**BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE**
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
*Bull. Int. Math. Virtual Inst.*, Vol. **10**(3)(2020), 535-545
DOI: 10.7251/BIMVI2003535K

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

# STRUCTURE OF WEAK IDEMPOTENT RINGS

**Venkateswarlu Kolluru, Dereje Wasihun, and Tilahun Abebaw**

Abstract. We construct a partial synthesis of a Weak idempotent ring and a subclass 2-Weak idempotent ring of Weak idempotent ring. Also we investigate the structure of a Weak idempotent ring with unity of 4 and 8 elements. Further we prove that every maximal ideal is nil whenever 0 and 1 are the only idempotent elements of the Weak idempotent ring with unity. We obtain certain properties of semiprime and submaximal ideals of a commutative Weak idempotent ring with unity. Finally we prove that the Weak idempotent ring satisfies the Köthe's conjecture.

## 1. Introduction

The notion of a BLR is due to A. L. Foster [**2**]. A BLR is a commutative ring R with unity of characteristic 2 in which $ab(1+a)(1+b) = 0$ for all $a$, $b$ in $R$. Clearly Foster's Boolean - like rings are a natural generalization of Boolean rings. Later V. Swaminathan [**6, 7, 8**] has extensively studied Boolean - like rings by considering both algebraic and geometric aspects of this class of rings. D. D. Anderson [**1**], H. A. Khuzam and A. Yaqub [**3**] and A. Yaqub [**10**] have introduced and studied certain generalizations of Boolean rings and Boolean - like rings in a different direction. Recently, K. Venkateswarlu, D. Wasihun, T. Abebaw and Y. Yitayew [**9**] have introduced the notion of a weak idempotent ring (WIR, for short) as a generalization of BLRs and have studied the basic properties of a WIR. Further results concerning certain properties of completely prime ideals and left(right) completely primary ideals were obtained in [**9**].

This paper continues the study of the theory of weak idempotent rings. We construct a partial synthesis of WIR. This means that, given a Boolean - like ring, we can construct a WIR taking the product of the set of all idempotent elements

2010 *Mathematics Subject Classification.* 16N40, 16S99, 16W99.

*Key words and phrases.* Weak idempotent ring, Semiprime, Submaximal ideals, Semiprime ideals, Quasi regular ideals.

of the ring and the set of all nilpotent elements of the ring. Also we prove that a non-commutative WIR with unity of eight elements has only one non-zero nilpotent element. We classify WIR of 4 elements and 8 elements in terms of Boolean rings and Boolean like rings. We study the relation between maximal and submaximal ideals. Finally we use quasi regular ideals to prove Köthe's conjecture.

Throughout this paper, ring R stands for weak idempotent ring (WIR, for short) unless otherwise specified.

## 2. Preliminaries

We recall certain definitions and results concerning Weak idempotent rings and its properties from [**9**].

Definition 2.1. A ring $(R, +, \cdot)$ is called a weak idempotent ring (WIR, for short) if R is of characteristic 2 and $a^4 = a^2$ for each $a \in R$

The following is a commutative WIR of cardinality 2 without unity:

$$0 + 0 = 1 + 1 = 0;\ 1 + 0 = 0 + 1 = 1;\ 0 \cdot 0 = 0 \cdot 1 = 1 \cdot 0 = 1 \cdot 1 = 0.$$

Since $1 \cdot 1 = 0$, then it is not a Boolean ring. By Prover9 (see [**5**]), the least WIR that is not a Boolean like ring has cardinality 16.

Let R be a ring. We say that $a \in R$ is a quadratic residue if $a = x^2$, for some $x \in R$.

Lemma 2.1. *Let $R$ be a WIR. Then $a \in R$ is idempotent if and only if it is a quadratic residue.*

Lemma 2.2. *Let $R$ be a WIR. Then for all $a \in R$*

(1) *$a^n = a,\ a^2$ or $a^3$ for any positive integer $n$.*
(2) *If $0 \neq a$ is a nilpotent element, then $a^2 = 0$.*
(3) *$a = a^2 + (a^2 + a)$, where $a^2$ is idempotent and $a^2 + a$ is nilpotent.*

Proof. 2. If $0 = a^n = a^3, then a^2 = a^4 = aa^3 = a0 = 0$.
3. $(a^2 + a)^2 = a^4 + a^2 = a^2 + a^2 = 0$. □

Remark 2.1. Every element $a$ of $R$ is the sum of a nilpotent and an idempotent element. In general this representation is not unique as in Boolean like rings.

Lemma 2.3 ([**4**], Proposition 19.2). *If $R$ is a local ring with unity, then the only idempotents are 0 and 1.*

Lemma 2.4. *Let $R$ be a WIR with unity. Then $a \in R$ is a unit if and only if $a^2 = 1$.*

Proof. Let $a$ be a unit with inverse $a^{-1}$. Since

$$a^2(1 + a^2) = a^2 + a^4 = a^2 + a^2 = 0,$$

then by multiplying by $a^{-2}$ on the left we get $1 + a^2 = 0$ that implies the conclusion. □

Lemma 2.5. *Let $R$ be a WIR with unity. Then $a \in R$ is a unit if and only if $1 + a$ is nilpotent.*

PROOF. $(1+a)^2 = 1 + a^2 = 0$ iff $a^2 = 1$. □

COROLLARY 2.1. *Every non-zero non-unit in a WIR with unity is a zero-divisor.*

PROOF. Let $a \neq 0$ be a non-unit. Then $1 + a^2 \neq 0$. If $a(1 + a^2) = 0$, then we get the conclusion. If $a(1 + a^2) \neq 0$, then we get the conclusion: $aa(1 + a^2) = a^2 + a^4 = a^2 + a^2 = 0$. □

If the ring is a commutative WIR, then the representation of each element as the sum of an idempotent and a nilpotent element is unique and we use the notation $a_B$ for the idempotent $a^2$ and $a_N$ for the nilpotent $a + a^2$ of the unique representation.

The following result is well known.

LEMMA 2.6. *If R is a commutative ring of characteristic 2, then the Frobenius map $\tau : R \to R$, defined by $\tau(x) = x^2$, is an endomorphism of R.*

We denote by $R_B$ the set of idempotent elements of a ring R and by N the set of nilpotent elements of R.

PROPOSITION 2.1. *Let R be a commutative WIR and $\tau$ be the Frobenius endomorphism. Then N is the kernel of $\tau$, $R_B$ is the image of $\tau$, and $R/N \cong R_B$ is a Boolean ring.*

If R is a commutative WIR, then $R_B$ is both a subring of R and a homomorphic image of R. Hence, $R_B$ is a retract of R.

PROPOSITION 2.2. *Let R be a commutative WIR with unity.*

(1) *An ideal is maximal iff it is prime.*
(2) *Every maximal ideal of R contains the ideal N of nilpotents of R.*
(3) *The ideal N is the intersection of all maximal ideals of R.*

PROOF. (1) Let P be a prime ideal. Then $R/P$ is an integral domain. By Corollary 2.1, $R/P$ is a field. Then P is maximal. The converse is trivial.

(2) By (1) and by Lemma 2.2.

(3) The lattice interval $[N, R]$ is isomorphic to the Boolean lattice of ideals of the Boolean ring $R/N$. Then the result follows from the fact that $\{0\}$ is the intersection of the maximal ideals of the the Boolean ring $R/N$. □

THEOREM 2.1. *If R is a WIR with unity such that $I(R) = \{0, 1\}$, then every proper ideal of R is a nil ideal.*

PROOF. For every $x(\neq 0, 1)$, Since $x^2$ is idempotent, then either $x^2 = 0$ or $x^2 = 1$. Thus $x$ is nilpotent or a unit. Let $M$ be proper ideal of $R$. If $x \in M$ then $x \neq 1$. Suppose $x \neq 0$. Then x is nilpotent or a unit. But $M$ does not contain a unit element. Thus $x$ is nilpotent and hence $M \subseteq N$. Hence, $M$ is nil. □

THEOREM 2.2. *Let R be WIR which is a local ring with unity. Then we have:*

1. *The set $N(R)$ of all nilpotent elements of R is the unique maximal ideal of R;*

2. *$R$ is a commutative ring.*

PROOF. (1) Let M be the unique maximal ideal. By Lemma 2.3 and by Theorem 2.1 we have that M is a nil ideal. Let $a$ be a nilpotent element such that $a \notin M$. Then there exist $r, s \neq 0$ such that $1 = ras$. This implies that $r = rras$ and $s = rass$. Hence, r and s are units (i.e., $r2 = s2 = 1$). By $1 = ras$ we derive $rs = rrass = a$. Thus $rsrs = 0$ and, multiplying by $s^{-1}r^{-1}$ on the left, we get $a = rs = 0$. Contradiction. We conclude that $M = N(R)$.

(2) The product of units is commutative: $ab = ba$ iff $a = abb = bab$ iff $1 = aa = abab$. This last condition is true because $ab$ and $ba$ are units. The product of nilpotents is commutative. If $a, b$ are nilpotents, then $1 + a$ and $1 + b$ are units. Then, $1+b+a+ab = (1+a)(1+b) = (1+b)(1+a) = 1+a+b+ba$. By subtracting $1 + a + b$ we get $ab = ba$. The product of a unit and a nilpotent is commutative. If $a$ is nilpotent and b is a unit, then we have $b + ab = (1 + a)b = b(1 + a) = b + ba$. Then subtracting b we get ab = ba. □

COROLLARY 2.2. *Every non-commutative WIR R with unity is not local.*

THEOREM 2.3. *Let R be a commutative WIR. Then $R_B$ is a Boolean ring isomorphic to $R/N$.*

PROPOSITION 2.3. *If R is a commutative WIR with unity such that the product of any two elements of N is zero, then R is a Boolean like ring.*

PROOF. By Lemma 2.2 $a(1 + a) = a^2 + a$ is nilpotent for every $a$. Then $a(1 + a)b(1 + b) = 0$ by hypothesis. □

PROPOSITION 2.4. *A field F of characteristic 2 is a WIR if and only if it is the 2-element field.*

PROOF. The polynomial $x^4 - x^2$ has exactly two double roots in a field of characteristic 2: $x = 0$ and $x = 1$. □

A commutative ring is semiprimitive if and only if it is a subdirect product of fields.

THEOREM 2.4. *If P is any completely prime ideal of a Weak idempotent ring R with unity, then $R/P$ is isomorphic to the 2- element field.*

DEFINITION 2.2. Let $S$ be an arbitrary ring with unity and $Q$ be an ideal of a ring $S$ . $Q$ is said to be:

(1) left completely primary ideal of $S$ if, for $a, b \in S$, $ab \in Q$ implies $a \in Q$ or $b^n \in Q$ for some $n \in \mathbb{N}$.
(2) right completely primary ideal of $S$ if, for $a, b \in S$, $ab \in Q$ implies $a^n \in Q$ or $b \in Q$ for some $n \in \mathbb{N}$.

THEOREM 2.5. *An ideal of a WIR R with unity is left completely primary if and only if for any idempotent element $b \in R$, either $b \in I$ or $1 + b \in I$.*

PROOF. ($\Rightarrow$) By $b(1+b) = b+b^2 = b+b = 0 \in I$ we get $b^2 \in I$ or $(1+b)^n \in I$, for some n. Since $1+b$ is idempotent, then $(1+b)^n = 1+b$ and we get the conclusion.

($\Leftarrow$) Let $ab \in I$ with $a \notin I$. Since $b^2$ is idempotent, then by hypothesis either $b^2 \in I$ or $1 + b^2 \in I$. If $1 + b^2 \in I$, then $a(1 + b^2) = a + ab^2 \in I$. Since $ab^2 \in I$, then $a = a + ab^2 + ab^2 \in I$ contradicting the assumption. Then $b^2 \in I$ and I is left completely primary. □

COROLLARY 2.3. *An ideal $I \neq R$ of a WIR with unity is left completely primary ideal if and only if $R/I$ has only two idempotents.*

THEOREM 2.6. *In a WIR with unity, a left completely primary ideal $I$ is completely prime if and only if all nilpotent elements of $R$ contained in $I$.*

## 3. 2-Weak Idempotent Rings

We introduce partial synthesis of Weak idempotent ring by the following theorem and construct the subclass of weak idempotent ring.

DEFINITION 3.1. Let $R$ be a WIR. Then

(1) For $a \in R$, if $ab = b$ ($ba = b$) for every $b \in R$, then $a$ is said to be left(right) unity of $R$.
(2) For any $a, b, c \in R$, if $abc = bac$ ($abc = acb$), then $R$ is said to be left(right) weak-commutative.

THEOREM 3.1. *Let $R$ be a Boolean like ring, $R_B$ be the set of all idempotent elements of $R$ and $N$ be the set of all nilpotent elements of $R$. Let $\bar{R} = R_B \times N = \{(b, n) : b \in R_B$ and $n \in N\}$. Define the operations of addition and multiplication as follows $(b_1, n_1)+(b_2, n_2) = (b_1+b_2, n_1+n_2)$ and $(b_1, n_1)*(b_2, n_2) = (b_1b_2, b_1n_2)$. Then $(\bar{R}, +, *)$ is a left weak-commutative WIR and has a left unity of the form $(1, n)$ for all $n \in N$.*

PROOF. It is a routine verification that $\bar{R}$ is a left weak-commutative WIR and it has a left unity of the form $(1, n)$ for all $n \in N$. □

NOTE 3.1. Similarly we can obtain right weak-commutative WIR $(\bar{R}, +, *)$ with having right identity element(s)of the form $(1, n)$ if we replace the operation of multiplication with $(b_1, n_1) * (b_2, n_2) = (b_2b_1, b_2n_1)$ in the above theorem. It can easily be seen that one sided unity is not necessarily unique.

THEOREM 3.2. *Let $\bar{R}$ be the left weak-commutative WIR defined in Theorem 3.1 . Let $\bar{\bar{R}} = \bar{R} \times \mathbb{Z}_2 = \{(r, b) : r \in \bar{R}$ and $b \in \mathbb{Z}_2\}$ and define the operations of addition and multiplication as follows: $(r_1, b_1) + (r_2, b_2) = (r_1 + r_2, b_1 + b_2)$ and $(r_1, b_1) * (r_2, b_2) = (r_1r_2 + b_1r_2 + b_2r_1, b_1b_2)$. Then $(\bar{\bar{R}}, +, *)$ is a non-commutative WIR with unity $(0, 1)$.*

PROOF. It is well known that $\bar{R}$ can be embedded into a ring with unity, that is, $\bar{\bar{R}}$ since $\bar{R}$ is characteristic 2. We claim that $\bar{\bar{R}}$ is a WIR. Let $(r, b) \in \bar{\bar{R}}$. Then $(r, b) + (r, b) = (r + r, b + b) = (0, 0)$ and $(r, b)^2 = (r^2 + br + br, b^2) = (r^2, b)$ and $(r, b)^4 = (r^2, b)^2 = (r^4 + br^2 + br^2, b^2) = (r^2, b)$. Thus, $(r, b)^4 = (r, b)^2$. Hence $\bar{\bar{R}}$ is a WIR with unity $(0,1)$. Let $(r,b)^2 = 0$. Then $(r^2,b) = 0$ implies that $b = 0$ and $r$ is nilpotent element of $\bar{R}$. Hence a nilpotent element of $\bar{\bar{R}}$ is of the form $(n,0)$ where $n$ is a nilpotent element of $\bar{R}$. □

NOTE 3.2. If we replace the ring $\bar{R}$ in Theorem 3.2 by the right weak-commutative WIR, we obtain a non-commutative WIR with unity.

DEFINITION 3.2. The ring $(\bar{\bar{R}},+,*)$ defined in Theorem 3.2 is called 2-Weak idempotent ring (2-WIR, for short)

THEOREM 3.3. *Let $\bar{\bar{R}}$ be a 2-WIR. Then the product of any two nilpotent elements of $\bar{\bar{R}}$ is zero.*

PROOF. Let $(n_1,0)$ and $(n_2,0)$ be two distinct nilpotent elements of $\bar{\bar{R}}$. Then $(n_1,0)(n_2,0) = (n_1n_2,0) = (0,0)$ since the product of any two nilpotent elements of $\bar{R}$ is zero. □

THEOREM 3.4. *Let $\bar{\bar{R}}$ be a 2-WIR and $N$ be the set of all nilpotent elements of $\bar{\bar{R}}$. Then $N$ is an ideal of $\bar{\bar{R}}$.*

PROOF. WE recall from Theorem 3.2 that $\bar{\bar{R}} = \bar{R}\times\mathbb{Z}_2$, where $\bar{R}$ is the left weak-commutative WIR defined in Theorem 3.1. Let $n_1,\ n_2 \in N$. Then $(n_1+n_2)^2 = n_1n_2 + n_2n_1$=0 since the product of any two nilpotent element is zero. Thus, $n_1+n_2 \in N$. For $r \in \bar{\bar{R}}$ and $n' \in N$, $r = (s,b)$ and $n' = (n,0)$, $n$ is nilpotent element of $S$. We have $rn' = (s,b)(n,0) = (sn+bn,0)$ and $(rn')^2 = ((sn+bn)^2,0) = (0,0)$ since $S$ is left weak-commutative. Thus, $\bar{\bar{R}}N \subseteq N$ and hence $N$ is a left ideal of $\bar{\bar{R}}$. In a similar way we prove that $N$ is a right ideal of $\bar{\bar{R}}$. □

THEOREM 3.5. *The set of all unit elements of a WIR $R$ with unity is precisely $\{1+n : n \in N\}$, where $N$ is the set of nilpotent elements of $R$.*

PROOF. Let $a$ be a unit element of $R$. Then $(1+a)^2 = 1+a+a+a^2 = 0$ since $a^2+1 = 0$. Hence, $1+a$ is nilpotent and $a = 1+(1+a)$. On the other hand, for any nilpotent element $n \in R$, $(1+n)^2 = 1+n+n+n^2 = 1$. Hence, $1+n$ is a unit. □

EXAMPLE 3.1. The BLR $(H_4,+,\star)$ with $H_4 = \{0,1,p,q\}$ and $+$ and $\star$ are defined as follows:

| + | 0 | 1 | p | q |
|---|---|---|---|---|
| 0 | 0 | 1 | p | q |
| 1 | 1 | 0 | q | p |
| p | p | q | 0 | 1 |
| q | q | p | 1 | 0 |

| $\star$ | 0 | 1 | p | q |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 1 | p | q |
| p | 0 | p | 0 | p |
| q | 0 | q | p | 1 |

THEOREM 3.6. *Let $R$ be a WIR with unity of four elements. Then up to isomorphism either $R$ is the Boolean like ring $H_4$ or $R$ is a Boolean ring.*

Proof. Let $R = \{0, 1, a, b\}$ be a WIR with unity 1. Clearly $(R, +)$ is a Boolean group with $a + b = 1$. Suppose $a$ is the only non-zero nilpotent element of $R$. Then $ab = a(1 + a) = a + a^2 = (1 + a)a = ba$. Hence $R$ is commutative. Thus, $R$ is the BLR $H_4$. If $N = \{0\}$, then $R$, which is isomorphic to $R/N = R_B$, is a Boolean ring. No other possibilities exist because an ideal of $R$ has a number of elements that is a divisor of 4. □

Theorem 3.7. *A non-commutative WIR with unity of eight elements has only one non-zero nilpotent element.*

Proof. Let $R = \{0, 1, a, b, c, d, e, f\}$ be a non-commutative WIR with unity of eight elements. If 0 is the only nilpotent, then $R$ is a Boolean ring which is commutative and hence a contradiction. Suppose $R$ has three distinct non-zero nilpotent elements, namely $a,\ b,\ c$. Then $1 + a$, $1 + b$ and $1 + c$ are distinct units. So, 0 and 1 are the only idempotent elements and thus $R$ is commutative which is a contradiction. Hence $R$ has no three distinct non-zero nilpotent elements. If $R$ has more than three distinct non-zero nilpotent elements, then the number of elements of $R$ is strictly greater than eight and it contradicts. Hence, $R$ has only one non-zero nilpotent element. □

Now we have the following case for commutativity

Theorem 3.8. *A commutative WIR with unity of eight elements is a Boolean like ring.*

Proof. Let $R = \{0, 1, a, b, c, d, e, f\}$ be a commutative WIR with unity of eight elements. If 0 is the only nilpotent, then R is a Boolean ring and hence a BLR. Suppose $a$ is the only non-zero nilpotent element of $R$. Then the product of any two nipotent element is zero. Thus, by Theorem 2.3 $R$ is a Boolean like ring. Suppose $R$ has three distinct non-zero nilpotent elements, namely $a,\ b,\ c$. Then $1 + a$, $1 + b$ and $1 + c$ are three distinct units. So, 0 and 1 are the only idempotent elements. Suppose $ab \neq 0$. Since $(ab)^2 = 0$, either $ab = a$, $ab = b$ or $ab = c$. If $ab = a$, then $a(1 + b) = a + ab = a + a = 0$. Thus, $a = 0$ which is a contradiction. If $ab = b$, then $b(1 + a) = b + ab = b + b = 0$. Thus, $b = 0$ which is a contradiction. If $ab = c$, then $ab + b = c + b = a$. Thus, $(ab + b)b = ab \Rightarrow ab = 0 = c$ which is a contradiction. Thus, the product of any two nilpotent element is 0 and by theorem 2.3, $R$ is a BLR. $R$ cannot have more than three distinct non-zero nilpotent elements. □

## 4. Maximal and Submaximal ideals

We introduce the properties of maximal ideals of a WIR and semiprime and submaximal ideals of a commutative WIR with unity.

Theorem 4.1. *Every completely prime ideal of a WIR $R$ with unity is a maximal ideal.*

Proof. Let $P$ be a completely prime ideal and $J$ be an ideal of $R$ such that $P \subsetneq J \subseteq R$. Let $a \in J \setminus P$. Then $a^2 \notin P$. Since $a^4 = a^2$, then $a^2(a^2 + 1) = 0 \in P$; hence $a^2 + 1 \in P \subseteq J$. Since $a \in J$, then $a^2 \in J$, so that $1 = a^2 + a^2 + 1 \in J$. Thus $J = R$. Hence, $P$ is a maximal ideal. □

Corollary 4.1. *If $M$ is a maximal ideal of a WIR $R$ with unity, then $R/M$ is isomorphic to the 2-element field.*

Theorem 4.2. *If $R$ is a WIR with unity such that $I(R) = \{0, 1\}$, then every proper ideal of $R$ is a nil ideal.*

Proof. For every $x(\neq 0, 1)$, Since $x^2$ is idempotent, then either $x^2 = 0$ or $x^2 = 1$. Thus $x$ is nilpotent or a unit. Let $M$ be proper ideal of $R$. If $x \in M$ then $x \neq 1$. Suppose $x \neq 0$. Then x is nilpotent or a unit. But $M$ does not contain a unit element. Thus $x$ is nilpotent and hence $M \subseteq N$. Hence, $M$ is nil. □

Theorem 4.3. *Let $R$ be a commutative WIR with unity and $I$ be an ideal of $R$. Let $x \in R$ be such that $x \notin I$.*

(1) *If $x_B \notin I$, then there exists a maximal ideal $J$ of $R$ such that $I \subset J$ and $x \notin J$.*
(2) *If $x_N \notin I$, then there exists a primary ideal $P$ of $R$ such that $I \subset P$ and $x \notin P$.*

Proof. (1) Let $\Sigma = \{J : J \text{ is an ideal of } R,\ I \subseteq J \text{ and } x_B \notin J\}$. Then by Zorn's lemma , $\Sigma$, has a maximal element say $J$. Clearly $x_B \notin J$. Let $a,\ b \in R$ such that $a \notin J$ and $b \notin J$. Then $x_B \in J + Ra$. Thus, $x_B = j_1 + r_1 a = j_2 + r_2 b$ where $j_1,\ j_2 \in J$ and $r_1,\ r_2 \in R$. That is $x_B = x_B^2 = j_3 + r_1 r_2 ab$ with $j_3 \in J$ and hence $ab \notin J$. Therefore, $J$ is prime and hence maximal such that $I \subset J$ and $x \notin J$.

(2) Let $\Sigma = \{J : J \text{ is an ideal of } R,\ I \subseteq J \text{ and } x_N \notin J\}$. $\Sigma$ is non-empty since $\{0\} \in \Sigma$. By Zorn's lemma, $\Sigma$ ordered by inclusion, has a maximal element. Let $P$ be a maximal element of $\Sigma$. Now claim that $P$ is primary. Let $xy \in P$ and $x \notin P$. Clearly, $x_N \notin P$. Since $x \notin P$, $P \subsetneq P + Rx$. Thus $P + Rx \notin \Sigma$ and hence $x_N \in P + Rx$. So, $x_N = i + rx$ for $i \in P$ and $r \in R$. Then $x_N y = iy + rxy \in P$. Assume no positive power of $y$ belongs to $P$. That is, $y^3 \notin P$. Hence, $x_N \in P + Ry^3$ since $P \subsetneq P + Ry^3$ and hence $P + Ry^3 \notin \Sigma$. Let $x_N = j + sy^3 = j + sy^2 + sy^3 + sy^4$ where $j \in P$ and $s \in R$ which implies $x_N = j + sy_B(1 + y_N)$. By multiplying both sides by $1 + y_N$, we get $x_N + x_N y_N = j(1 + y_N) + sy_B(1 + y_N)^2 = j(1 + y_N) + sy_B = k + sy_B$ where $j(1 + y_N) = k \in P$. In addition to that $x_N = j + sy^3$ implies $x_N y = jy + sy_B$. Using the two results, we obtained $x_N + x_N y_N + x_N y = k + jy \in P$, but $x_N y \in P$ and $x_N y_N \in P$. Thus, $x_N \in P$ which is a contradiction. Therefore, $y^m \in P$ for some positive integer $m$. Thus, $P$ is primary. Since $x_N \notin P \in \Sigma$, $x \notin P$. □

The following theorem gives a characterization of semiprime ideals of a commutative WIR with unity.

Theorem 4.4. *Let $I$ be an ideal of a commutative WIR $R$ with unity. Then the following statements are equivalent.*

1. *$I$ is semiprime*
2. *The nilradical $N$ of $R$ is contained in $I$*
3. *$R/I$ is a Boolean ring*

Proof. ($1 \Rightarrow 2$) Let $I$ be semiprime and $a \in N$. Then $a^n = 0 \in I$ for some n. Then $a \in r(I) = I$. Hence, the nilradical $N$ of $R$ is contained in $I$.

($2 \Rightarrow 3$) Suppose the nilradical $N$ of $R$ is contained in $I$. By Theorem 2.3, $R/N$ is a Boolean ring. Then $R/I$, isomorphic to a Boolean ring, is a Boolean ring.

($3 \Rightarrow 1$) Let $R/I$ be a Boolean ring and $x \in r(I)$. This implies that $x^n \in I$ for some positive integer $n$ and hence $x^n + I = I \in R/I$. Thus, $I = (x+I)^n = x + I$. Hence, $r(I) = I$, that is, $I$ is semiprime. □

Theorem 4.5. *Every proper semiprime ideal $I$ of a commutative WIR $R$ with unity is the intersection of all maximal ideals of $R$ containing $I$.*

Proof. A semiprime ideal I is intersection of all prime ideals containing I. Since in a WIR an ideal is maximal if and only if it is prime, then we get the conclusion. □

In a commutative ring $R$, an ideal $Q$ is primary if, for $a, b \in R$, $ab \in Q$ implies $a \in Q$ or $b^n \in Q$ for some $n \in \mathbb{N}$. It coincide with the left completely primary ideal in an arbitrary ring.

Theorem 4.6. *Let $I$ be an ideal of a commutative WIR $R$ with unity. Then $I$ is contained in at least two maximal ideals of $R$ if and only if $I$ is not primary.*

Proof. $I$ is contained in only one maximal ideal of $R$ if and only if the quotient ring $R/I$ is a local ring if and only if $R/I$ has only two idempotent elements if and only if $I$ is primary. □

Definition 4.1. An ideal $I$ of a WIR $R$ is called submaximal if $I$ is covered by a maximal ideal of $R$ i.e. there exists a maximal ideal $M$ of $R$ such that $I \subsetneq M$ and for any ideal $J$ of $R$ such that $I \subset J \subset M$ we have that $J = I$ or $J = M$.

Theorem 4.7 and Theorem 4.8 are trivial, because by Proposition 2.2 you can read these theorems in the Boolean ring $R/N$.

Theorem 4.7. *The intersection of any two distinct maximal ideals of a commutative WIR $R$ with unity is submaximal and it is covered by both of the maximal ideals. Further, there exists no other maximal ideal containing it.*

Theorem 4.8. *If an ideal of a commutative WIR $R$ with unity is submaximal, then $R/I$ is the four element Boolean ring.*

## 5. Quasi-regular ideals

Definition 5.1. Let R be a ring.

(1) An element $a \in R$ is said to be quasi-regular if and only if there exists $b \in R$ such that $a + b - ab = 0$ and we call $b$ the quasi inverse of $a$.
(2) An ideal $I$ of $R$ is said to be quasi-regular if every element of $I$ is quasi-regular.

Lemma 5.1. *If $R$ is a ring with unity, then $a$ is quasi-regular if and only if $1 - a$ is a unit.*

Proof. $(1-a)(1-b) = 1-a-b+ab = 1-(a+b-ab) = 1$ □

For any ring $R$ with unity, the Jacobson radical of $R$ is the largest quasi-regular ideal of $R$ and denoted by $J(R)$. The set of all nilpotent elements in $R$ denoted by $N$. Every nilpotent element is quasi-regular, so $N \subseteq Q(R)$ where $Q(R)$ is the set of all quasi-regular elements of the ring $R$. In fact if $x^n = 0$, then $-x-x^2-...-x^{n-1}$ is the quasi inverse of $x$. The upper nilradical of $R$ is the largest nil ideal of $R$ and denoted by $Nil^*(R)$. If $R$ is commutative, then $Nil^*(R) = N$. For any ring $R$, $Nil^*(R) \subseteq J(R)$. The following theorem shows that the set of all nilpotent elements of a WIR $R$ with unity is precisely the set of all quasi-regular elements of $R$.

Theorem 5.1. *Let $R$ be a WIR with unity and $N$ be the set of all nilpotent elements of $R$. Then $N$ is the set of all quasi-regular elements of $R$.*

Proof. $a$ is quasi-regular if and only if $1+a$ is a unit if and only if $a$ is nilpotent. □

In an arbitrary WIR $R$, $N$ need not be an ideal of $R$. But we have the following

Corollary 5.1. *If $R$ is a WIR with unity, then $J(R) = Nil^*(R)$.*

Note 5.1. [**4**] **Köthe's Conjecture**: If $Nil^*(R) = 0$, then $R$ has no non-zero nil one sided ideals. An equivalent formulation of the same conjecture is: Every nil left or right ideal of a ring $R$ is contained in $Nil^*(R)$.

Theorem 5.2. *Let $R$ be a WIR with unity. Then $R$ satisfies the Köthe's conjecture.*

Proof. By corollary 5.1, $Nil^*(R) = J(R)$ for a ring $R$, then $R$ satisfies the Köthe's conjecture since $J(R)$ contains every nil one-sided ideal. □

Theorem 5.3. *Every 2-weak idempotent ring $R$ has at least one idempotent element which is neither 0 nor 1.*

Proof. Let $R$ be a 2-weak idempotent ring and $N$ be the set of all nilpotent elements of $R$. By theorem 3.4, $N$ is an ideal of $R$. Thus, $Nil^*R = N$. By corollary 5.1, $J(R) = Nil^*R = N$. So, $N \subseteq M$ for every maximal ideal $M$. Suppose 0 and 1 are the only idempotents of $R$. By Theorem 4.2, every maximal ideal is nil. So $M = N$, that is, $R$ is local which is a contradiction to Theorem 2.2. Hence the theorem holds. □

**Acknowledgement** The authors are thankful to the referee for his/her valuable suggestions.

## References

[1] D. D. Anderson. Generalization of Boolean, Boolean - like and Von Neumann regular rings. *Comment. Math. Univ. St. Pauli*, **35**(1)(1986), 69–76.

[2] A. L. Foster. The theory of Boolean - like rings. *Trans. Am. Math. Soc.*, **59**(1)(1946), 166–187.

[3] H. A. Khuzam and A. M. Yaqub. Generalized Boolean and Boolean - like rings. *Int. J. Algebra*, **7**(9)(2013), 429 - 438.
[4] T.-Y. Lam. *A First Course in Noncommutative Rings.* Second Edition, Graduate Texts in Math., Vol. 131, Springer-Verlag, Berlin-Heideiberg-New York, 2001.
[5] W. McCune. Prover9. http://www.mcs.anl.gov/ mccune/prover9/2005.
[6] V. Swaminathan On Foster's Boolean - like rings. *Math. Sem. Notes Kobe Univ.*, **8**(2)(1980), 347–367.
[7] V. Swaminathan. Geometry of Boolean like rings. *Math. Sem. Notes Kobe Univ.*, **9**(1)(1981), 195–214.
[8] V. Swaminathan. *Boolean - like rings.* PhD dissertation, Andhra University, India, 1982.
[9] K. Venkateswarlu, D. Wasihun, T. Abebaw and Y. Yitayew. Weak Idempotent Rings. communicated to *Iran. J. Math. Sci. Inform.*
[10] A. M. Yaqub. A generalization of Boolean rings. *Int. J. Algebra*, **1**(8)(2007), 353-362.

Received by editors 30.09.2019; Revised version 15.03.2020 and 02.04.2020; Available online 13.04.2020.

V. Kolluru,
Department of Mathematics, Dr.Lankapalli Bullayya College, Visakhapatnam, India
*E-mail address*: drkvenkateswarlu@gmail.com

D. Wasihun,
Department of Mathematics, Addis Ababa University, Addis Ababa, Ethiopia.
*E-mail address*: derawase@gmail.com

T. Abebaw,
Department of Mathematics, Addis Ababa University, Addis Ababa, Ethiopia.
*E-mail address*: tilahun.abebaw@aau.edu.et