h-ideal of S. Then it is sufficient to show that the condition (ii) is satisfied. Let $x \in S$. If x can be expressed as $x + \sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z$, for $z, a_i, b_i, c_i, d_i \in S$ and $\gamma_i, \delta_i \in \Gamma$, for i=1,...,n, then we have

$$\begin{aligned}(\widetilde{\zeta}_{\chi_S} \circ_{ch} \widetilde{\mu})(x) &= \underset{x+\sum_{i=1}^{n} a_i\gamma_i b_i+z=\sum_{i=1}^{n} c_i\delta_i d_i+z}{\cup[\bigcap_i\{\cap\{\widetilde{\zeta}_{\chi_S}(a_i), \widetilde{\zeta}_{\chi_S}(c_i), \widetilde{\mu}(b_i),\ \widetilde{\mu}(d_i)\}\}]} \\ &= \underset{x+\sum_{i=1}^{n} a_i\gamma_i b_i+z=\sum_{i=1}^{n} c_i\delta_i d_i+z}{\cup[\bigcap_i\{\cap\{\widetilde{\mu}(b_i), \widetilde{\mu}(d_i)\}\}]}\end{aligned}$$

and

$$(\widetilde{\zeta}_{\chi_S} \circ_{ch} \widetilde{\mu})(x) \begin{array}{l} \subseteq \bigcup\limits_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z} [\bigcap_i \{\cap\{\widetilde{\mu}(a_i\gamma_i b_i), \widetilde{\mu}(c_i\delta_i d_i)\}\}] \\ \subseteq \bigcup\limits_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z} [\cap\{\widetilde{\mu}(\sum_{i=1}^{n} a_i\gamma_i b_i), \widetilde{\mu}(\sum_{i=1}^{n} c_i\delta_i d_i)\}] \\ \subseteq \bigcup\limits_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z} \widetilde{\mu}(x) = \widetilde{\mu}(x). \end{array}$$

$$\begin{aligned} (\eta_{\chi_S} \circ_{ch} f)(x) &= \inf_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z} [\max_i \{\max\{\eta_{\chi_S}(a_i), \eta_{\chi_S}(c_i), f(b_i), f(d_i)\}\}] \\ &= \inf_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z} [\max_i \{\max\{f(b_i), f(d_i)\}\}] \\ &\geqslant \inf_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z} [\max_i \{\max\{f(a_i\gamma_i b_i), f(c_i\delta_i d_i)\}\}] \\ &\geqslant \inf_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z} [\max\{f(\sum_{i=1}^{n} a_i\gamma_i b_i), f(\sum_{i=1}^{n} c_i\delta_i d_i)\}] \\ &\geqslant \inf_{x+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z} f(x) = f(x). \end{aligned}$$

This implies that $\widetilde{\zeta}_{\chi_S} \circ_{ch} \widetilde{\mu} \subseteq \widetilde{\mu}$, $\eta_{\chi_S} \circ_{ch} f \supseteq f$.

Conversely, assume that the given conditions hold. Then it is sufficient to show the second condition of the definition of cubic h-ideal. Let $x, y \in S$ and $\gamma \in \Gamma$. Then we have

$$\begin{aligned} \widetilde{\mu}(x\gamma y) &\supseteq (\widetilde{\zeta}_{\chi_S} \circ_{ch} \widetilde{\mu})(x\gamma y) = \bigcup_{x\gamma y+\sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z} [\bigcap_i \{\cap\{\widetilde{\mu}(b_i), \widetilde{\mu}(d_i)\}\}] \\ &\supseteq \widetilde{\mu}(y) (\text{since } \ x\gamma y + 0\gamma y + 0 = x\gamma y + 0). \end{aligned}$$

$$
\begin{aligned}
f(x\gamma y) \quad &\leqslant (\eta_{\chi_S} \circ_{ch} f)(x\gamma y) = \inf_{x\gamma y + \sum_{i=1}^{n} a_i\gamma_i b_i + z = \sum_{i=1}^{n} c_i\delta_i d_i + z} [\max_i\{\max\{f(b_i), f(d_i)\}\}] \\
&\leqslant f(y)(\text{since } x\gamma y + 0\gamma y + 0 = x\gamma y + 0).
\end{aligned}
$$

Hence $< \widetilde{\mu}, f >$ is a cubic left h-ideal of S. □

THEOREM 4.2. *Let $A =< \widetilde{\mu}, f >$ and $B =< \widetilde{\nu}, g >$ be a cubic right h-ideal and a cubic left h-ideal of a Γ-hemiring S, respectively. Then $A \cap B$ is a cubic h-quasi-ideal of S.*

PROOF. Let x, y be any element of S. Then

$$
\begin{aligned}
(\widetilde{\mu} \cap \widetilde{\nu})(x + y) \quad &= \cap\{\widetilde{\mu}(x + y), \widetilde{\nu}(x + y)\} \\
&\supseteq \cap\{\cap\{\widetilde{\mu}(x), \widetilde{\mu}(y)\}, \cap\{\widetilde{\nu}(x), \widetilde{\nu}(y)\}\} \\
&= \cap\{\cap\{\widetilde{\mu}(x), \widetilde{\nu}(x)\}, \cap\{\widetilde{\mu}(y), \widetilde{\nu}(y)\}\} \\
&= \cap\{(\widetilde{\mu} \cap \widetilde{\nu})(x), (\widetilde{\mu} \cap \widetilde{\nu})(y)\}.
\end{aligned}
$$

$$
\begin{aligned}
(f \cup g)(x + y) \quad &= \max\{f(x + y), g(x + y)\} \\
&\leqslant \max\{\max\{f(x), f(y)), \max(g(x), g(y)\}\} \\
&= \max\{\max\{f(x), g(x)\}, \cap\{f(y), g(y)\}\} \\
&= \max\{(f \cup g)(x), (f \cup g)(y)\}.
\end{aligned}
$$

Now let $a, b, x, z \in S$ such that $x + a + z = b + z$. Then

$$
\begin{aligned}
(\widetilde{\mu} \cap \widetilde{\nu})(x) \quad &= \cap\{\widetilde{\mu}(x), \widetilde{\nu}(x)\} \\
&\supseteq \cap\{\cap\{\widetilde{\mu}(a), \widetilde{\mu}(b)\}, \cap\{\widetilde{\nu}(a), \widetilde{\nu}(b)\}\} \\
&= \cap\{\cap\{\widetilde{\mu}(a), \widetilde{\nu}(a)\}, \cap\{\widetilde{\mu}(b), \widetilde{\nu}(b)\}\} \\
&= \cap\{(\widetilde{\mu} \cap \widetilde{\nu})(a), (\widetilde{\mu} \cap \widetilde{\nu})(b)\}
\end{aligned}
$$

$$
\begin{aligned}
(f \cup g)(x) \quad &= \max\{f(x), g(x)\} \\
&\leqslant \max\{\max\{f(a), f(b)\}, \max\{g(a), g(b)\}\} \\
&= \max\{\max\{f(a), g(a)\}, \max\{f(b), g(b)\}\} \\
&= \max\{(f \cup g)(a), (f \cup g)(b)\}
\end{aligned}
$$

On the other hand, we have

$$((A \cap B)o_{ch}\chi_S) \cap (\chi_S o_{ch}(A \cap B)) \subseteq (A o_{ch}\chi_S) \cap (\chi_S o_{ch} B) \subseteq (A \cap B).$$

This completes the proof. □

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