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$\Gamma-$INCLINE AND QUOTIENT $\Gamma-$INCLINE

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Abstract. In this paper, we introduce the notion of quotient $\Gamma-$incline, prime ideal, maximal ideal, filter, irreducible ideal, strongly irreducible ideal in $\Gamma-$incline. We study the properties of ideals in $\Gamma-$incline and quotient $\Gamma-$incline and the relations between them. We prove that every ideal in a mono regular $\Gamma-$incline is a prime ideal.

1. Introduction

The non trivial example of semiring first appeared in the work of German mathematician Richard Dedikind in 1894 in connection with the algebra of ideals of a commutative ring. A natural example of semiring is the set of all natural numbers under usual addition and multiplication of numbers. In particular, if I is the unit interval on the real line, then $(I, \max, \min)$ is a semiring in which 0 is the additive identity and 1 is the mutiplicative identity. The theory of rings and the theory of semigroups have considerable impact on the development of the theory of semirings. In structure, semiring lies between semigroup and ring. Many semirings have order structure in addition to their algebraic structure.

The notion of a semiring was introduced by Vandiver [**30**] in 1934, but semirings had appeared in earlier studies on the theory of ideals of rings. Semiring is a generalization of ring but also of a generalization of distributive lattice. Semirings are structually similar to semigroups than to rings. Semiring theory has many applications in other branches of mathematics.

In 1995, M. M. K. Rao [**20, 18, 19, 8, 9, 10, 12**] introduced the notion of $\Gamma-$semiring as a generalization of $\Gamma-$ ring, ternary semiring and semiring. After the paper [**8**] was published, many mathematicians obtained interesting results on Γ-semirings. M. M. K. Rao et. al [**13, 14, 11, 15, 16, 17, 21, 24, 25, 22, 26, 27,**

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Key words and phrases. $\Gamma-$incline, quotient $\Gamma-$incline, integral $\Gamma-$incline, regular $\Gamma-$incline, mono $\Gamma-$incline, prime ideal, maximal ideal, filter, irreducible ideal, strongly irreducible.ordered semiring, derivation, semiring, integral ordered semiring.

23] studied regular $\Gamma-$incline, field $\Gamma-$semiring, ideals in semigroups, semirings, $\Gamma-$semigroups, $\Gamma-$incline and $\Gamma-$semiring.

The concept of incline was first introduced by Z. Q. Cao [**3**] in 1984. Inclines are additively idempotent semiring in which products are less than or equal to either factor. Products reduce the values of quantities and make them go down which is why the structures were named inclines. Idempotent semirings and Kleene algebras have recently been established as fundamentals structures in computer sciences. An incline is a generalization of Boolean algebra, fuzzy algebra and distributive lattice and incline is a special type of semiring. An incline has both semiring structure and the poset structure. Every distributive lattice and every Boolean algebra is an incline but an incline need not be a distributive lattice. Set of all idempotent elements in an incline is a distributive lattice. W. Yao and S. C. Han [**31**] studied the relations between ideals, filters and congruences in inclines and it is shown that there is a one to one correspondence between the set of ideals and the set of all regular congruences. Kim and Rowsh [**5**] have studied matrices over an incline. Many research scholars have been researched the theory of incline matrices. Few research scholars studied the algebraic structure of inclines. Inclines and matrices over inclines are useful tools in diverse areas such as automata theory, design of switching circuits, graph theory, information systems, modeling, decision making, dynamical programming, control theory, classical and non classical path finding problems in graphs, fuzzy set theory, data analysis, medical diagnosis, nervous system, probable reasoning, physical measurement and so on. S. H. Ahn, Y. B. Jun and H. S. Kim [**1, 2**] studied ideals in incline and quotient incline.

The notion of Γ-ring was introduced by N. Nobusawa [**28**] as a generalization of ring in 1964. M. K. Sen [**29**] introduced the notion of $\Gamma-$semigroup in 1981. The notion of ternary algebraic system was introduced by Lehmer [**6**] in 1932, Lister [**7**] introduced ternary ring.

In this paper, we introduce the notion of quotient $\Gamma-$incline, prime ideal, maximal ideal, filter, irreducible ideal, strongly irreducible ideal in $\Gamma-$incline. We study the properties of ideals in Γ-incline and quotient Γ-incline. We prove that every ideal in a mono regular $\Gamma-$incline is a prime ideal.

2. Preliminaries

In this section, we will recall some of the fundamental concepts and definitions [**3, 18, 19**], which are necessary for this paper.

Definition 2.1. ([**3**]) A commutative incline M with additive identity 0 and multiplicative identity 1 is a nonempty set M with operations addition $(+)$ and multiplication $(.)$ defined on $M \times M \to M$ such that satisfying the following conditions for all $x, y, z \in M$

(i) $x + y = y + x$ (ii) $x + x = x$, (iii) $x + xy = x$.
(iv) $y + xy = y$, (v)$x + (y + z) = (x + y) + z$, (vi) $x(yz) = x(yz)$
(vii) $x(y + z) = xy + xz$. (viii) $(x + y)z = xz + yz$, (ix) $x1 = 1x = x$.
(x) $x + 0 = 0 + x = x$, (xi) $xy = yx$.

Definition 2.2. ([**18**]) Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. Then we call M as a $\Gamma-$semiring,if there exists a mapping $M \times \Gamma \times M \to M$ is written (x, α, y) as $x\alpha y$ such that it satisfies the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$

(i) $x\alpha(y+z) = x\alpha y + x\alpha z$, (ii) $(x+y)\alpha z = x\alpha z + y\alpha z$
(iii) $x(\alpha+\beta)y = x\alpha y + x\beta y$, (iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Every semiring R is a $\Gamma-$ semiring with $\Gamma = R$ and ternary operation $x\gamma y$ as the usual semiring multiplication.

Definition 2.3. ([**19**]) Let $(M, +)$ and $(\Gamma, +)$ be commutative semigroups. If there exists a mapping $M \times \Gamma \times M \to M\big((x, \alpha, y) \to x\alpha y\big)$ such that it satisfies the following axioms for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$

(i) $x\alpha(y+z) = x\alpha y + x\alpha z$, (ii) $(x+y)\alpha z = x\alpha z + y\alpha z$.
(iii) $x(\alpha+\beta)y = x\alpha y + x\beta y$, (iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$
(v) $x + x = x$, (vi) $x + x\alpha y = x$, (vii) $y + x\alpha y = y$.

Then M is called a $\Gamma-$incline.

Every incline M is a $\Gamma-$incline with $\Gamma = M$ and ternary operation $x\gamma y$ as the usual incline multiplication. In an $\Gamma-$incline define the order relation such that for all $x, y \in M, y \leqslant x$ if and only if $y + x = x$. Obviously $\leqslant$ is a partially order relation over M.

Example 2.1. Let S be a semiring and $M_{p,q}(S)$ denote the additive abelian semigroup of all $p \times q$ matrices with identity element whose entries are from S. Then $M_{p,q}(S)$ is a $\Gamma-$ semiring with $\Gamma = M_{p,q}(S)$ ternary operation is defined by $x\alpha z = x(\alpha^t)z$ as the usual matrix multiplication, where α^t denote the transpose of the matrix α; for all x, y and $\alpha \in M_{p,q}(S)$.

Example 2.2. If $M = [0,\ 1]$ and $\Gamma = N$, a binary operation $+$ is defined as $a + b = max\{a, b\}$ and ternary operation is defined as $xry = min\{x, r, y\}$ for all $x, y \in M, r \in \Gamma$ then M is a $\Gamma-$ incline .

Example 2.3. If $M = [0, 1], \Gamma = \{0, 1\}$,a binary operation $+$ is maximum, ternary operation $a\alpha b$ is the usual multiplication for all $a, b \in M, \alpha \in \Gamma$ then M is $\Gamma-$incline with unity 1.

Definition 2.4. A $\Gamma-$incline M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0$ for all $x \in M$ and $\alpha \in \Gamma$."

Definition 2.5. A $\Gamma-$incline M is said to be commutative $\Gamma-$incline if $x\alpha y = y\alpha x$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 2.6. A $\Gamma-$ subincline I of a $\Gamma-$incline M is a non-empty subset of M which is closed under the $\Gamma-$incline operations addition and ternary operation.

Definition 2.7. An element $a \in M$, is said to be idempotent of M if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$ and a is also said to be α idempotent.

Definition 2.8. Every element of M, is idempotent of M then M is said to be an idempotent $\Gamma-$incline M.

Definition 2.9. Every element of M, is a regular element of M then M is said to be a regular $\Gamma-$incline M.

Definition 2.10. If $x \leqslant y$ for all $y \in M$ then x is called the least element of M and denoted as 0. If $x \geqslant y$ for all $y \in M$ then x is called the greatest element of M and denoted as 1.

Definition 2.11. A $\Gamma-$incline M is said to be linearly ordered if $x, y \in M$ then either $x \leqslant y$ or $y \leqslant x$, where $\leqslant$ is an incline order relation.

Definition 2.12. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.

Definition 2.13. In a $\Gamma-$incline M with unity 1, an element $a \in M$ is said to be invertible if there exists $b \in M, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 1$.

Definition 2.14. A non zero element a in a $\Gamma-$incline M is said to be zero divisor if there exists non-zero element $b \in M,\ \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 0$.

Definition 2.15. A $\Gamma-$incline M is said to be field $\Gamma-$incline if M is a commutative $\Gamma-$incline with unity 1 and every non- zero element of M is invertible.

Definition 2.16. A $\Gamma-$incline M with unity 1 and zero element 0 is called an integral $\Gamma-$incline if it has no zero divisors.

Definition 2.17. A $\Gamma-$incline M with zero element 0 is said to be hold cancellation laws if $a \neq 0$, $a\alpha b = a\alpha c$, $b\alpha a = c\alpha a$, where $a, b, c \in M$, $\alpha \in \Gamma$, then $b = c$.

Definition 2.18. Let M be a $\Gamma-$incline with unity 1 and zero element 0. M is said to be pre -integral incline if M holds cancellation law.

3. Ideals in $\Gamma-$incline

In this section, we introduce the notion of prime ideal, maximal ideal, filter, irreducible ideal, strongly irreducible ideal and homomorphism in $\Gamma-$incline and we study the properties of ideals in $\Gamma-$incline and relations between them.

Definition 3.1. Let M and N be $\Gamma-$inclines. A mapping $f : M \to N$ is called a homomorphism if

(i) $f(a + b) = f(a) + f(b)$
(ii) $f(a\alpha b) = f(a)\alpha f(b)$, for all $a, b \in M, \alpha \in \Gamma$.

Definition 3.2. A $\Gamma-$incline M is said to be zero sum free $\Gamma-$incline if $x+y = 0 \Rightarrow x = 0$ and $y = 0$, for all $x, y \in M$.

Definition 3.3. A subincline $I(F)$ of a $\Gamma-$incline M is called an ideal (filter) if it is a lower (upper) set. i.e., for any $x \in I, y \in M$ and $y \leqslant x \Rightarrow y \in I$. ($x \in F$, $y \in M$ and $x \leqslant y \Rightarrow y \in F$.)

Definition 3.4. A proper ideal P of a $\Gamma-$incline M is said to be prime ideal if for all $x, y \in M, \alpha \in \Gamma, x\alpha y \in P \Rightarrow x \in P$ or $y \in P$.

Definition 3.5. An ideal K of a $\Gamma-$incline M is said to be maximal ideal if $K \neq M$ and for every ideal I of M with $K \subseteq I \subseteq M$ then either $I = M$ or $I = M$.

DEFINITION 3.6. A proper ideal I of a $\Gamma-$incline M is said to be irreducible ideal if $I = A \cap B$ then $I = A$ or $I = B$.

DEFINITION 3.7. An ideal I of a $\Gamma-$incline M is strongly irreducible ideal if for ideals J and K of M, $J \cap K \subseteq I$ then $J \subseteq I$ or $K \subseteq I$.

DEFINITION 3.8. A subincline I of a $\Gamma-$incline M is said to be $k-$ideal if $x + y \in I, y \in I$ then $x \in I$.

DEFINITION 3.9. A $\Gamma-$incline M is said to be simple if it has no proper ideals.

THEOREM 3.1. *Let I be a subincline of a $\Gamma-$incline M. Then I is an ideal of M if and only if I is a $k-$ideal of M.*

PROOF. Let I be a subincline of the $\Gamma-$incline M and $x + y \in I, y \in I$. Then

$$x + y = (x + x) + y = x + (x + y).$$

Thus $x \leqslant x + y$. Therefore, by Definition of ideal, $x \in I$. Hence I is a $k-$ideal.

Conversely, suppose that $x \in I$.. Then from $y \leqslant x$ it follows $y + x = x$ and $y + x \in I$. Thus $y \in I$ since I is a $k-$ideal of $\Gamma-$incline M. Hence I is an ideal of the $\Gamma-$incline M. □

THEOREM 3.2. *In a $\Gamma-$incline M, every maximal ideal of M is an irreducible ideal.*

PROOF. Let S be a maximal ideal of the $\Gamma-$incline M. Suppose S is not irreducible and $S = U \cap V$. Then $S \neq U$ and $S \neq V$. Thus $S \subset U \subset M$ and $S \subset V \subset M$ which is a contradiction. Hence S is an irreducible ideal of the $\Gamma-$incline M. □

THEOREM 3.3. *Let I be an ideal of a $\Gamma-$incline M.*

(i) *If I is a prime ideal then I is a strongly irreducible ideal.*
(ii) *If I is a strongly irreducible ideal then I is an irreducible ideal.*

PROOF. Let I be an ideal of the $\Gamma-$incline M.

(i) Suppose I is a prime ideal of the $\Gamma-$incline M, J and K are ideals of the $\Gamma-$incline M such that $J \cap K \subseteq I$. Then $J\Gamma K \subseteq I \Rightarrow J \subseteq I$ or $K \subseteq I$, since I is a prime ideal. Hence I is a strongly irreducible ideal.

(ii) Suppose I is a strongly irreducible ideal of $\Gamma-$incline M, J and K are ideals of $\Gamma-$incline M such that $J \cap K = I$. Then from $J \cap K \subseteq I$ it follows $J \subseteq I$ or $K \subseteq I$. Hence $J = I$ or $K = I$. Therefore I is an irreducible ideal. □

COROLLARY 3.1. *In a $\Gamma-$incline M, If I is a prime ideal then I is an irreducible ideal.*

EXAMPLE 3.1. Let $M = [0, 1]$ and $\Gamma = N$. A binary operation $+$ is defined as $a + b = \max\{a, b\}$, for all $a, b \in M$, $x + y = \max\{x, y\}$, for all $x, y \in N$ and ternary operation is defined as $x\gamma y = xy$ (usual product), for all $x, y \in M$ and $\gamma \in N$. Then M is a $\Gamma-$incline. All ideals of M are closed intervals $[0, a]$ for some $a \in M$. Let $I = [0, 0.2]$. Then I is an irreducible ideal but not a prime ideal. Therefore converse of the above Corollary 3.1 is not true.

The following theorem is a straightforward verification.

Theorem 3.4. *If F is a non-empty subset of a $\Gamma-$incline M then the following are equivalent*
(i) *F is a filter,*
(ii) *$a+b \in F$, for all $a \in F$ and $b \in M$.*

Theorem 3.5. *Let M be a $\Gamma-$incline. F is a filter of a $\Gamma-$incline M if and only if F^c is a prime ideal of a $\Gamma-$incline M.*

Proof. Let F be a filter of the $\Gamma-$incline M and $a, b \in F^c$. Then from $a, b \notin F$ it is implies $a\alpha b \notin F$ and $a+b \notin F$ for all $\alpha \in \Gamma$. Thus $a\alpha b \in F^c$ and $a+b \in F^c$ for all $\alpha \in \Gamma$. Let $a, b \in M$ and $a\alpha b \in F^c, \alpha \in \Gamma$. Suppose $a, b \notin F^c$. Then $a, b \in F$ and $a\alpha b \in F$, $\alpha \in \Gamma$. Which is a contradicts to our assumption. Therefore F^c is a prime ideal of the $\Gamma-$incline M.

Conversely suppose that F^c is a prime ideal of the $\Gamma-$incline M. Let $a, b \in F, \alpha \in \Gamma \Rightarrow a \leqslant a+b$. If $a+b \in F^c$ then $a \in F^c$. Hence $a+b \notin F^c$. Therefore $a+b \in F$. If $a\alpha b \notin F$. Then from $a\alpha b \in F^c$ it follows $a \in F^c$ or $b \in F^c$, which is a contradiction. Hence $a\alpha b \in F$. Let $a \in F$, $a \leqslant b$, and $b \in M$. Suppose $b \notin F$. Then $b \in F^c$. Thus $a \in F^c$ which is a contradiction. Therefore $b \in F$. Hence F is a filter of the $\Gamma-$incline M. $\square$

Theorem 3.6. *Let $f : K \to L$ be a homomorphism of $\Gamma-$inclines. If J is an ideal of L then $f^{-1}(J)$ is an ideal of the $\Gamma-$incline K.*

Proof. Let $x, y \in f^{-1}(J), \alpha \in \Gamma$. Then $f(x), f(y) \in J$ and $f(x)+f(y) = f(x+y) \in J$. Thus $x+y \in f^{-1}(J)$. On the other side, $x, y \in f^{-1}(J)$ it means $f(x), f(y) \in J$. Then $f(x)\alpha f(y) \in J$ i.e. $f(x\alpha y) \in J$. Thus $x\alpha y \in f^{-1}(J)$.

Hence $f^{-1}(J)$ is a subincline of K. Let $x \in K, y \in f^{-1}(J)$ such that $x \leqslant y$. Then from $x+y=y$ it follows $f(x+y) = f(y)$, i.e. $f(x)+f(y) = f(y) \in J$. This means $f(x) \leqslant f(y)$. Thus $f(x) \in J$. So, $x \in f^{-1}(J)$. Hence $f^{-1}(J)$ is an ideal of the $\Gamma-$ incline K. $\square$

Theorem 3.7. *Let M be a $\Gamma-$incline with unity* 1 *and zero element* 0. *If I is an ideal containing a unit element then $I = M$.*

Proof. Let I be an ideal of the $\Gamma-$incline M containing an unit element u and $x \in M$. Then there exists $\alpha \in \Gamma$, such that $x\alpha 1 = x$. Since I is an ideal, $x\alpha u \in I$. Since u is a unit, there exist $\delta \in \Gamma$, $t \in M$ such that $u\delta t = 1$. Thus $x\alpha u\delta t = x\alpha 1 = x$. So, $x \in I$ Hence $I = M$. $\square$

Theorem 3.8. *A field $\Gamma-$incline M is simple.*

Proof. Let I be a proper ideal of thefield $\Gamma-$incline M. Every nonzero element of I is a unit. By Theorem 3.7, we have $I = M$. Hence field $\Gamma-$incline is simple. $\square$

4. Ideals in quotient $\Gamma-$incline

In this section, we introduce the notion of quotient $\Gamma-$incline and study the properties of ideals of quotient $\Gamma-$incline. Suppose that I is an ideal of $\Gamma-$incline M with zero element 0. We define a relation $\sim$ on the $\Gamma-$incline M by $x \sim y$ if and only if $x + i_1 = y + i_2$ for some $i_1, i_2 \in I$ and $x, y \in M$. Obviously $\sim$ is an equivalence relation.

Let M be a $\Gamma-$incline. The equivalence class of $x \in M$ is determined by an ideal I is denoted by $x + I$. The set of all equivalence classes $\{x + I \mid x \in M\}$ is denoted by M/I. We define two operations on M/I by

(i) $(x + I) + (y + I) = x + y + I$ and

(ii) $(x + I)\alpha(y + I) = x\alpha y + I$

for all $x, y \in M$ and $\alpha \in \Gamma$. Then M/I is a $\Gamma-$incline. The $\Gamma-$incline M/I is called a quotient $\Gamma-$incline. If M is a commutative $\Gamma-$incline then M/I is a commutative $\Gamma-$incline. Define $\phi : M \to M/I$ by $\phi(x) = x + I,$ for all $x \in M$. Clearly ϕ is a homomorphism. We define the order relation on $\Gamma-$incline M/I by $a + I \leqslant b + I$ if and only if $a \leqslant b$. i.e. $a + b = b$. Obviously $\leqslant$ is a partial order relation on M/I.

Theorem 4.1. *Let a, b be in a $\Gamma-$incline M. If $a \sim b$ then $a\alpha x \sim b\alpha x$ and $a + x \sim b + x,$ for all $x \in M, \alpha \in \Gamma$. i.e., relation $\sim$ is a congruence relation.*

Proof. Let M be the $\Gamma-$incline. If $a \sim b,\ \alpha \in \Gamma, x \in M$ there exist $i_1, i_2 \in I$ such that $a + i_1 = b + i_2$. Then $(a + i_1)\alpha x = (b + i_2)\alpha x$ implies $a\alpha x + i_1\alpha x = b\alpha x + i_2\alpha x$ since $i_1\alpha x, i_2\alpha x \in I$ Thus $a\alpha x \sim b\alpha x$. Now, from $a + i_1 = b + i_2$ it follows $+i_1 + x = b + i_2 + x$ and $(a + x) + i_1 = (b + x) + i_2$. Thus $a + x \sim b + x$. Hence relation $\sim$ on $\Gamma-$incline M is a congruence relation. □

Theorem 4.2. *Let I be an ideal of a $\Gamma-$incline M and $a \in I$. Then $a+I = b+I$ for every $b \in M$ if and only if $b \in I$. In particular $c + I = I$ if and only if $c \in I$.*

Proof. Let $a + I = b + I$ then $a + u = b + v$ for some $u, v \in I$. Then $b + v \in I$ it follows $b \leqslant b + v \in I$ and it means $b \in I$.

Converse is obvious. □

The proofs of the following theorems are similar to incline Theorems 2.4. and 2.7 in [**1**]

Theorem 4.3. *If I and J are any ideals of a $\Gamma-$incline M and $I \subseteq J$ then*

(i) *I is also an ideal of the subincline J.*

(ii) *J/I is an ideal of the quotient $\Gamma-$incline M/I.*

Theorem 4.4. *Let I be an ideal of a $\Gamma-$incline M. If A is an ideal of a quotient $\Gamma-$incline M/I then $\phi^{-1}(A)$ is an ideal of $\Gamma-$incline M containing I when ϕ is a natural homomorphism from M onto M/I*

Theorem 4.5 ([**19**]). *Let M be a $\Gamma-$incline with unity 1 and zero element 0. If $a \in M$ then $0 \leqslant a \leqslant 1$.*

Theorem 4.6. *Let M/I be a quotient $\Gamma-$incline. Then*

(i) *M/I is a zero sum free quotient $\Gamma-$incline.*

(ii) *I is the least element of M/I.*

(iii) *$1 + I$ is the greatest element of M/I.*

(iv) *$I \leqslant a + I \leqslant 1 + I,$ for all $a \in M$.*

Proof. Let M/I be the quotient $\Gamma-$incline.

(i) Suppose $(a + I) + (b + I) = I$. Then $a + b + I = I$ and $a + b \in I$. We have $a \leqslant a + b$ and $b \leqslant a + b$. Thus $a, b \in I$. Therefore $a + I + b + I = I$. So, $a + I = I$ or $b + I = I$. Hence M/I is a zero sum free quotient $\Gamma-$incline.

(ii) Let $a, b \in \Gamma-$incline M and $a \leqslant b$.. Then from $a+b=b$ it follows $a+b+I=b+I$ and $(a+I)+(b+I)=b+I$. Thus $a+I \leqslant b+I$. We have $0 \leqslant a$ for all $a \in M$. Hence $0+I \leqslant a+I$. This means $I \leqslant a+I$ for all $a+I \in M$. So, I is the least element of M/I.

(iii) We have $a \leqslant 1 \Rightarrow a+I \leqslant 1+I,$ for all $a \in M$. Hence $1+I$ is the greatest element of M/I.

(iv) Obvious. □

Theorem 4.7. *If $a+I$ is a regular element of a $\Gamma-$ incline M/I then there exist $x+I \in M/I, \alpha, \beta \in \Gamma$ such that*

$$a+I=(a+I)\alpha(x+I)=(x+I)\beta(a+I).$$

Proof. Suppose $a+I$ is a regular element of the $\Gamma-$ incline M/I. Then there exist $x+I \in M/I, \alpha, \beta \in \Gamma$ such that

$$a+I=(a+I)\alpha(x+I)\beta(a+I) \leqslant \ (a+I)\alpha(x+I) \leqslant a+I.$$

Therefore $(a+I)\alpha(x+I)=a+I$. Now

$$a+I=(a+I)\alpha(x+I)\beta(a+I) \leqslant (x+I)\beta(a+I) \leqslant (a+I).$$

Therefore $a+I=(x+I)\beta(a+I)$. Hence $a+I=(a+I)\alpha(x+I)=(x+I)\beta(a+I)$. □

Theorem 4.8. *Let M/I be a $\Gamma-$incline. M/I is a regular $\Gamma-$incline if and only if M/I is an idempotent $\Gamma-$incline*

Proof. Suppose M/I is a regular $\Gamma-$incline and $a+I \in M/I$. Then $a+I$ is a regular element. Since $a+I$ is a regular element, there exists $x+I \in M/I, \alpha, \beta \in \Gamma$ such that $a+I=(a+I)\alpha(x+I)\beta(a+I)$. By Theorem 4.7, $a+I=(a+I)\alpha(x+I)=(x+I)\beta(a+I)$. Now $a+I=(a+I)\alpha(x+I)\beta(a+I)=(a+I)\alpha(a+I)$. Therefore $a+I$ is an idempotent. Hence M/I is the idempotent $\Gamma-$incline.

Conversely suppose that $a+I$ is a $\alpha-$idempotent of $M/I, \alpha \in \Gamma$.
$a+I=(a+I)\alpha(a+I)=(a+I)\alpha(a+I)\alpha(a+I)$. Hence M/I is a regular $\Gamma-$incline. □

Theorem 4.9. *Let M/I be a commutative $\Gamma-$incline. If $b+I, c+I \in M/I$ are α, β idempotents respectively, $\alpha, \beta \in \Gamma$ then*

$$(b+I)\alpha(c+I)=(b+I)\beta(c+I).$$

Proof. Let M/I be a commutative $\Gamma-$incline and $b+I, c+I \in M/I$ be α, β idempotents respectively , $\alpha, \beta \in \Gamma$. Then we have

$$(b+I)\alpha(b+I)=b+I, \text{ and } (c+I)\beta(c+I)=c+I.$$

Now

$$\begin{aligned}
(b+I)\alpha(c+I) &= \big((b+I)\alpha(b+I)\big)\alpha\big((c+I)\beta(c+I)\big) \\
&= (b+I)\alpha\big((b+I)\alpha(c+I)\beta(c+I)\big) \\
&= (b+I)\alpha\big((c+I)\alpha(b+I)\big)\beta(c+I) \\
&= \big((b+I)\alpha(c+I)\alpha(b+I)\beta(c+I)\big) \\
&\leqslant (b+I)\beta(c+I).
\end{aligned}$$

Similarly we can prove $(b+I)\beta(c+I) \leqslant (b+I)\alpha(c+I)$. Hence

$$(b+I)\alpha(c+I) = (b+I)\beta(c+I).$$

□

Theorem 4.10. *Let M/I be a commutative $\Gamma-$incline. If $b+I, c+I \in M/I$ are α, β idempotents respectively, $\alpha, \beta \in \Gamma$, $a+I \in M/I$, $(a+I)+(b+I) = (a+I)+(c+I)$ and $(b+I)\alpha(a+I) = (c+I)\beta(a+I)$ then $b+I = c+I$.*

Proof. Suppose M/I is a commutative $\Gamma-$incline and $b+I, c+I \in M/I$ are α, β idempotents respectively, $\alpha, \beta \in \Gamma$. Then $a+I \in M/I$, $(a+I)+(b+I) = (a+I)+(c+I)$ and $(b+I)\alpha(a+I) = (c+I)\beta(a+I)$. We have $(b+I)\alpha(b+I) = (b+I), (c+I)\beta(c+I) = c+I$. By Theorem 4.9, we have $(b+I)\alpha(c+I) = (b+I)\beta(c+I)$.

$$\begin{aligned}
&(a+I)+(b+I) = (a+I)+(c+I).\\
\Rightarrow\ &(b+I)\alpha\big((a+I)+(b+I)\big) = (b+I)\alpha\big((a+I)+(c+I)\big)\\
\Rightarrow\ &(b+I)\alpha(a+I)+(b+I) = (b+I)\alpha(a+I)+(b+I)\alpha(c+I)\\
\Rightarrow\ &(b+I) = (c+I)\beta(a+I)+(b+I)\beta(c+I) \text{ (By incline definition.)}\\
\Rightarrow\ &(b+I) = (a+I)\beta(c+I)+(b+I)\beta(c+I)\\
\Rightarrow\ &(b+I) = \big((a+I)+(b+I)\big)\beta(c+I)\\
\Rightarrow\ &(b+I) = \big((a+I)+(c+I)\big)\beta(c+I)\\
\Rightarrow\ &(b+I) = (a+I)\beta(c+I)+(c+I)\beta(c+I)\\
\Rightarrow\ &(b+I) = (a+I)\beta(c+I)+(c+I)\\
\Rightarrow\ &(b+I) = (c+I).
\end{aligned}$$

Hence the theorem. □

Definition 4.1. Let M/I be a $\Gamma-$incline is said to be mono $\Gamma-$incline. If $a+I, c+I \in M/I, \alpha \in \Gamma$ and $a+I$ is α idempotent then $(a+I)\alpha(c+I) = (a+I)+(c+I)$.

Theorem 4.11. *If M/I is a mono regular $\Gamma-$incline then M/I is a pre-integral $\Gamma-$incline.*

Proof. Let M/I be a mono regular $\Gamma-$incline and $a+I, b+I, c+I \in M/I$ and $\gamma \in \Gamma$.

Suppose $(b+I)\gamma(a+I) = (c+I)\gamma(a+I)$ and $(b+I)\alpha(b+I) = (b+I)$. Then

$$\begin{aligned}
&(c+I)\beta(c+I) = (c+I),\ (\alpha, \beta \in \Gamma)\\
\Rightarrow&\big((b+I)\alpha(b+I)\big)\gamma(a+I) = \big((c+I)\beta(c+I)\big)\gamma(a+I)\\
\Rightarrow&(b+I)\alpha\big((b+I)\gamma(a+I)\big) = (c+I)\beta\big((c+I)\gamma(a+I)\big)\\
\Rightarrow&(b+I)+(b+I)\gamma(a+I) = (c+I)+(c+I)\gamma(a+I)
\end{aligned}$$

since M/I is a mono Γ-incline. Therefore by definition of $\Gamma-$incline, $(b+I) = (c+I)$. Hence M/I is a pre-integral $\Gamma-$incline. □

Theorem 4.12. *Let M be a $\Gamma-$incline. If I is an ideal of M and J is a strongly irreducible ideal with $I \subseteq J$ then J/I is a strongly irreducible ideal of M/I.*

PROOF. Let M be a $\Gamma-$incline, I be an ideal of M and J be a strongly irreducible ideal with $I \subseteq J$. By Theorem 4.3, J/I is an ideal of the quotient $\Gamma-$incline M/I. K/I and H/I ideals of M/I such that $K/I \cap H/I \subseteq J/I \Rightarrow K \cap H \subseteq J$. Since J is a strongly irreducible ideal of M/I we have $K \subseteq J$ or $H \subseteq J$.. Then $K/I \subseteq J/I$ or $H/I \subseteq J/I$. Hence J/I is a strongly irreducible ideal of $\Gamma-$incline M/I. □

THEOREM 4.13. *Let I and J be ideals of a $\Gamma-$incline M with $I \subseteq J$. Then following*

(i) *If $1 + I \in J/I$ then $M/I = J/I$.*
(ii) *If $a+I$ is an invertible element of M/I with $a+I \in J/I$ then $M/I = J/I$.*

PROOF. Let I and J be ideals of the $\Gamma-$incline M with $I \subseteq J$.
(i) Let $x+I \in M/I$ and $1+I \in J/I$. Then $(x+I)\alpha(1+I) \in J/I$, for all $\alpha \in \Gamma$. Thus $x\alpha 1 + I \in J/I$ for all $\alpha \in \Gamma$ and $x + I \in J/I$. Hence $M/I = J/I$.
(ii) Since $a+I$ is an invertible, there exist $\alpha \in \Gamma, b+I$ such that $(a+I)\alpha(b+I) = 1 + I$. Then $1 + I \in J/I$. By (i) $M/I = J/I$. □

THEOREM 4.14. *In a $\Gamma-$incline M with unity 1 and zero element 0, P is a prime ideal if and only if quotient $\Gamma-$incline M/P is an integral $\Gamma-$incline.*

PROOF. Suppose M/P is a quotient integral $\Gamma-$incline, P is a prime ideal, $a + P, b + P \in M/P$, $\alpha \in \Gamma$ and $(a + P)\alpha(b + P) = P$. Then from $a\alpha b + P = P$ it follows $a\alpha b \in P$. Thus $a \in P$ or $b \in P$. Hence $a + P = P$ or $b + P = P$. Therefore M/P is the integral $\Gamma-$incline.
Conversely suppose that M/P is the integral $\Gamma-$incline and $a\alpha b \in P$, $a, b \in M$, $\alpha \in \Gamma$. Then from $a\alpha b+P = P$ it follows $(a+P)\alpha(b+P) = P$. Thus $a+P = P$ or $b + P = P$. So, $a \in P$ or $b \in P$. Hence P is a prime ideal of the $\Gamma-$incline M.. □

The following proof of the theorem is similar to proof of Theorem 3.20 in [**1**]

THEOREM 4.15. *In an $\Gamma-$incline with unity 1 and zero element 0, an ideal I is a maximal if and only if quotient $\Gamma-$incline M/I is simple.*

THEOREM 4.16. *Let P be a proper ideal of commutative $\Gamma-$incline M with unity. P is a maximal ideal if and only if M/P is a field $\Gamma-$incline.*

PROOF. Let P be a maximal ideal of the commutative $\Gamma-$incline M with unity and $P \neq a + P \in M/P$. Then $a \notin P$ implies $P + M\Gamma a = M$ by maximality of P. There exist $r \in M$, $\alpha \in \Gamma$, $p \in P$ such that $p + r\alpha a = 1$. Thus $(r + P)\alpha(a + P) = 1 + P$. Hence $a + P$ is an invertible. Hence M/P is the field $\Gamma-$incline.
Conversely suppose that M/P is the field $\Gamma-$incline and $P \subseteq J$. Then there exists $b \in J \setminus P$ such that $P \neq b + P \in M/P$. Thus $b + P$ is an invertible and $\alpha \in \Gamma$, $c + P \in M/P$. From here it follows $(b + P)\alpha(c + P) = 1 + P$ and $b\alpha c + P = 1 + P \in J/P$. Hence $J/P = M/P$ by Theorem 4.14. Finally $J = M$. □

THEOREM 4.17. *Every pre-integral $\Gamma-$inclineM/I is an integral $\Gamma-$incline.*

PROOF. Let M/I be the pre-integral $\Gamma-$incline. Suppose $(a + I)\alpha(b + I) = I$, $a + I, b + I \in M/I$, $\alpha \in \Gamma$ and $b + I \neq I$. Then $(a + I)\alpha(b + I) = I\alpha(b + I)$. Thus $a + I = I$ since M/I is a pre-integral. □

THEOREM 4.18. *Any commutative finite pre-integral quotient* $\Gamma-$*incline* M/I *with unity is a field quotient* $\Gamma-$*incline.*

PROOF. Let M/I be a commutative finite pre-integral quotient $\Gamma-$incline with unity. Suppose $M/I = \{a_1+I, a_2+I, \cdots, a_n+I\}$, $I \neq a+I \in M/I$ and $\alpha \in \Gamma$. Then $a\alpha a_1 + I$, $a\alpha a_2 + I$, $\cdots$, $a\alpha a_n + I$ are distinct element in M/I. From $a\alpha a_i + I = a\alpha a_j + I$ it follows $a + I)\alpha(a_i + I) = (a + I)\alpha(a_j + I)$. Then $a_i + I = a_j + J$. Since $1 + I$ is a unity, there exists $a\alpha a_k + I$ such that $a\alpha a_k + I = 1 + I$. Therefore $(a + I)\alpha(a_k + I) = 1 + I$. □

THEOREM 4.19. *Every ideal in a mono regular* $\Gamma-$*inclineM is a prime ideal.*

PROOF. Let M be a mono regular $\Gamma-$incline and I be an ideal of M. Obviously M/I is a mono regular $\Gamma-$incline. By Theorem 4.11, M/I is a pre-integral $\Gamma-$incline. By Theorem 4.17, M/I is an integral $\Gamma-$incline. By Theorem 4.14, I is a prime ideal of the $\Gamma-$incline M. Hence every ideal of mono regular $\Gamma-$incline is a prime ideal. □

5. Conclusion

In this paper, we introduced the notion of quotient $\Gamma-$incline, prime ideal, maximal ideal, filter, irreducible ideal, strongly irreducible ideal in $\Gamma-$incline. We studyied the properties of ideals in $\Gamma-$incline and quotient $\Gamma-$incline and the relations between them. and we proved that every ideal in a mono regular $\Gamma-$incline is a prime ideal and field $\Gamma-$incline is simple.

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