**BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE**
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
*Bull. Int. Math. Virtual Inst.*, Vol. **10**(3)(2020), 559-566
DOI: 10.7251/BIMVI2003559J

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

# SUPER VERTEX SUM LABELING OF COMBINATIONS OF STAR WITH PATHS - CATEGORY I & II

**Gerard Rozario Joseph and Joseph Varghese Kureethara**

ABSTRACT. A graph which accepts super vertex sum labeling is Super Vertex Sum Graph. In this paper, we combine stars and paths under Category I & II of combinations paving way to formation of new graphs, analyze and obtain optimal super vertex sum labeling for the new graphs and their super subdivided graphs so formed.

## 1. Introduction

In this study simple, finite and undirected graphs only are considered. We refer [**1**] and [**2**] for all terminologies and notations related to graphs. *Super Vertex Sum Labeling* (SVSL) was introduced by Joseph [**3**] and it was proved that the lower bound of the super vertex sum number is $\sigma_{sv}(G) = 2$. Optimal SVSL for super subdivision of path, cycle, star and spider with $\sigma_{sv}(G) = 2$ was also provided in [**3**]. In [**4**], algorithm for the optimal SVSL of super subdivision of combination of graphs, namely star with spider and star with star under all the possible categories was provided.

Let $G$ be a graph with $q$ edges. A graph $H$ is called a *super subdivision* of $G$ if $H$ is obtained from $G$ by replacing every edge $e_i$ of $G$ by a complete bipartite graph $K_{2,m_i}$ for some $m_i$, $1 \leqslant i \leqslant q$ in such a way that the end vertices of each $e_i$ are identified with the two vertices of the 2-vertex part of $K_{2,m_i}$ after removing the edge $e_i$ from graph $G$. If $m_i$ is varying arbitrarily for each edge $e_i$ then super subdivision is called an *arbitrary super subdivision* of $G$ [**5**]. The gracefulness of the combination of spiders with stars were studied in [**6**].

---

2010 *Mathematics Subject Classification.* Primary 05C78.
*Key words and phrases.* super vertex sum labeling, combination of graphs, super subdivision.

## 2. Combination of graphs

The combinations two graphs $G_1$ and $G_2$ are as follows.

(1) Category I: If a center vertex of $G_1$ of any degree is attached to center vertex of $G_2$ of any degree then the combination $G_1 * G_2$ is Category I.

(2) Category II:

(a) If a non center vertex of $G_1$ of degree 1 is attached to center vertex of $G_2$ of any degree then the combination $G_1 * G_2$ is Category II, type I.

(b) If a non center vertex of $G_1$ of degree 2 is attached to center vertex of $G_2$ of any degree then the combination $G_1 * G_2$ is Category II, type II.

(c) If a non center vertex of $G_1$ of degree m is attached to center vertex of $G_2$ of any degree then the combination $G_1 * G_2$ is Category II, type III.

(3) Category III

(1) If a center vertex of $G_1$ of any degree is attached to non center vertex of $G_2$ of degree 1 then the combination $G_1 * G_2$ is Category III, type I.

(2) If a center vertex of $G_1$ of any degree is attached to non center vertex of $G_2$ of degree 2 then the combination $G_1 * G_2$ is Category III, type II.

(3)If a center vertex of $G_1$ of any degree is attached to non center vertex of $G_2$ of degree m then the combination $G_1 * G_2$ is Category III, type III.

(4) Category IV

(1) If a non center vertex of $G_1$ of degree 1 is attached to non center vertex of $G_2$ of degree 1 then the combination $G_1 * G_2$ is Category IV, type I.

(2) If a non center vertex of $G_1$ of degree 2 is attached to non center vertex of $G_2$ of degree 1 then the combination $G_1 * G_2$ is Category IV, type II.

(3) If a non center vertex of $G_1$ of degree m is attached to non center vertex of $G_2$ of degree 1 then the combination $G_1 * G_2$ is Category IV, type III.

## 3. SVSL of Combination of graphs $G_1 * G_2$

In this section, we study the combination of graphs $G_1 * G_2$ under the consideration that Path $P_m$ $[G_1]$ and Star $K_{1,n}$ $[G_2]$ under the possible categories as defined in [**6**] and obtain the optimal SVSL for new graphs formed and their super subdivided graphs so formed. Unlike star, the path does not always have a center vertex and hence we consider the combinations under category II and category IV for the study.

Theorem 3.1. *The combined graph $G_1 * G_2$ admits SVSL with $\sigma_{sv}(G) = 1$ under the Category II [Type I].*

Proof. By definition, a non center vertex v with degree 1 of [$G_1$] is attached to the center vertex of [$G_2$]. The graph so obtained is a coconut tree graph which has (m + n) vertices and (m + n − 1) edges. Consider x as the isolate required to SVSL the combined graph $G_1 * G_2$. Define $f : V (G_1 * G_2) \to \{1, 2, 3, ..., (m + n + 1)\}$ Label the first vertex with degree $(n + 1)$ as $f (v_1) = 1$ and the vertex with degree 2 and adjacent to it as $f (v_2) = m$.

Initialize $i = 1$.

$$for\ i = 3\ to\ m$$

$$\begin{cases} for\ a\ unlabeled\ v\ of \text{ degree 2 or 1 } and\ adjacent\ to\ v_{i-1} \\ f(v_i) = \begin{cases} f(v_{i-2}) + 1, if\ i\ is\ odd \\ f(v_{i-2}) - 1, if\ i\ is \text{ even} \end{cases} \\ i = i + 1 \end{cases}$$

Reinitialize $i = 1$

$$for\ i = 1\ to\ n$$

$$\begin{cases} for\ a\ unlabeled \text{ pendent vertex } v\ adjacent\ to\ v_1 \\ f(v_{m+i}) = m + i \\ i = i + 1 \end{cases}$$

$$f(x) = m + n + 1$$

Hence, the combined graph $G_1 * G_2$ is Super Vertex Sum Graph with $\sigma_{sv}(G) = 1$ under the Category II [Type I]. □

THEOREM 3.2. *The combined graph $G_1 * G_2$ admits SVSL with $\sigma_{sv}(G) = 1$ under the Category II [Type II].*

PROOF. By definition, Paths's [$G_1$] non center vertex of degree 2 is attached to the center vertex of the star [$G_2$]. The combined graph has $(m + n)$ vertices and $(m + n - -1)$ edges. Consider $x$ as the isolate required to SVSL the combined graph $G_1 * G_2$. Define $f : V(G_1 * G_2) \to \{1, 2, 3, ..., (m + n + 1)\}$

**Case (i)**: adjacent vertex of the end vertex of the path with deg (2) is attached to center vertex of star.

The first vertex with degree $(m + 2)$ receives the label as $f(v_1) = 1$. Label the vertex with $\deg(2)$ $and\ d(v, v_1) = 1$ as $f(v_2) = m - 1$. All the other vertices are labeled as per category II type I by replacing $m$ by $(m - -1)$ and $n$ by $(n + 1)$ in the respective for loops. ∴ The combined graph $G_1 * G_2$ is Super Vertex Sum Graph with $\sigma_{sv}(G) = 1$.

**Case (ii)**: midvertex of path is attached to center vertex of star.

This case can be studied only if $(n \geqslant 2)$ and $(m \geqslant 3)$ in order to identify a midvertex of path.

**(i) If $m$ is even**

$G_1 * G_2$ constitute a spider with n legs of length 1 and two leg of length $\frac{m}{2}$ and $(\frac{n}{2} - 1)$ respectively. The first vertex with degree $n + 2$ receives the label as $f(c) = 1$ For a vertex v with $\deg(1)$ & $d(c, v) = \frac{m}{2} - 1$, $f(v_1) = m$ a vertex v with $\deg(2)$ & $d(v_1, v) = 1$, $f(v_2) = 2$

$$for\ i = 3\ to\ \left(\frac{m}{2} - 1\right)$$

$$\begin{cases} for\ a\ unlabeled\ v\ of \text{ degree 2 } and\ d(v, v_1) = i - 1 \\ f(v_i) = \begin{cases} f(v_{i-2}) - 1, if\ i\ is\ odd \\ f(v_{i-2}) + 1, if\ i\ is \text{ even} \end{cases} \\ i = i + 1 \end{cases}$$

For a vertex v with $\deg(1)$ & $d(c, v) = \frac{m}{2}$, $f\left(v_{\frac{m}{2}}\right) = \frac{m}{2} + 1$

For a vertex v with $\deg(2)\ \&\ d\left(v_{\frac{m}{2}},v\right)=1,\ f\left(v_{\frac{m}{2}+1}\right)=\frac{m}{2}$

$$for\ i=1\ to\ \left(\frac{m}{2}-2\right)$$
$$\begin{cases} for\ a\ unlabeled\ v\ of\ \text{degree 2}\ and\ d\left(v,v_{\frac{m}{2}}\right)=i \\ f\left(v_{\frac{m}{2}+i+1}\right)=\begin{cases} f\left(v_{\frac{m}{2}+i-1}\right)+1, if\ i\ is\ odd \\ f\left(v_{\frac{m}{2}+i+1}\right)-1, if\ i\ is\ \text{even} \end{cases} \\ i=i+1 \end{cases}$$

**(ii) If m is odd**

$G_1 * G_2$ constitute a spider with $n$ legs of length 1 and two legs of length $\frac{m-1}{2}$ each. The first vertex with degree $(n+2)$ receives the label as $f(c)=1$ For the two vertices with $\deg(1)\ \&\ d(c,v)=\frac{m-1}{2}$, label one vertex as$f(v_1)=m$ and the other as $f\left(v_{\frac{m+1}{2}}\right)=\frac{m+1}{2}$ For a vertex v with $\deg(2)\ \&\ d(v_1,v)=1,\ f(v_2)=2$ For a vertex v with $\deg(2)\ \&\ d\left(v_{\frac{m+1}{2}},v\right)=1,\ f\left(v_{\frac{m+3}{2}}\right)=\frac{m+3}{2}$

$$for\ i=3\ to\ \left(\frac{m-1}{2}\right)$$
$$\begin{cases} for\ a\ unlabeled\ v\ of\ \text{degree 2}\ and\ d(v,v_1)=i-1 \\ f(v_i)=\begin{cases} f(v_{i-2})-1, if\ i\ is\ odd \\ f(v_{i-2})+1, if\ i\ is\ \text{even} \end{cases} \\ i=i+1 \end{cases}$$

$$for\ i=1\ to\ \left(\frac{m-1}{2}-2\right)$$
$$\begin{cases} for\ a\ unlabeled\ v\ of\ \text{degree 2}\ and\ d\left(v,v_{\frac{m+1}{2}}\right)=i \\ f\left(v_{\frac{m+3}{2}+i}\right)=\begin{cases} f\left(v_{\frac{m+1}{2}+i-1}\right)-1, if\ i\ is\ odd \\ f\left(v_{\frac{m+1}{2}+i+1}\right)+1, if\ i\ is\ \text{even} \end{cases} \\ i=i+1 \end{cases}$$

To label the remaining unvisited vertices in both cases

$$for\ i=1\ to\ \text{n}$$
$$\begin{cases} for\ a\ unlabeled\ \text{pendent vertex}\ v\ adjacent\ to\ c \\ f(v_{m+i})=m+i \\ i=i+1 \end{cases}$$
$$f(x)=n+m+1$$

$\therefore$ The combined graph $G_1 * G_2$ is Super Vertex Sum Graph with$\sigma_{sv}(G)=1$.

**Case (iii)**: r$^{th}$ vertex of degree 2 other than midvertex or adjacent vertex to end vertices is attached to center vertex of the star.

This case can be studied only if $n\geqslant 2$ & $m\geqslant 7$. If $n<7$ then the combination can be discussed in case (i) or (ii). The combined graph in this case will be a spider with $n$ legs of length 1 and two legs of length $(m-r)$ and $(r-1)$ respectively. The first vertex with degree $(n+2)$ receives the label as $f(c)=1$ For a vertex v with $\deg(1)\ \&\ d(c,v)=m-r,\ f(v_1)=m$ For a vertex v with $\deg(2)\ \&\ d(v_1,v)=1$, $f(v_2)=2$.

For a vertex v with $\deg(1)\ \&\ d(c,v)=r-1$,

$$f\left(v_{m-r+1}\right)=\begin{cases}\frac{m+1}{2}; m\ is\ odd\ \&\ for\ any\ r\\ \frac{m}{2}; m\ is\ \text{even}\ \&\ r\ is\ odd\\ \frac{m}{2}+1; m\ is\ \text{even}\ \&\ r\ is\ \text{even}\end{cases}$$

For a vertex v with $\deg(2)\ \&\ d\left(v, v_{m-r+1}\right)=1$,

$$f\left(v_{m-r+2}\right)=\begin{cases}\frac{m+3}{2}; m\ is\ odd\ \&\ r\ is\ odd\\ \frac{m-1}{2}; m\ is\ odd\ \&\ r\ is\ \text{even}\\ \frac{m}{2}+1; m\ is\ \text{even}\ \&\ r\ is\ odd\\ \frac{m}{2}; m\ is\ \text{even}\ \&\ r\ is\ \text{even}\end{cases}$$

$$for\ i=3\ to\ (m-r)$$
$$\begin{cases}for\ a\ unlabeled\ v\ of\ \text{degree 2}\ and\ d\left(v, v_1\right)=i-1\\ f\left(v_i\right)=\begin{cases}f\left(v_{i-2}\right)-1, if\ i\ is\ odd\\ f\left(v_{i-2}\right)+1, if\ i\ is\ \text{even}\end{cases}\\ i=i+1\end{cases}$$

$$for\ i=1\ to\ (r-3)$$
$$\begin{cases}for\ a\ unlabeled\ v\ of\ \text{degree 2}\ and\ d\left(v, v_{m-r+1}\right)=i+1\\ f\left(v_{m-r+i-2}\right)=\begin{cases}f\left(v_{m-r+i}\right)-1;\\ if\ r\ is\ odd\ \&\ i\ is\ odd/r\ is\ even\ \&\ i\ is\ even\\ f\left(v_{m-r+i}\right)+1,\\ if\ r\ is\ odd\ \&\ i\ is\ even/r\ is\ even\ \&\ i\ is\ odd\end{cases}\\ i=i+1\end{cases}$$

To label the remaining unvisited vertices

$$for\ i=1\ to\ \text{n}$$
$$\begin{cases}for\ a\ unlabeled\ \text{pendent vertex}\ v\ adjacent\ to\ c\\ f\left(v_{m+i}\right)=m+i\\ i=i+1\end{cases}$$
$$f(x)=n+m+1$$

$\therefore$ The combined graph $G_1 * G_2$ of Category II [Type II] is Super Vertex Sum Graph with $\sigma_{sv}(G)=1$. □

Theorem 3.3. *The combined graph $G_1 * G_2$ admits SVSL with $\sigma_{sv}(G)=1$ under the Category IV [Type I].*

Proof. By definition, end vertex $v$ with degree 1 of [$G_1$] is attached to the non center vertex of degree 1 of [$G_2$]. The graph $G_1 * G_2$ is again a coconut tree graph which has $(m+n-1)$ vertices. Consider $x$ as the isolate required to SVSL the combined graph $G_1 * G_2$. Define $f: V\left(G_1 * G_2\right) \rightarrow\{1,2,3, \ldots,(m+n+1)\}$

The first vertex with degree n receives the label as $f(v_1) = 1$ and the vertex with degree 2 and adjacent to $v_1$ as $f(v_2) = n$

$$for\ i = 3\ to\ m+1$$
$$\begin{cases} for\ a\ unlabeled\ v\ of\ \text{degree 2 or 1}\ and\ adjacent\ to\ v_{i-1} \\ f(v_i) = \begin{cases} f(v_{i-2}) + 1, if\ i\ is\ odd \\ f(v_{i-2}) - 1, if\ i\ is\ \text{even} \end{cases} \\ i = i+1 \end{cases}$$
$$for\ i = 1\ to\ \text{n} - 1$$
$$\begin{cases} for\ a\ unlabeled\ \text{pendent vertex}\ v\ adjacent\ to\ v_1 \\ f(v_{n+i}) = n+i \\ i = i+1 \end{cases}$$
$$f(x) = n+m+1$$

Hence, the combined graph $G_1 * G_2$ of Category IV [Type I] is Super Vertex Sum Graph with $\sigma_{sv}(G) = 1$. □

## 4. SVSL of Super subdivision of Combination of graphs $G_1 * G_2$

THEOREM 4.1. *The Super subdivision of combined graph $G_1 * G_2$ admits SVSL with $\sigma_{sv}(G) = 2$ under the Category II [Type I].*

PROOF. As per definition, a non center vertex $v$ with degree 1 of $G_1$ is attached to the center vertex of $G_2$. The graph constitute a coconut tree graph which has $(m+n)$ vertices and $(m+n--1)$ edges. The graph can also be considered as spider with $n$ paths of length 1 and 1 path of length $(m-1)$. Since Super subdivision of spider is SVSG with $\sigma_{sv}(G) = 2$, Super subdivision of the combined graph $G_1 * G_2$ of Category II [Type I] is also Super Vertex Sum Graph with $\sigma_{sv}(G) = 2$. □

THEOREM 4.2. *The Super subdivision of combined graph $G_1 * G_2$ admits SVSL with $\sigma_{sv}(G) = 2$ under the Category II [Type II].*

PROOF. By definition, a non center vertex $v$ with degree 2 of $G_1$ is attached to the center vertex of $G_2$.

**Case (i)**: adjacent vertex of deg (2) to end vertex is attached to center vertex of $G_2$.
The resulting graph is also a spider with $(n+1)$ paths of length 1 and 1 path of length $(m-2)$.

**Case (ii)**: midvertex of $G_1$ is attached to center vertex of $G_2$.

- If m is even, the resulting graph is spider with n paths of length 1, 1 path of length $\frac{m}{2}$ and 1 path of length $\frac{m}{2} - 1$.
- If m is odd, the resulting graph is spider with n paths of length 1 and 2 path of length $\frac{m-1}{2}$.

**Case (iii)**: the $r^{th}$ vertex of degree 2 other than midvertex or adjacent vertex to pendent vertex of the $G_1$ is attached to center vertex of $G_2$.
The resulting graph is also a spider with $(n-1)$ paths of length 1 and 1 path of length $(m+1)$.

In all cases the combined graph is a spider. Since Super subdivision of spider is SVSG with $\sigma_{sv}(G) = 2$, Super subdivision of the combined graph $G_1 * G_2$ of Category II [Type II] is also Super Vertex Sum Graph with $\sigma_{sv}(G) = 2$. □

THEOREM 4.3. *The Super subdivision of combined graph $G_1 * G_2$ admits SVSL with $\sigma_{sv}(G) = 2$ under the Category IV [Type I].*

PROOF. As per definition, a non center vertex v with degree 1 of $G_2$ is attached to the non center vertex of degree 1 of $G_1$. The graph form a coconut tree graph which is also a spider with $(n-1)$ paths of length 1 and 1 path of length $(m+1)$. Since Super subdivision of spider is SVSG with $\sigma_{sv}(G) = 2$, Super subdivision of the combined graph $G_1 * G_2$ of Category IV [Type I] is also Super Vertex Sum Graph with $\sigma_{sv}(G) = 2$. □

THEOREM 4.4. *The Super subdivision of combined graph $G_1 * G_2$ admits SVSL with $\sigma_{sv}(G) = 2$ under the Category IV [Type II].*

PROOF. As per definition, a non center vertex $v$ with degree 2 of $G_1$ is attached to the non center vertex of degree 1 of $G_2$. Considering the vertex of degree 3 as centre vertex, the resulting combined graph in the cases below will constitute a spider similar to Combination of Spider with star in Category II Type I.

**Case (i):** adjacent vertex of deg (2) to end vertex of $G_1$ is attached to non center vertex of $G_2$.

**Case (ii)**: midvertex of $G_1$ is attached to non center vertex of $G_2$.

**Case (iii)**: the $r^{th}$ vertex of degree 2 other than midvertex or adjacent vertex to pendent vertex of the $G_1$ is attached to non center vertex of $G_2$.

The super subdivision of Combination of Spider with star in Category II Type I is already proved to be SVSG in [**4**]. Since Super subdivision of spider is SVSG with $\sigma_{sv}(G) = 2$, Super subdivision of the combined graph $G_1 * G_2$ of Category IV [Type II] is also Super Vertex Sum Graph with $\sigma_{sv}(G) = 2$. □

## References


[1] J. A. Gallian. *A dynamic survey of graph labeling*. Dynamic Surveys. Special issues of *Electron. J. Comb.*, DS6: Dec 15, 2018. Available at: https://www.combinatorics.org/ojs/index.php/eljc/article/view/DS6/pdf

[2] F. Harary. *Graph theory*. Addison Wesley, Reading, Massachusetts, 1969.

[3] G. R. Joseph and J. V. Kureethara. Super vertex sum labeling of graphs. *Int. J. Pure Appl. Math.*, **118**(10)(2018), 293–300.

[4] G. R. Joseph and J. V. Kureethara. Super vertex sum labeling on combinations of star and spider graphs. *International Journal of Scientific Research in Computer Science Applications and Management Studies*, **7**(5)(2018), Article 223. Available at http://www.ijsrcsams.com/images/stories/Past.Issue.Docs/ijsrcsamsv7i5p223.pdf.

[5] G. Sethuraman and P. Selvaraju, Gracefulness of arbitrary super subdivisions of graphs. *Indian J. Pure Appl. Math.*, **32**(7)(2001), 1059–1064.

[6] A. N. Mohamad. The combination of spider graphs with star graphs forms graceful. *International Journal of Advanced Research in Engineering and Applied Sciences*,**2**(5)(2013), 81–103.


Received by editors 30.10.2019; Revised version 17.04.2020; Available online 26.04.2020.

Gerard Rozario Joseph,
Department of Mathematics, IFET College of Engineering, Villupuram, India
*E-mail address*: jgerardrozario@gmail.com

Joseph Varghese Kureethara,
CHRIST (Deemed to be University), Department of Mathematics, Bangalore, India
*E-mail address*: frjoseph@christuniversity.in