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# COMPLETE RESIDUATED ALMOST DISTRIBUTIVE LATTICES

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Abstract. The concepts of the radical of an element and a $p$-primary element in a complete ADL $L$ with a maximal element $m$ are introduced and important properties of radical of an element in a complete residuated ADL are derived.

## 1. Introduction

Swamy, U.M. and Rao, G.C. [**9**] introduced the concept of Almost Distributive Lattices (ADL) as a common abstraction of almost all the existing ring theoretic generalizations of a Boolean algebra (like regular rings, $p-$rings, bi-regular rings, associate rings, $P_1-$rings and etc.) on one hand and distributive lattices on the other.

In [**1**], Dilworth, R.P., has introduced the concept of a residuation in lattices and Ward, M. and Dilworth, R.P., have studied residuated lattices in [**10, 11**]. In [**12**], Ward, M., has studied residuated distributive lattices. We introduced the concepts of a residuation and a multiplication in an ADL and the concept of a residuated ADL in our earlier paper [**6**]. We have proved some important properties of residuation ':' and multiplication '$\cdot$' in a residuated ADL $L$ in [**7**].

In this paper, we introduce the concepts of the radical of an element and a $p$-primary element in a complete ADL $L$ with a maximal element $m$ and derive some properties of radical of an element in a complete residuated ADL. We prove important results in a complete residuated ADL with a maximal element $m$. In Section 2, we recall the definition of an Almost Distributive Lattice (ADL), complete ADL and certain elementary properties of an ADL from Swamy, U.M. and Rao, G.C. [**9**], Rao, G.C. [**2**], Rao, G. C., and Venugopalam Undurthi [**8**] and some

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important results on a residuated almost distributive lattice from our earlier papers [**6, 7**]. In Section 3, we introduce the concepts of the radical $r(a)$ of an element $a$ in a complete ADL $L$ and derive its important properties in a complete residuated ADL $L$. We prove that the radical of a primary element in a complete ADL $L$ is a prime element of $L$. If $p$ is a prime element of a complete ADL $L$, then we define the concept of a $p$-primary element in $L$. We prove important results in a complete residuated ADL $L$ with a maximal element $m$.

## 2. Preliminaries

In this section we collect a few important definitions and results which are already known and which will be used more frequently in the paper.

We begin with the definition of an ADL :

Definition 2.1. ([**2**]). An *Almost Distributive Lattice* (ADL) is an algebra $(L, \vee, \wedge)$ of type $(2, 2)$ satisfying

(1) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
(2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(3) $(a \vee b) \wedge b = b$
(4) $(a \vee b) \wedge a = a$
(5) $a \vee (a \wedge b) = a$,

for all $a, b, c \in L$.

It can be seen directly that every distributive lattice is an ADL.

If there is an element $0 \in L$ such that $0 \wedge a = 0$ for all $a \in L$, then $(L, \vee, \wedge, 0)$ is called *ADL with* 0.

Example 2.1. ([**2**]). Let $X$ be a non-empty set. Fix $x_0 \in X$. For any $x, y \in X$, define

$$x \wedge y = \begin{cases} x_0, & \text{if } x = x_0 \\ y, & \text{if } x \neq x_0 \end{cases} \qquad x \vee y = \begin{cases} y, & \text{if } x = x_0 \\ x, & \text{if } x \neq x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL, with $x_0$ as its zero element. This ADL is called a *discrete ADL*.

For any $a, b \in L$, we say that $a$ is *less than or equals to* $b$ and write $a \leqslant b$, if $a \wedge b = a$. Then "$\leqslant$" is a partial ordering on $L$.

Theorem 2.1 ([**2**]). *Let $(L, \vee, \wedge, 0)$ be an ADL with '0'. Then, for any $a, b \in L$, we have*

(1) $a \wedge 0 = 0$ *and* $0 \vee a = a$
(2) $a \wedge a = a = a \vee a$
(3) $(a \wedge b) \vee b = b$, $a \vee (b \wedge a) = a$ *and* $a \wedge (a \vee b) = a$
(4) $a \wedge b = a \Longleftrightarrow a \vee b = b$ *and* $a \wedge b = b \Longleftrightarrow a \vee b = a$
(5) $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$ *whenever* $a \leqslant b$
(6) $a \wedge b \leqslant b$ *and* $a \leqslant a \vee b$
(7) $\wedge$ *is associative in* $L$
(8) $a \wedge b \wedge c = b \wedge a \wedge c$

(9) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(10) $a \wedge b = 0 \Longleftrightarrow b \wedge a = 0$
(11) $a \vee (b \vee a) = a \vee b$.

It can be observed that an ADL $L$ satisfies almost all the properties of a distributive lattice except, possible the right distributivity of $\vee$ over $\wedge$, the commutativity of $\vee$, the commutativity of $\wedge$ and the absorption law $(a \wedge b) \vee a = a$. Any one of these properties convert $L$ into a distributive lattice.

THEOREM 2.2 ([**2**]). *Let $(L, \vee, \wedge, 0)$ be an ADL with 0. Then the following are equivalent:*
(1) *$(L, \vee, \wedge, 0)$ is a distributive lattice;*
(2) *$a \vee b = b \vee a$, for all $a, b \in L$;*
(3) *$a \wedge b = b \wedge a$, for all $a, b \in L$;*
(4) *$(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$, for all $a, b, c \in L$.*

PROPOSITION 2.1 ([**2**]). *Let $(L, \vee, \wedge)$ be an ADL. Then for any $a, b, c \in L$ with $a \leqslant b$, we have*
(1) *$a \wedge c \leqslant b \wedge c$*
(2) *$c \wedge a \leqslant c \wedge b$*
(3) *$c \vee a \leqslant c \vee b$.*

DEFINITION 2.2. ([2]). An element $m \in L$ is called *maximal* if it is maximal as in the partially ordered set $(L, \leqslant)$. That is, for any $a \in L$, $m \leqslant a$ implies $m = a$.

THEOREM 2.3 ([**2**]). *Let $L$ be an ADL and $m \in L$. Then the following are equivalent:*
(1) *$m$ is maximal with respect to $\leqslant$;*
(2) *$m \vee a = m$ for all $a \in L$; and*
(3) *$m \wedge a = a$ for all $a \in L$.*

LEMMA 2.1 ([**2**]). *Let $L$ be an ADL with a maximal element $m$ and $x, y \in L$. If $x \wedge y = y$ and $y \wedge x = x$, then $x$ is maximal if and only if $y$ is maximal. Also, the following conditions are equivalent:*
(i) *$x \wedge y = y$ and $y \wedge x = x$; and*
(ii) *$x \wedge m = y \wedge m$.*

DEFINITION 2.3. ([**2**]) If $(L, \vee, \wedge, 0, m)$ is an ADL with 0 and with a maximal element $m$, then the set $I(L)$ of all ideals of $L$ is a complete lattice under set inclusion. In this lattice, for any $I, J \in I(L)$, the l.u.b. and g.l.b. of $I, J$ are given by $I \vee J = \{(x \vee y) \wedge m \mid x \in I, y \in J\}$ and $I \wedge J = I \cap J$. The set $PI(L) = \{(a] \mid a \in L\}$ of all principal ideals of $L$ forms a sublattice of $I(L)$. (Since $(a] \vee (b] = (a \vee b]$ *and* $(a] \cap (b] = (a \wedge b]$.)

DEFINITION 2.4. ([**8**]) An ADL $L = (L, \vee, \wedge, 0, m)$ with a maximal element $m$ is said to be a *complete* ADL, if $PI(L)$ is a complete sublattice of the lattice $I(L)$.

It can be noted that this concept of complete ADL generalizes the concept of a complete lattice in the sense that if $(L, \vee, \wedge, 0, m)$ is complete ADL and if $(L, \vee, \wedge)$ is a lattice, then it is a complete lattice.

Theorem 2.4 ([**8**]). *Let* $L = (L, \vee, \wedge, 0, m)$ *be an ADL with a maximal element* $m$*. Then* $L$ *is a complete ADL if and only if the lattice* $([0, m], \vee, \wedge)$ *is a complete lattice.*

In the following, we give the concepts of residuation and multiplication in an almost distributive lattice $(ADL)$ $L$ and the definition of a residuated almost distributive lattice taken from our earlier paper [**6**].

Definition 2.5. ([**6**]) Let $L$ be an ADL with a maximal element $m$. A binary operation : on an ADL $L$ is called a *residuation* over $L$ if, for $a, b, c \in L$ the following conditions are satisfied:
$(R1)$ $a \wedge b = b$ if and only if $a : b$ is maximal;
$(R2)$ $a \wedge b = b \implies$ (i) $(a : c) \wedge (b : c) = b : c$ and (ii) $(c : b) \wedge (c : a) = c : a$;
$(R3)$ $[(a : b) : c] \wedge m = [(a : c) : b] \wedge m$;
$(R4)$ $[(a \wedge b) : c] \wedge m = (a : c) \wedge (b : c) \wedge m$;
$(R5)$ $[c : (a \vee b)] \wedge m = (c : a) \wedge (c : b) \wedge m$.

Definition 2.6. ([**6**]) Let $L$ be an ADL with a maximal element $m$. A binary operation $\cdot$ on an ADL $L$ is called a *multiplication* over L if, for $a, b, c \in L$ the following conditions are satisfied:
$(M1)$ $(a.b) \wedge m = (b.a) \wedge m$;
$(M2)$ $[(a.b).c] \wedge m = [a.(b.c)] \wedge m$;
$(M3)$ $(a.m) \wedge m = a \wedge m$; and $(M4)$ $[a.(b \vee c)] \wedge m = [(a.b) \vee (a.c)] \wedge m$.

Definition 2.7. ([**6**]) An ADL $L$ with a maximal element $m$ is said to be a *residuated almost distributive lattice (residuated ADL)*, if there exists two binary operations ':' and '$\cdot$' on $L$ satisfying conditions R1 to R5, M1 to M4 and the following condition
(A) $(x : a) \wedge b = b$ if and only if $x \wedge (a.b) = a.b$, for any $x, a, b \in L$.

We use the following properties frequently later in the results.

Lemma 2.2 ([**6**]). *Let* $L$ *be an ADL with a maximal element* $m$ *and* $\cdot$ *a binary operation on* $L$ *satisfying the conditions* $M1 - -M4$*. Then for any* $a, b, c, d \in L$,
(i) $a \wedge (a.b) = a \cdot b$ *and* $b \wedge (a.b) = a \cdot b$;
(ii) $a \wedge b = b \implies (c \cdot a) \wedge (c \cdot b) = c \cdot b$ *and* $(a \cdot c) \wedge (b \cdot c) = b \cdot c$;
(iii) $d \wedge [(a \cdot b) \cdot c] = (a \cdot b) \cdot c$ *if and only if* $d \wedge [a \cdot (b \cdot c)] = a(b \cdot c)$;
(iv) $(a \cdot c) \wedge (b \cdot c) \wedge [(a \wedge b) \cdot c] = (a \wedge b) \cdot c$;
(v) $d \wedge (a \cdot c) \wedge (b \cdot c) = (a \cdot c) \wedge (b \cdot c) \implies d \wedge [(a \wedge b) \cdot c] = (a \wedge b) \cdot c$;
(vi) $d \wedge [(a \cdot c) \vee (b \cdot c)] = (a \cdot c) \vee (b \cdot c) \Leftrightarrow d \wedge [(a \vee b) \cdot c] = (a \vee b) \cdot c$;
(vii) $a \wedge m = b \wedge m \implies (a \cdot c) \wedge m = (b \cdot c) \wedge m$; *and*
(viii) $(a \wedge m = b \wedge m$ *and* $c \wedge m = d \wedge m) \implies (a \cdot c) \wedge m = (b \cdot d) \wedge m$.

The following result is a direct consequence of M1 of Definition 2.6.

Lemma 2.3 ([**6**]). *Let* $L$ *be an ADL with a maximal element* $m$ *and* $\cdot$ *a binary operation on* $L$ *satisfying the condition* $M1$*. For* $a, b, x \in L$, $a \wedge (x \cdot b) = x \cdot b$ *if and only if* $a \wedge (b \cdot x) = b \cdot x$.

In the following, we give some important properties of residuation ': ' and multiplication '$\cdot$ ' in a residuated ADL $L$. These are taken from our earlier paper [**7**].

LEMMA 2.4 ([**7**]). *Let $L$ be a residuated ADL with a maximal element $m$. For $a, b, c, d \in L$, the following hold in $L$:*

(1) $(a : b) \wedge a = a$;
(2) $[a : (a : b)] \wedge (a \vee b) = a \vee b$;
(3) $[(a : b) : c] \wedge [a : (b.c)] = a : (b.c)$;
(4) $[a : (b \cdot c)] \wedge [(a : b) : c] = (a : b) : c$;
(5) $[(a \wedge b) : b] \wedge (a : b) = a : b$;
(6) $(a : b) \wedge [(a \wedge b) : b] = (a \wedge b) : b$;
(7) $[a : (a \vee b)] \wedge m = (a : b) \wedge m$;
(8) $[c : (a \wedge b)] \wedge [(c : a) \vee (c : b)] = (c : a) \vee (c : b)$;
(9) *If* $a : b = a$ *then* $a \wedge (b \cdot d) = b \cdot d \Longrightarrow a \wedge d = d$;
(10) $\{a : [a : (a : b)]\} \wedge (a : b) = a : b$;
(11) $[(a \vee b) : c] \wedge [(a : c) \vee (b : c)] = (a : c) \vee (b : c)$;
(12) $a \wedge m \geqslant b \wedge m \Longrightarrow (a : c) \wedge m \geqslant (b : c) \wedge m$;
(13) $(a : b) \wedge \{a : [a : (a : b)]\} = a : [a : (a : b)]$;
(14) $a \wedge b = b \Longrightarrow (a \cdot c) \wedge (b \cdot c) = b \cdot c$;
(15) $a \wedge b \wedge (a \cdot b) = a \cdot b$;
(16) $[(a \cdot b) : a] \wedge b = b$;
(17) $(a \cdot b) \wedge [(a \wedge b) \cdot (a \vee b)] = (a \wedge b) \cdot (a \vee b)$;
(18) $a \vee b$ *is maximal* $\Longrightarrow (a \cdot b) \wedge a \wedge b = a \wedge b$;
(19) $(x_1 \vee x_2)^{n+1} \wedge m = [x_1^{n+1} \vee (x_1^n \cdot x_2) \vee (x_1^{n-1} \cdot x_2^2) \vee ... \vee (x_1 \cdot x_2^n) \vee x_2^{n+1}] \wedge m$, *for any* $x_1, x_2 \in L$ *and* $n \in Z^+$; *and*
(20) $(x_1 \vee x_2)^{k_1+k_2} \wedge m \leqslant (x_1^{k_1} \vee x_2^{k_2}) \wedge m$, *for any* $x_1, x_2 \in L$ *and* $k_1, k_2 \in Z^+$.

## 3. Complete residuated Almost Distributive Lattices

In this section, we introduce the concepts of the radical of an element and a $p$-primary element in a complete ADL $L$ with a maximal element $m$ and derive some properties of radical of an element in a complete residuated ADL. We prove important results in a complete residuated ADL with a maximal element $m$.

We recall the following concepts on a residuated ADL $L$

DEFINITION 3.1. ([**4**]] An element $p$ of a residuated ADL $L$ is called
(i) *prime*, if $p$ is not a maximal element of $L$ and for any $a, b \in L$ holds

$$p \wedge (a \cdot b) = a \cdot b \Longrightarrow \text{ either } p \wedge a = a \text{ or } p \wedge b = b.$$

(ii) *primary*, if $p$ is not a maximal element of $L$ and for any $a, b \in L$ holds

$$p \wedge (a \cdot b) = a \cdot b \text{ and } p \wedge a \neq a \Longrightarrow p \wedge b^s = b^s$$

for some $s \in Z^+$.

NOTE 3.1. Clearly, every prime element in a residuated ADL is primary.

DEFINITION 3.2. ([**4**]) An ADL $L$ is said to satisfy the *ascending chain condition (a.c.c.)* if for every increasing sequence $x_1 \leqslant x_2 \leqslant x_3 \leqslant ...$ in $L$, there exists a positive integer $n$ such that $x_n = x_{n+1} = x_{n+2} = ....$

In this case, we say that $L$ is an *ADL with a.c.c.*

DEFINITION 3.3. Let $L$ be a residuated ADL with a maximal element $m$ and with a.c.c. and $a \in L$. We define $a^n$, by induction, as follows $a^1 = a$ and $a^{n+1} = a^n \cdot a$ for all $n \in Z^+ - \{1\}$. By convention, we take $a^0 = m$.

In the following theorem we define a binary operation ':' on $L$ from the multiplication '$\cdot$'. Using only the properties M1, M2, M3 and M4 of '$\cdot$', we prove some properties of ':'.

THEOREM 3.1. *Let $L$ be a complete ADL with a maximal element $m$. Suppose '$\cdot$' is a multiplication on $L$. For $a, b \in L$, write $B_{a,b} = \{x \in L \mid a \wedge (x \cdot b) = x \cdot b\}$ and define $a : b = \bigvee_{x \in B_{a,b}} (x \wedge m)$. Then for $a, b, x, c \in L$, we have*

(i) $a \wedge b = b \Longrightarrow c : a \leqslant c : b$;
(ii) $x \wedge (a \cdot b) = a.b \Longrightarrow (x : a) \wedge b = b$; *and*
(iii) $[a : (b \cdot c)] \leqslant (a : b) : c$.

PROOF. Let $a, b, c, x \in L$.
(i) Suppose $a \wedge b = b$. Let $x \in B_{c,a}$. Then $c \wedge (x \cdot a) = x \cdot a$ and $c \wedge (x \cdot a) \wedge (x \cdot b) = (x \cdot a) \wedge (x \cdot b)$. Thus $c \wedge (x \cdot b) = x \cdot b$ by Lemma 2.2 (ii). From here, it follows $x \in B_{c,b}$ $x \wedge m \leqslant c : b$. Thus $x \wedge m \leqslant c : b$, for all $x \in B_{c,a}$. So that $\bigvee_{x \in B_{c,a}} (x \wedge m) \leqslant c : b$. Therefore, $c : a \leqslant c : b$ since $c : a \leqslant m$ and $c : b \leqslant m$).
(ii) Suppose $x \wedge (a \cdot b) = a \cdot b$ Then $x \wedge (b \cdot a) = b \cdot a$ by Lemma 2.3 and $b \in B_{x,a}$. Thus $b \wedge m \leqslant x : a$ and $(x : a) \wedge b \wedge m = b \wedge m$. From here, it follows $(x : a) \wedge b = b$.
(iii) Let $x \in B_{a,(b \cdot c)}$. Then $a \wedge [x \cdot (c \cdot b)] = x \cdot (c \cdot b)$ and $a \wedge [(x \cdot c) \cdot b] = (x \cdot c) \cdot b$ By Lemma 2.2 (iii). From here, it follows $a \wedge [b \cdot (x \cdot c)] = b \cdot (x \cdot c)$ by Lemma 2.3 and $(a : b) \wedge (x \cdot c) = x \cdot c$ By (ii), above. from here, it follows $(a : b) \wedge (c \cdot x) = c \cdot x$ by Lemma 2.3 and $[(a : b) : c] \wedge x = x$ again by (ii), above. Thus $x \wedge m \leqslant (a : b) : c$ for all $x \in B_{a,(b \cdot c)}$. So that $\bigvee_{x \in B_{a,(b \cdot c)}} (x \wedge m) \leqslant (a : b) : c$. Therefore,

$$a : (b \cdot c) \leqslant (a : b) : c.$$

□

The following Lemma is taken from our earlier paper [**5**].

LEMMA 3.1. *Let $L$ be an ADL with a maximal element $m$, '$\cdot$' a multiplication on $L$ and $a, b \in L$ such that $a \wedge b = b$. Then $a^n \wedge b^n = b^n$, for any $n \in Z^+$.*

In the following, we introduce the concept of *radical of an element* in a complete ADL with a maximal element $m$.

DEFINITION 3.4. Let $L$ be a complete ADL with a maximal element $m$. Suppose '$\cdot$' is a multiplication on $L$ and $a \in L$. Let $R_a = \{x \in L \mid a \wedge x^k = x^k$ for some $k \in Z^+\}$. Then $\bigvee_{x \in R_a} (x \wedge m)$ is called *radical of* $a$ and it is denoted by $\mathbf{r(a)}$.

We derive important properties of radical of an element in a complete residuated ADL $L$ with a maximal element $m$.

THEOREM 3.2. *Let $L$ be a complete residuated ADL with a maximal element $m$ and $a, b \in L$. Then*

(1) $r(a) \wedge a = a$ *and* $r(a) \leqslant r(r(a))$;

(2) *If $a$ is a maximal element of $L$, then $r(a)$ is a maximal element of $L$;*

(3) $a \wedge b = b \Longrightarrow r(b) \leqslant r(a)$ *and hence* $b \leqslant a \Longrightarrow r(b) \leqslant r(a)$;

(4) $r(a \cdot b) = r(a \wedge b) \leqslant r(a) \wedge r(b)$;

(5) $r(a) \vee r(b) \leqslant r(a \vee b) \leqslant r[r(a) \vee r(b)]$;

(6) *If $a \wedge b^k = b^k$ for some $k \in Z^+$, then $r(b) \leqslant r(a)$ and hence*

$$b^k \leqslant a \Longrightarrow r(b) \leqslant r(a);$$

(7) *If $p$ is a prime element of $L$, then $r(p) = p \wedge m = r(p \wedge m)$ and*

$$r(p^n) = p \wedge m, \quad \textit{for all } n \in Z^+;$$

(8) $r(m) = m$.

PROOF. Let $a, b \in L$ and $R_a = \{x \in L \mid a \wedge x^k = x^k, for\ some\ k \in Z^+\}$. Define radical of an element $a$ by $r(a) = \bigvee\limits_{x \in R_a} (x \wedge m)$.

(1) Since $a \wedge a = a$, we get that $a \in R_a$. Therefore, $a \wedge m \leqslant r(a)$. Hence $r(a) \wedge a = a$. Now, replacing $a$ by $r(a)$ in the above, we get that $r(a) \leqslant r(r(a))$.

(2) Suppose that $a$ is a maximal element of $L$. By (1), we have $r(a) \wedge a = a$. Then $r(a) \wedge a \wedge t = a \wedge t$, for any $t \in L$. Since $a$ is a maximal element of $L$, we get that $r(a) \wedge t = t$, for any $t \in L$. Therefore, $r(a)$ is a maximal element of $L$.

(3) Suppose $a \wedge b = b$. Let $x \in R_b$. Then $b \wedge x^k = x^k$, for some $k \in Z^+$. Thus $x^k = a \wedge b \wedge x^k$ (since $a \wedge b = b$ ) $= a \wedge x^k$ (since $b \wedge x^k = x^k$ ). Hence, $x \in R_a$. Therefore, $R_b \subseteq R_a$. So that $\bigvee\limits_{x \in R_b} (x \wedge m) \leqslant \bigvee\limits_{x \in R_a} (x \wedge m)$. Hence $r(b) \leqslant r(a)$. Therefore, $a \wedge b = b \Longrightarrow r(b) \leqslant r(a)$ and hence $b \leqslant a \Longrightarrow r(b) \leqslant r(a)$.

(4) By property (15) of Lemma 2.4, we have $a \wedge b \wedge (a \cdot b) = a \cdot b$. Now, by (3) above, we get that $r(a \cdot b) \leqslant r(a \wedge b)$. Let $x \in R_{a \wedge b}$. Then $a \wedge b \wedge x^k = x^k$, for some $k \in Z^+$. Thus
$a \wedge x^k = a \wedge b \wedge x^k = x^k$ and $b \wedge x^k = a \wedge b \wedge x^k = x^k$, for some $k \in Z^+$.
$\Longrightarrow (a \cdot b) \wedge (x^k \cdot b) = x^k \cdot b$ and $(x^k \cdot b) \wedge (x^k \cdot x^k) = x^k \cdot x^k$ (by Lemma 2.1.6 (ii) )
$\Longrightarrow (a \cdot b) \wedge (x^k \cdot b) \wedge (x^k \cdot \cdot b) \wedge (x^k \cdot .x^k) = (x^k \cdot b) \wedge (x^k \cdot x^k)$
$\Longrightarrow (a \cdot b) \wedge (x^k \cdot x^k) = x^k \cdot x^k$
$\Longrightarrow (a \cdot b) \wedge x^{2k} = x^{2k}$
$\Longrightarrow x \in R_{a \cdot b}$
Therefore, $R_{a \wedge b} \subseteq R_{a \cdot b}$. Hence $\bigvee\limits_{x \in R_{a \wedge b}} (x \wedge m) \leqslant \bigvee\limits_{x \in R_{a \cdot b}} (x \wedge m)$. Therefore, $r(a \wedge b) \leqslant r(a \cdot b)$. Thus $r(a \cdot b) = r(a \wedge b)$. Since $a \wedge a \wedge b = a \wedge b$ and $b \wedge a \wedge b = a \wedge b$. By (3), above, we get that $r(a \wedge b) \leqslant r(a)$ and $r(a \wedge b) \leqslant r(b)$ and $r(a \wedge b) \leqslant r(a) \wedge r(b)$. Hence $r(a \cdot b) = r(a \wedge b) \leqslant r(a) \wedge r(b)$.

(5) Since $(a \vee b) \wedge a = a$ and $(a \vee b) \wedge b = b$. By (3), above, we get that $r(a) \leqslant r(a \vee b)$ and $r(b) \leqslant r(a \vee b)$. Therefore, $r(a) \vee r(b) \leqslant r(a \vee b)$. Now, by (1) above, we have $r(a) \wedge a = a$ and $r(b) \wedge b = b$. Then $[r(a) \wedge a] \vee [r(b) \wedge b] = a \vee b$. Now,

$[r(a) \vee r(b)] \wedge (a \vee b) = [r(a) \vee r(b)] \wedge [\{r(a) \wedge a\} \vee \{r(b) \wedge b\}]$
$= [\{r(a) \vee r(b)\} \wedge r(a) \wedge a] \vee [\{r(a) \vee r(b)\} \wedge r(b) \wedge b] = [r(a) \wedge a] \vee [r(b) \wedge b] = a \vee b$ since $r(a) \wedge a = a$ and $r(b) \wedge b = b$ ). By (3) above, we get that $r(a \vee b) \leqslant r[r(a) \vee r(b)]$. Hence $r(a) \vee r(b) \leqslant r(a \vee b) \leqslant r[r(a) \vee r(b)]$.

(6) Suppose $a \wedge b^k = b^k$, for some $k \in Z^+$. Let $x \in R_b$. Then $b \wedge x^m = x^m$, for some $m \in Z^+$. Thus $b^k \wedge x^{m.k} = x^{m.k}$, for any $k \in Z^+$ (b Lemma 3.1). From here, it follows $a \wedge x^{m.k} = a \wedge b^k \wedge x^{m.k} = b^k \wedge x^{m.k}$ (since $a \wedge b^k = b^k$) $= x^{m.k}$ (since $b^k \wedge x^{m.k} = x^{m.k}$ ). Hence, $x \in R_a$. Therefore, $R_b \subseteq R_a$. So, $\bigvee_{x \in R_b} (x \wedge m) \leqslant \bigvee_{x \in R_a} (x \wedge m)$ and $r(b) \leqslant r(a)$. Therefore, $a \wedge b^k = b^k$, for some $k \in Z^+$ which it follows $r(b) \leqslant r(a)$ and hence $b^k \leqslant a \Longrightarrow r(b) \leqslant r(a)$.

(7) Suppose $p$ is a prime element of $L$. By (1) above, we have $r(p) \wedge p = p$. Them $p \wedge m \leqslant r(p)$. Now, let $x \in R_p$. Then $p \wedge x^k = x^k$, for some $k \in Z^+$. Since $p$ is a prime element of $L$, we get that $p \wedge x = x$. This is true for any $x \in R_p$. Therefore, $x \wedge m \leqslant p \wedge m$. So that $\bigvee_{x \in R_p} (x \wedge m) \leqslant p \wedge m$. Therefore, $r(p) \leqslant p \wedge m$. Hence $r(p) = p \wedge m$. Since $p$ is a prime element of $L$, we get that $p \wedge m$ is a prime element of $L$. By last equality, we get that $r(p \wedge m) = p \wedge m$. Thus $r(p) = p \wedge m = r(p \wedge m)$. Now, we prove that $r(p^n) = p \wedge m$, for all $n \in Z^+$. By Lemma 2.2 (i), we have $p^n \wedge p^{n+1} = p^{n+1}$. Then $p \in R_{p^n}$. Therefore, $p \wedge m \leqslant r(p^n)$ Again, by Lemma 2.2 (i), we get that $p \wedge p^n = p^n$. By (3) above, we get that $r(p^n) \leqslant r(p) = p \wedge m$ (since $r(p) = p \wedge m$). Thus $r(p^n) \leqslant p \wedge m$. Finally, we get that $r(p^n) = p \wedge m$ for all $n \in Z^+$.

(8) We have that every maximal element of $L$ is a prime element of $L$. Then, $m$ is a prime element of $L$. By (7), above, we get that $r(m) = m$. □

The following result is taken from our earlier paper [**3**].

THEOREM 3.3. *If an ADL $L$ satisfies the ascending chain condition, then every non empty subset of $L$ has a maximal element.*

In the following, we prove an important result in a complete ADL $L$ with a multiplication '$\cdot$'. We do not use M5 and define a binary operation '$:$' on $L$ which satisfies many properties of residuation.

THEOREM 3.4. *Let $L$ be a complete ADL with a maximal element $m$ satisfying the a.c.c. and '$\cdot$' a multiplication on $L$. Then, for any $a \in L$, there exists $k \in Z^+$ such that $a \wedge (r(a))^k = (r(a))^k$ and $a \wedge (r(a))^{k-1} \neq (r(a))^{k-1}$.*

PROOF. Let $a \in L$ and write $R_a = \{x \in L \mid a \wedge x^s = x^s, for\ some\ s \in Z^+\}$. Then $r(a) = \bigvee_{x \in R_a} (x \wedge m)$.

First we prove that $R_a$ is closed under the operation '$\vee$'. Now, for $x_1, x_2 \in R_a$ we have $a \wedge x_1^s = x_1^s$, $a \wedge x_2^t = x_2^t$, for some $s, t \in Z^+$. Then $(a \wedge x_1^s) \vee (a \wedge x_2^t) = x_1^s \vee x_2^t$ and $a \wedge (x_1^s \vee x_2^t) = x_1^s \vee x_2^t$. Thus $a \wedge m \geqslant (x_1^s \vee x_2^t) \wedge m \geqslant (x_1 \vee x_2)^{s+t} \wedge m$ (by property 20 of Lemma 2.4 . Hence $a \wedge (x_1 \vee x_2)^{s+t} = (x_1 \vee x_2)^{s+t}$ and $x_1 \vee x_2 \in R_a$. Therefore, $R_a$ is closed under the operation '$\vee$'.

Since $L$ satisfies the a.c.c., by Theorem 3.3, we get that $R_a$ has a maximal element, say $x_a$. Then $x_a \wedge m \leqslant r(a)$. For any $x \in R_a$, since $x_a \leqslant x_a \vee x$, we get that $x_a = x_a \vee x$. Therefore, $x \wedge m \leqslant x_a \wedge m$. This is true for all $x \in R_a$. Hence $r(a) = \bigvee_{x \in R_a} (x \wedge m) \leqslant x_a \wedge m$. Thus $r(a) = x_a \wedge m$. By Lemma 3.1, we get that $x_a^n \wedge m = (r(a))^n \wedge m$, for all $n \in Z^+$. Now, for $x_a \in R_a$ we have $a \wedge x_a^s = x_a^s$, for some $s \in Z^+$. Then $a \wedge x_a^s \wedge m = x_a^s \wedge m$ and $a \wedge (r(a))^s \wedge m = (r(a))^s \wedge m$. Thus $q \wedge (r(a))^s = (r(a))^s$, for some $s \in Z^+$. Now, $q \wedge (r(a))^s = (r(a))^s$, for some $s \in Z^+$. Then $q \wedge (r(a))^n = (r(a))^n$, for any $n \geqslant s$. Choose a smallest positive integer $k$ such that $q \wedge (r(a))^k = (r(a))^k$. It is possible, since $a \wedge (r(a))^0 = a \wedge m = a \neq (r(a))^0$. Thus $a \wedge (r(a))^{k-1} \neq (r(a))^{k-1}$. □

Now, we prove the following theorem.

Theorem 3.5. *Let $L$ be a complete ADL with a maximal element $m$. Suppose '$\cdot$' is a multiplication on $L$, $q$ be a primary element of $L$ and $a, b \in L$. Then $r(q) \wedge a \neq a$ if and only if $q : a = q \wedge m$, where $a : b$ is defined as in Theorem 3.1.*

Proof. Assume that $r(q) \wedge a \neq a$. For any $x \in B_{q,a}$, we have $q \wedge (x \cdot a) = x \cdot a$. Since $q$ is a primary element of $L$, we get that either $q \wedge x = x$ or $q \wedge a^t = a^t$ for some $t \in Z^+$. If $q \wedge a^t = a^t$ for some $t \in Z^+$, then $a \in R_q$. Therefore, $r(q) \wedge a \wedge m = a \wedge m$ or $r(q) \wedge a = a$. This is a contradiction to our assumption that $r(q) \wedge a \neq a$. Therefore, $q \wedge a^t \neq a^t$, for any $t \in Z^+$. Hence $q \wedge x = x$. This is true for any $x \in B_{q,a}$. Thus $x \wedge m \leqslant q \wedge m$, for all $x \in B_{q,a}$. Hence $q : a \leqslant q \wedge m$.

By Lemma 2.2 (i), we have $q \wedge (q \cdot a) = q \cdot a$, so that $q \in A_{q,a}$. Hence $q \wedge m \leqslant q : a$. Thus $q : a = q \wedge m$.

Now, suppose that $q : a = q \wedge m$ and suppose, if possible, $r(q) \wedge a = a$. Then, by condition (i) of Theorem 3.1, we get that $q : r(q) \leqslant q : a = q \wedge m$. Thus $q : r(q) = q \wedge m \wedge (q : r(q)) = q \wedge (q : r(q))$.

Now, by Lemma 2.2 (i), we have $q \wedge (r(q) \cdot q) = r(q) \cdot q$. Hence, by Theorem 3.1 (ii), we get that $(q : r(q)) \wedge q = q$. Therefore, $(q : r(q)) \wedge m = q \wedge m$. By Theorem 3.4, there exists $k \in Z^+$ such that $q \wedge (r(q))^k = (r(q))^k$ and $q \wedge (r(q))^{k-1} \neq (r(q))^{k-1}$. Now, $q \wedge (r(q))^k = (r(q))^k$ implies $q \wedge (r(q).(r(q))^{k-1}) = r(q).(r(q))^{k-1}$ and $(q : r(q)) \wedge (r(q))^{k-1} = (r(q))^{k-1}$ (by condition (ii) of Theorem 3.1 ). Then $(q : r(q)) \wedge m \wedge (r(q))^{k-1} = (r(q))^{k-1}$ and $q \wedge m \wedge (r(q))^{k-1} = (r(q))^{k-1}$ (since $(q : r(q)) \wedge m = q \wedge m$). Thus $q \wedge (r(q))^{k-1} = (r(q))^{k-1}$. This is a contradiction to $q \wedge (r(q))^{k-1} \neq (r(q))^{k-1}$. Therefore, have to be $r(q) \wedge a \neq a$. □

Theorem 3.6. *Let $L$ be a complete ADL with a maximal element $m$ satisfying the a.c.c and '$\cdot$' a multiplication on $L$. If $q$ is a primary element of $L$, then $r(q)$ is a prime element of $L$.*

Proof. Let $R_q = \{x \in L \mid q \wedge x^s = x^s, \text{ for some } s \in Z^+\}$. Define $r(q) = \bigvee_{x \in R_q} (x \wedge m)$. By Theorem 3.4, we get that $q \wedge r(q)^k = r(q)^k$ and $q \wedge r(q)^{k-1} \neq r(q)^{k-1}$, for some $k \in Z^+$. Let $a \in L$ and $A_{q,a} = \{x \in L \mid q \wedge (x.a) = x.a\}$. Define $q : a = \bigvee_{x \in A_{q,a}} (x \wedge m)$. By Theorem 3.5, we get that $r(q) \wedge a \neq a$ if and only if $q : a = q \wedge m$. Now, we will prove that $r(q)$ is a prime element of $L$. Let $a, b \in L$ such that $r(q) \wedge a \neq a$ and $r(q) \wedge b \neq b$.

First we prove that $q : b = (q \wedge m) : b$. If $x \in A_{q,b}$, it means that $q \wedge (x \cdot b) = x \cdot b$. Then $q \wedge m \wedge (x \cdot b) = x \cdot b$ and $x \in A_{q \wedge m, b}$. Therefore, we get that $q : b = (q \wedge m) : b$. Since $r(q) \wedge a \neq a$ and $r(q) \wedge b \neq b$, by Theorem 3.5, we get that $q : a = q \wedge m$ and $q : b = q \wedge m$. Now, $q : (a.b) = (q : a) : b$ (by Theorem 3.1 ) $= (q \wedge m) : b$ (since $q : a = q \wedge m$) $= q : b$ (since $(q \wedge m) : b = q : b$) $= q \wedge m$ (since $q : b = q \wedge m$). Hence $r(q) \wedge (a \cdot b) \neq a \cdot b$ (by Theorem 3.5, again). Thus $r(q)$ is a prime element of $L$. □

In the following, we introduce the concept of *p-primary element* in a complete ADL $L$ with a maximal element $m$.

Definition 3.5. Let $L$ be a complete ADL with a maximal element $m$ and let $p$ be a prime element of $L$. An element $q$ of $L$ is called *p-primary*, if $q$ is a primary element of $L$ and $r(q) = p$.

The following Corollary is a directly consequences of Theorem 3.5.

Corollary 3.1. *Let $L$ be a complete ADL with a maximal element $m$ and $a \in L$. Suppose '$\cdot$' is a multiplication on $L$ and $q$ is a $p$-primary element of $L$. Then $p \wedge a \neq a$ if and only if $q : a = q \wedge m$.*

Theorem 3.7. *Let $L$ be a complete residuated ADL with a maximal element $m$ satisfying the a.c.c. If $q$ is a $p$-primary element of $L$ and $a$ is any element of $L$ such that $q \wedge a \neq a$, then $q : a$ is a $p$-primary element of $L$ such that $(q : a) \wedge [r(q)]^k = [r(q)]^k$ and $(q : a) \wedge [r(q)]^{k-1} \neq [r(q)]^{k-1}$, for some $k \in Z^+$.*

Proof. Suppose $q$ is a $p$-primary element of $L$ and $a \in L$ such that $q \wedge a \neq a$.

First we will prove that $q : a$ is primary. Let $b, c \in L$ such that $(q : a) \wedge (b \cdot c) = b \cdot c$ and $(q : a) \wedge b \neq b$. Now, $(q : a) \wedge (b \cdot c) = b \cdot c$ and $q \wedge [a \cdot (b \cdot c)] = a \cdot (b \cdot c)$ (by condition (A) of Definition 2.7). Then $q \wedge [(a.b).c] = (a.b).c$ ( By (iii) of Lemma 2.2)

Since $q$ is a primary element of $L$, we get that either $q \wedge (a \cdot b) = a \cdot b$ or $q \wedge c^k = c^k$ for some $k \in Z^+$. If $q \wedge (a \cdot b) = a \cdot b$, then, by condition (A) of Definition 2.7, we get that $(q : a) \wedge b = b$ which is not true. Therefore, have to be $q \wedge c^k = c^k$ for some $k \in Z^+$. By R2 (i) of Definition 2.5, we get that $(q : a) \wedge (c^k : a) = c^k : a$. So $(q : a) \wedge (c^k : a) \wedge c^k = (c^k : a) \wedge c^k$. Hence, by Lemma 2.4 (1), we have $(q : a) \wedge c^k = c^k$. Thus $q : a$ is a primary element of $L$. Since $q$ is primary and $r(q) = p$, by Theorem 3.4, Theorem 3.2 (1), we get that $p \wedge q = q$, $q \wedge p^k = p^k$ and $q \wedge p^{k-1} \neq p^{k-1}$ for some $k \in Z^+$. Now, since $(q : a) \wedge (q : a) = q : a$, by condition (A) of Definition 2.7, we get that $q \wedge [a \cdot (q : a)] = a \cdot (q : a)$. Since $q$ is a primary element of $L$ and $q \wedge a \neq a$, we get that $q \wedge (q : a)^s = (q : a)^s$ for some $s \in Z^+$. Now $p \wedge (q : a)^s = p \wedge [q \wedge (q : a)^s] = q \wedge (q : a)^s$ (since $p \wedge q = q$) $= (q : a)^s$. Since $p$ is a prime element of $L$, we get that $p \wedge (q : a) = q : a$. So that $r(q : a) \leqslant r(p) = p \wedge m$. Therefore, $p \wedge r(q : a) = r(q : a)$. Also, since $(q : a) \wedge q = q$, by Theorem 3.2 (3), we get that $p = r(q) \leqslant r(q : a)$. Therefore, $r(q : a) = p$. Thus $q : a$ is a $p$-primary element of $L$.

Now, by Lemma 2.4 (1), we have $(p^k : a) \wedge p^k = p^k$. So that $(q : a) \wedge p^k = (q : a) \wedge (p^k : a) \wedge p^k = [(q \wedge p^k) : a] \wedge p^k$ (by R4 of Definition 2.4) $= (p^k : a) \wedge p^k = p^k$. Therefore, $(q : a) \wedge [r(q)]^k = [r(q)]^k$, for some $k \in Z^+$. Choose least positive integer $l$ such that $(q : a) \wedge p^l = p^l$, where $l \leqslant k$. Therefore, $(q : a) \wedge p^{l-1} \neq p^{l-1}$. Hence $(q : a) \wedge [r(q)]^{l-1} \neq [r(q)]^{l-1}$, for some $l \in Z^+$. □

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