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FILTERS IN RESIDUATED RELATIONAL SYSTEM ORDERED UNDER QUASI-ORDER

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Abstract. The concept of residuated relational systems ordered under a quasi-order relation was introduced in 2018 by S. Bonzio and I. Chajda as a structure $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$, where $(A, \cdot)$ is a commutative monoid with the identity 1 as the top element in this ordered monoid under a quasi-order R. In this article, we introduce and analyze the concept of filters in that relational systems. The motive for determining the filters in this way are the properties of the left classes of the quasi-order relations R. Some of the fundamental features of the filters designed on this way are proven.

1. Introduction

Let $\mathfrak{A} = \langle A, \mathfrak{R}\rangle$ be a system, where $(A, =)$ be a set and $\mathfrak{R}$ be a family of relations on A. For this system we say that it is a *relational system.* For set A, we say that it is the *carrier* of a relational system $\mathfrak{A}$. For ease of writing, we assume that $\mathfrak{R} = \{R\}$, where R is a binary relation on the set A. The analysis of such a binary relational system includes, inter alia, the observation of the properties of left aR and right Rb classes of R generated by element a and b respectively. If the relation R has some one or more additional properties, then the class characteristics become more complex. Analogously, let A be not only a set, but an algebra structure has been built on it. In this case, the number of questions about the characteristics of the relational system to which some answers should be offered is significantly increased. It is quite justifiable to expect to recognize, describe, and possibly prove the observed features of such a designated relational system built on an algebraic structure.

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In this article we will describe the properties of a class of substructures (so-called filters) of a residual relational system ordered under quasi-order.

2. Preliminaries

Many authors associate the study of binary relational systems with Riguet's article [**5**]. Also, many authors link Mal'cev's article [**4**] and the first attempts to research of relational systems into the algebraic frameworks. In a recently published article [**2**], S. Bonzio and I. Chajda introduced and analyzed the concept of 'residual relational systems'.

Definition 2.1. ([**2**], Definition 2.1) A *residuated relational system* is a structure $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$, where $\langle A, \cdot, \rightarrow, 1\rangle$ is an algebra of type $\langle 2, 2, 0\rangle$ and R is a binary relation on A and satisfying the following properties:

(1) $(A, \cdot, 1)$ is a commutative monoid;

(2) $(\forall x \in A)((x, 1) \in R)$;

(3) $(\forall x, y, z \in A)((x \cdot y, z) \in R \Longleftrightarrow (x, y \rightarrow z) \in R)$.

We will refer to the operation $\cdot$ as multiplication, to $\rightarrow$ as its residuum and to condition (3) as residuation.

The basic properties for residuated relational systems are subsumed in the following

Theorem 2.1 ([**2**], Proposition 2.1). *Let* $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, R\rangle$ *be a residuated relational system. Then*

(4) $(\forall x, y \in A)(x \rightarrow y = 1 \Longrightarrow (x, y) \in R)$,

(5) $(\forall x \in A)((x, 1 \rightarrow 1) \in R)$,

(6) $(\forall x \in A)((1, x \rightarrow 1) \in R)$,

(7) $(\forall x, y, z \in A)(x \rightarrow y = 1 \Longrightarrow (z \cdot x, y) \in R)$,

(8) $(\forall x, y \in A)((x, y \rightarrow 1) \in R)$.

Recall that a *quasi-order relation* $'\preccurlyeq'$ on a set A is a binary relation which is reflexive and transitive (Some authors use the term pre-order relation).

Definition 2.2. ([**2**], Definition 3.1) A *quasi-ordered residuated system* is a residuated relational system $\mathfrak{A} = \langle A, \cdot, \rightarrow, 1, \preccurlyeq\rangle$, where $\preccurlyeq$ is a quasi-order relation in the monoid $(A, \cdot)$

The following proposition shows the basic properties of quasi-ordered residuated systems.

Proposition 2.1 ([**2**], Proposition 3.1). *Let* A *be a quasi-ordered residuated system. Then*

(9) $(\forall x, y, z \in A)(x \preccurlyeq y \Longrightarrow (x \cdot z \preccurlyeq y \cdot z \wedge z \cdot x \preccurlyeq z \cdot y))$;

(10) $(\forall x, y, z \in A)(x \preccurlyeq y \Longrightarrow (y \rightarrow z \preccurlyeq x \rightarrow z \wedge z \rightarrow x \preccurlyeq z \rightarrow y))$;

(11) $(\forall x, y \in A)(x \cdot y \preccurlyeq x \wedge x \cdot y \preccurlyeq y)$.

Estimating that this topic is interesting ([**1, 2, 3**]), it is certain that there is interest in the development of the concept of filters in these systems.

3. The main results

3.1. Motivation. Let $L(a) = \{y \in A : a \preccurlyeq y\}$ be the left class and $R(b) = \{x \in A : x \preccurlyeq b\}$ be the right class of the relation $\preccurlyeq$ generated by the elements a and b respectively. Then $R(1) = A$. Some authors use the notation $U(a)$ instead of $L(a)$ (see, for example [**2, 3**]).

Lemma 3.1. *If $\preccurlyeq$ is an antisymmetric relation, then $L(1) = \{1\}$.*

In the following propositions, we give the basic properties of classes $L(a)$:

Proposition 3.1. *The following holds*
(12) $a \in L(a) \wedge 1 \in L(a)$;
(13) $(\forall u, v \in A)((u \in L(a) \wedge u \preccurlyeq v) \Longrightarrow v \in L(a))$;
(14) $(\forall u, v \in A)(u \cdot v \in L(a) \Longrightarrow (u \in L(a) \wedge v \in L(a)))$;
(15) $L(a) \cap L(b) \subseteq L(a \cdot b)$;
(16) $(\forall u, v \in A)(v \in L(a) \Longrightarrow u \rightarrow v \in L(a))$;
(17) $(\forall u, v \in A)(u \preccurlyeq v \Longrightarrow u \rightarrow v \in L(a))$.

Proof. (12) It is obvious that $a \in L(a)$ and $1 \in L(a)$ are valid according to reflexivity of the relation $\preccurlyeq$ and (2).

(13) Let $u, v \in A$ arbitrary elements such that $u \in L(a)$ and $u \preccurlyeq v$. Then $a \preccurlyeq u \wedge u \preccurlyeq v$. Thus $a \preccurlyeq v$ by transitivity of $\preccurlyeq$. So, $a \preccurlyeq v$ and $v \in L(a)$.

(14) Let $u, v \in A$ be arbitrary elements such that $u \cdot v \in L(a)$. Then $a \preccurlyeq u \cdot v$. Thus $a \preccurlyeq u$ and $a \preccurlyeq v$ according to the statement (11). So, we have $u \in L(a)$ and $v \in L(a)$. Therefore, the assertion (14) has been proved.

(15) Let $u \in L(a) \cap L(b)$ be an arbitrary element. Then $u \in L(a)$ and $u \in L(b)$ and $a \preccurlyeq u$ and $b \preccurlyeq u$. Thus $a \cdot b \preccurlyeq u$ by claim (11) in the Proposition 2.1 and by transitivity of relation $\preccurlyeq$. So, $u \in L(a \cdot b)$. In this way, the assertion (15) has been proved.

(16) Let us assume that $u, v \in A$ be arbitrary element such that $v \in L(a)$. Then $a \preccurlyeq v$. Thus $a \cdot u \preccurlyeq v$ since $a \cdot u \preccurlyeq a$ by (11) and and transitivity of $\preccurlyeq$. So, we have $a \preccurlyeq u \rightarrow v$ by (3). Therefore, $u \rightarrow v \in L(a)$ and the condition (16) is proven.

(17) Suppose that $u \preccurlyeq v$. Since $a \cdot u \preccurlyeq u$ holds by (11), we have $a \cdot u \preccurlyeq v$ by transitivity of relation $\preccurlyeq$. Thus $a \preccurlyeq u \rightarrow v$ by (3). So, $u \rightarrow v \in L(a)$. □

Corollary 3.1. *Let $\mathfrak{A}$ be a quasi-ordered residuated system. If the implication*
(H) $(\forall u, v \in A)((u \in L(a) \wedge u \rightarrow v \in L(a)) \Longrightarrow v \in L(a))$

is valid, the the formula (13) *is valid too.*

Proof. Let $u, v \in A$ be elements such that $u \in L(a)$ and $u \preccurlyeq v$. Then $u \rightarrow v \in L(a)$ by (17). Thus $v \in L(a)$ by (H). Therefore, formula (13) is proved. □

3.2. Concepts of filters. The properties of class $L(a)$ are the motivation for introducing the concept of filters in a quasi-ordered residuated system. Before that, let us analyze the interrelations of the following formulas
(F1) $(\forall u, v \in A)((u \cdot v \in F \Longrightarrow (u \in F \wedge v \in F))$;
(F2) $(\forall u, v \in A)((u \in F \wedge u \preccurlyeq v) \Longrightarrow v \in F)$; and
(F3) $(\forall u, v \in A)((u \in F \wedge u \rightarrow v \in F) \Longrightarrow v \in F)$.

The recognizing of the essence of the interrelation between the conditions (F1), (F2) and (F3) are exhibited successively.

PROPOSITION 3.2. $(F2) \Longrightarrow (F1)$.

PROOF. Let $u, v \in A$ be elements such that $u \cdot v \in F$. Since $u \cdot v \preccurlyeq u$ and $u \cdot v \preccurlyeq v$ by (11) we conclude that $u \in F$ and $v \in F$ by (F2). □

The formula (F1) says that F is a prime subset of the monoid $(A, \cdot)$.

PROPOSITION 3.3. *Let F be a subset of a quasi-ordered residuated system $\mathfrak{A}$. If $u, v \in A$ are elements such that $(F2)$ holds, then $u \to v \in F$.*

PROOF. Let $u, v \in A$ such that $u \in F$ and $u \preccurlyeq v$. Then from $u \cdot u \preccurlyeq u$ (by (11)) follows $u \cdot u \preccurlyeq v$ by transitivity of relation $\preccurlyeq$. Thus $u \preccurlyeq u \to v$ by (3). So, we have $u \to v \in F$ by (F2). □

PROPOSITION 3.4. *Let F be a nonempty subset of a quasi-ordered residuated system A. Then $(F2)$ implies $1 \in F$.*

PROOF. Suppose $F \neq \emptyset$. Then there exists an element $u \in F$. Since it is $u \preccurlyeq 1$, according to (2), we have $1 \in F$ by (F2). □

REMARK 3.1. Assuming $1 \in F$ and (F2), we can reinforce the claim in Proposition 3.3 and prove $u \preccurlyeq v \Longrightarrow u \to v \in F$ in the following way:

PROOF. Let $u, v \in A$ be elements such that $u \preccurlyeq v$. Then $1 \in F$ by hypothesis and $1 \preccurlyeq u \to v$ by (3). Thus $u \to v \in F$ by (F2). □

PROPOSITION 3.5. *Let F be a subset of a quasi-ordered residuated system $\mathfrak{A}$. Then the condition $(F2)$ is equivalent to the condition*
(F4) $(\forall u, v, z \in A)((u \cdot v \in F \wedge u \preccurlyeq v \to z) \Longrightarrow z \in F)$.

PROOF. $(F2) \Longrightarrow (F4)$. Let $u, v, z \in A$ such that $x \cdot v \in F$ and $u \preccurlyeq v \to z$. Then $u \cdot v \in F \wedge u \cdot v \preccurlyeq z$ by (3). Thus $z \in F$ by (F2).
$(F4) \Longrightarrow (F2)$. Opposite, let us assume that (F4) holds. Let $u, v \in A$ such that $u \in F \wedge u \preccurlyeq v$. Then $u \cdot 1 \in F \wedge u \preccurlyeq 1 \to v$. Thus $v \in F$ according (F4). So, the formula (F2) is proven. □

A interrelation between conditions (F2) and (F3) we describes in the following proposition.

PROPOSITION 3.6. *Let F be a submonoid of the monoid $(A, \cdot)$ in a quasi-ordered residuated system $\mathfrak{A} = \langle (A, \cdot, \to, 1, \preccurlyeq \rangle$. Then $F(2) \Longrightarrow F(3)$.*

PROOF. Suppose $u, v \in A$ elements such that $u \in F$ and $u \to v \in F$. Then $(x \to v) \cdot u \in F$ by the hypothesis. On the other side, since it is $u \to v \preccurlyeq u \to v$, we have $(u \to v) \cdot u \preccurlyeq v$ by (3). Now, from $(x \to v) \cdot u \in F$ and $(u \to v) \cdot u \preccurlyeq v$ follows $v \in F$ according to (F2). So, the condition (F3) is proven. □

Based on our previous analysis of the interrelationship between conditions (F1), F(2), (F3) and (F4) in a quasi-ordered residual system, we introduce the concept of filters in the following definition.

Definition 3.1. For a subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that it is a *filter* in A if it satisfies conditions (F2) and (F3).

The sets $\emptyset$ and A are trivial filters in $\mathfrak{A}$. Therefore, the family $\mathfrak{F}(A)$ of all filters in a quasi-ordered resuduated system $\mathfrak{A}$ is not empty.

Theorem 3.1. *The family $\mathfrak{F}(A)$ of all filters in a quasi-ordered residuated system forms a completely lattice.*

Proof. Let $\{F_k\}_{k\in K}$ be a family of filters in a quasi-ordered residuated system $\mathfrak{A}$.

(i) Let us prove that $\bigcap_{k\in K} F_k$ is a filter in $\mathfrak{A}$. In this objective, we prove that the intersection $\bigcap_{k\in K} F_k$ satisfies the formulas (F2) and (F3).

Let $u, v \in A$ be arbitrary elements such that $u \in \bigcap_{k\in K} F_k$ and $u \preccurlyeq v$. Then for any index $i \in K$ holds $u \in F_i$. Thus $v \in F_i$ because F_i is a filter in $\mathfrak{A}$. So, $v \in \bigcap_{k\in K} F_k$. Therefore, the set $\bigcap_{k\in K} F_k$ satisfies an axiom (F2).

Let $u, v \in A$ arbitrary element such that $x \in \bigcap_{k\in K} F_k$ and $x \to y \in \bigcap_{k\in K} F_k$. Then for any index $i \in K$ holds $x \in F_i \wedge x \to y \in F_i$ holds. Thus $y \in F_i$ since F_i is a filter in $\mathfrak{A}$. So, $y \in \bigcap_{k\in K} F_k$. Therefore, the set $\bigcap_{k\in K} F_k$ satisfies the condition (F3).

(ii) Let $\mathfrak{B}$ be the family of all filter in $\mathfrak{A}$ which contain $\bigcup_{k\in K} F_k$. Then the intersection $\cap\mathfrak{B}$ is the minimal filter which contains $\bigcup_{k\in K} F_k$ according to the first part of this evidence.

(iii) If we put $\sqcup_{k\in K} F_k = \cap\mathfrak{B}$ and $\sqcap_{k\in K} F_k = \bigcap_{k\in K} F_k$, then $(\mathfrak{F}(A), \sqcup, \sqcap)$ is a completely lattice. □

Corollary 3.2. *Let $\mathfrak{A}$ be a quasi-ordered residuated system and let B be a subset in A. Then there exists the minimal filter in $\mathfrak{A}$ which contains B.*

Proof. The claim follows directly from the first part of the proof of the previous theorem. □

Corollary 3.3. *For any element a in A there exists the minimal filter F_a in $\mathfrak{A}$ such that $a \in F_a$.*

Proof. The claim follows directly from the previous corollary if we put $B = \{a\}$. □

In the following definition we introduce the concept of 2-filters, which is somewhat more complex than the concept of filters.

Definition 3.2. For a subset F of a quasi-ordered residuated system $\mathfrak{A}$ we say that the *2-filter* in $\mathfrak{A}$ if (F2) and the following

(F5) $(\forall u, v, z \in A)(((u \to v) \to z \in F \,\wedge\, u \to z \in F) \implies v \cdot z \in F)$

are valid.

It is immediately seen that $1 \in F$ if F is not an empty set and that, besides, the 2-filter satisfies condition (F1). Since a non-empty 2-filter F satisfies condition (F2), the claim of the Proposition 3.3 is valid for F also. Let us show that every 2-filter in $\mathfrak{A}$ is a filer in $\mathfrak{A}$.

THEOREM 3.2. *Let $\mathfrak{A}$ be a quasi-ordered residuated system. Then every 2-filter in $\mathfrak{A}$ is a filter u $\mathfrak{A}$.*

PROOF. Let F be a 2-filter in a quasi-ordered residuated system $\mathfrak{A}$. To prove that F is a filter in $\mathfrak{A}$, it suffices to prove that F satisfies condition (F3). Let $u, v \in A$ be arbitrary elements such that $u \to v \in F$ and $u \in F$. Then $(u \to v) \to 1 \in F$ and $u \to 1 \in F$ by Proposition 3.3 since $u \to v \preccurlyeq 1$ and $u \preccurlyeq 1$. Thus $v = v \cdot 1 \in F$ by (F5). So, the set F is a filter in $\mathfrak{A}$. □

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References

[1] S. Bonzio. *Algebraic structures from quantum and fuzzy logics.* Ph.D Thesis. Cagliari: Universit'a degli studi di Cagliari, 2015.

[2] S. Bonzio and I. Chajda. Residuated relational systems. *Asian-European J. Math.*, **11**(2)(2018), 1850024

[3] I. Chajda and H. Langer. Relatively residuated lattices and posets. arXiv:1901.06664v1

[4] A. I. Malcev. On the general theory of algebraic systems. *Mat. Sb.*, **35**(1)(1954), 3-20 (In Rusian).

[5] J. Riguet. Relations binaires, fermetures, correspondances de Galois. *Bull. Soc. Math. France*, **76**(1948), 114-155 (in French).

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