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# $\alpha$-DOT CUBIC NEW IDEAL OF PU-ALGEBRA

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Abstract. The notions of $\alpha$-dot cubic new-ideal of PU-algebras is introduced, and several related properties are investigated. Characterizations of $\alpha$-dot cubic new ideal on PU-algebras are established. The relations between $\alpha$-dot cubic sub algebras and $\alpha$-dot cubic new-ideal of PU-algebras are discussed. Moreover, the homomorphic image (pre image) of $\alpha$-dot cubic new-ideal of a PU-algebra under homomorphism of a PU-algebras are discussed.

## 1. Introduction

Imai and Isèki [**3, 4, 6, 5**] introduced two classes of abstract algebras: BCK-algebras and BCI-algebras. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [**2, 1**], Hu and Li introduced a wide class of abstract algebras: BCH-algebras. They are shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. Neggers et al. [**10**] defined the notion of Q-algebras, which is a generalization of BCH/BCI/BCK-algebras. Megalai and Tamilarasi [**9**] introduced the notion of a TM-algebra which is a generalization of BCK/BCI/BCH-algebras and several results are presented.

Mostafa, Abdel Naby and Elkabany [**11**] introduced a new algebraic structure called PU-algebra, which is a dual for TM-algebra and investigated several basic properties. Moreover they derived new view of several ideals on PU-algebra and studied some properties of them.

The concept of fuzzy sets was introduced by Zadeh [**14**]. In 1991, Xi [**13**] applied the concept of fuzzy sets to BCI, BCK, MV-algebras. Since its inception, the theory of fuzzy sets, ideal theory and its fuzzification has developed in many directions and is finding applications in a wide variety of fields. Mostafa, Abdel Naby and Elkabany [**12**] presented the notion $\alpha$-fuzzy new-ideal of a PU-algebra. They

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discussed the homomorphic image (pre image) of $\alpha$-fuzzy new-ideal of a PU-algebra under homomorphism of a PU-algebras. Jun, Kim, and Kang [**7**] introduced the notion of cubic sub-algebras /ideals in BCK/BCI-algebras, and then they investigated several properties. They discussed the relationship between a cubic sub-algebra and a cubic ideal. Also, they provided characterizations of a cubic sub-algebra/ideal and considered a method to make a new cubic sub algebra from an old one.

In this paper, we modify the ideas of Jun, Kim, and Kang [**8**] to introduce the notion $\alpha$-dot cubic new-ideal of PU-algebra. The homomorphic image (preimage) of $\alpha$-dot cubic new-ideal of PU-algebra under homomorphism of a PU-algebras are discussed. Many related results have been derived.

## 2. Preliminaries

Now, we will recall some known concepts related to PU-algebra from the literature, which will be helpful in further study of this article

Definition 2.1. ([**11**]) A PU-algebra is a non-empty set $X$ with a constant $0 \in X$ and a binary operation $*$ satisfying the following conditions:

(I) $0 * x = x$,

(II) $(x * z) * (y * z) = y * x$,

for all $x, y, z \in X$.

On $X$ one can define a binary relation " $\leqslant$ " by:

$$x \leqslant y \text{ if and only if } y * x = 0.$$

Example 2.1. ([**11**]) Let $X = \{0, 1, 2, 3, 4\}$ in which $*$ is defined by Table 2.1.

| $*$ | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 4 |
| 1 | 4 | 0 | 1 | 2 | 3 |
| 2 | 3 | 4 | 0 | 1 | 2 |
| 3 | 2 | 3 | 4 | 0 | 1 |
| 4 | 1 | 2 | 3 | 4 | 0 |

Table 2.1

Then $(X, *, 0)$ is a PU-algebra.

Proposition 2.1 ([**11**]). *In a PU-algebra $(X, *, 0)$ the following statements are valid for all $x, y, z, u \in X$.*

(a) $x * x = 0$.

(b) $(x * z) * z = x$.

(c) $x * (y * z) = y * (x * z)$.

(d) $x * (y * x) = y * 0$.

(e) $(x * y) * 0 = y * x$.

(f) *If* $x \leqslant y$, *then* $x * 0 = y * 0$.

(g) $(x * y) * 0 = (x * z) * (y * z)$.

(h) $x * y \leqslant z$ *if and only if* $z * y \leqslant x$.

(i) $x \leqslant y$ *if and only if* $y * z \leqslant x * z$.

(j) *In a PU-algebra* $(X, *, 0)$, *the following are equivalent:*
(1) $x = y$, (2) $x * z = y * z$, (3) $z * x = z * y$.

(k) *The right and the left cancellation laws hold in* $X$.

(l) $(z * x) * (z * y) = x * y$.

(m) $(x * y) * z = (z * y) * x$.

(n) $(x * y) * (z * u) = (x * z) * (y * u)$.

Definition 2.2. ([**11**]) A non-empty subset $S$ of a PU-algebra $(X, *, 0)$ is called a sub-algebra of $X$ if $x * y \in S$ whenever $x, y \in S$.

Definition 2.3. ([**11**]) A non-empty subset $\mathcal{I}$ of a PU-algebra $(X, *, 0)$ is called a new-ideal of $X$ if,

(i) $0 \in \mathcal{I}$,

(ii) $(a * (b * x)) * x \in \mathcal{I}$, for all $a, b \in \mathcal{I}$ and $x \in X$.

Example 2.2. [**11**] Let $X$=$\{0, a, b, c\}$ in which $*$ is defined by Table 2.2.

| $*$ | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | a | b | c |
| a | a | 0 | c | b |
| b | b | c | 0 | a |
| c | c | b | a | 0 |

Table 2.2.

Then $(X, *, 0)$ is a PU-algebra. It is easy to show that $\mathcal{I}_1 = \{0, a\}$, $\mathcal{I}_2 = \{0, b\}$, $\mathcal{I}_3 = \{0, c\}$ are new-ideals of $X$.

Lemma 2.1 ([**11**]). *If* $(X, *, 0)$ *is a PU-algebra, then*

$$(x * (y * z)) * z = (y * 0) * x$$

*for all* $x, y, z \in X$.

Theorem 2.1 ([**11**]). *Any sub-algebra* $S$ *of a PU-algebra* $X$ *is a new-ideal of* $X$.

Definition 2.4. ([**11**]) Let $(X, *, 0)$ and $(\grave{X}, \grave{*}, \grave{0})$ be PU-algebras. A map $f : X \longrightarrow \grave{X}$ is called a homomorphism if

$$f(x * y) = f(x) \grave{*} f(y)$$

for all $x, y \in X$.

Proposition 2.2 ([**11**]). *Let* $(X, *, 0)$ *and* $(\grave{X}, \grave{*}, \grave{0})$ *be PU-algebras. If* $f : X \longrightarrow \grave{X}$ *is a homomorphism, then* $Ker(f)$ *is a new-ideal of* $X$.

A fuzzy set $\mu$ in a set $X$ is a function $\mu$: $X \longrightarrow I$, where $I = [0, 1]$. Denote by $I^X$ the collection of all fuzzy sets in a set $X$. Define a relation $\leqslant$ on $I^X$ as follows:

$$(\forall \mu, \lambda \in I^X)(\forall x \in X)(\mu \leqslant \lambda \Longleftrightarrow \mu(x) \leqslant \lambda(x)).$$

REMARK 2.1. An interval-valued fuzzy subset (briefly, i-v fuzzy subset) $A$ defined in the set $X$ is given by $A=\{((x, [\mu_A^L(x), \mu_A^U(x)]\}$, for all $x \in X$ (briefly, it is denoted by $A = [\mu_A^L(x), \mu_A^U(x)]$ where $\mu_A^L(x)$, $\mu_A^U(x)$ are any two fuzzy subsets in $X$ such that $\mu_A^L(x) \leqslant \mu_A^U(x)$ for all $x \in X$. Let $\widetilde{\mu}_A(x) = [\mu_A^L(x), \mu_A^U(x)]$ for all $x \in X$ and $D[0, 1]$ be denotes the family of all closed sub-intervals of $[0, 1]$. It is clear that if $\mu_A^L(x) = \mu_A^U(x) = c$, where $0 \leqslant c \leqslant 1$, then $\widetilde{\mu}_A(x) = [c, c]$ in $D[0, 1]$, then $\widetilde{\mu}_A(x) \in D[0, 1]$, for all $x \in X$. Therefore the i-v fuzzy subset $A$ is given by: $A = \{\langle x, \widetilde{\mu}_A(x)\rangle : x \in X\}$ where $\widetilde{\mu}_A(x) : X \longrightarrow D[0, 1]$. Now we define the refined minimum (briefly r min) and order " $\leqslant$ " on elements $D_1 = [a_1, b_1]$ and $D_2 = [a_2, b_2]$ of $D[0, 1]$ as follows:

$$r\,min(D_1, D_2) = [min\{a_1, a_2\}, min\{b_1, b_2\}],$$

$$D_1 \leqslant D_2 \Longleftrightarrow a_1 \leqslant a_2 \text{ and } b_1 \leqslant b_2.$$

Similarly we can define ($\geqslant$) and ($=$). Also we can define $D_1 + D_2=[a_1 + a_2, b_1 + b_2]$, and if $c\in[0, 1]$, then $cD_1=[ca_1, cb_1]$. Also if $D_i=[a_i, b_i]$, $i\in I$, then we define

$$r\,sup(D_i) = [supa_i, supb_i] \text{ and } r\,inf(D_i) = [infa_i, infb_i].$$

We will consider that $\widetilde{1}=[1, 1]$ and $\widetilde{0}=[0, 0]$.

Jun et al. [**7**], introduced the concept of cubic sets defined on a non-empty set $X$ as objects having the form: $A = \{\langle x, \widetilde{\mu}_A(x), \lambda_A(x)\rangle$: $x \in X\}$ which is briefly denoted by $A = \langle\widetilde{\mu}_A, \lambda_A\rangle$ where the functions $\widetilde{\mu}_A : X \longrightarrow D[0, 1]$ and $\lambda_A : X \longrightarrow [0, 1]$.

## 3. $\alpha$-dot cubic new-ideal of PU-algebra

In what follows, let $(X, *, 0)$ or simply $X$ denote a PU-algebra unless otherwise specified. In this section, we shall introduce a new notion called $\alpha$-dot cubic new-ideal of a PU-algebra and study several properties of it.

DEFINITION 3.1. Let $X$ be a PU-algebra. If $\widetilde{\mu}_A : X \longrightarrow D[0, 1]$ and $\lambda_A : X \longrightarrow [0, 1]$, then a cubic set $A = \langle\widetilde{\mu}_A, \lambda_A\rangle$ is called a cubic sub-algebra of $X$ if the following statements hold for all $x, y \in X$

(SC1) $\widetilde{\mu}_A(x * y) \geqslant r\,min\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\}$,

(SC2) $\lambda_A(x * y) \leqslant max\{\lambda_A(x), \lambda_A(y)\}$.

DEFINITION 3.2. Let $A=\langle\widetilde{\mu}_A, \lambda_A\rangle$ be a cubic set of a PU-algebra $X$. If $\alpha\in[0, 1]$, then the cubic set $A^\alpha=\langle\widetilde{\mu}_A^\alpha, \lambda_A^\alpha\rangle$ of $X$ is called the $\alpha$-dot cubic subset of $X$ (w.r.t. cubic set $A=\langle\widetilde{\mu}_A, \lambda_A\rangle$) if

(a) $\widetilde{\mu}_A^\alpha(x) = \widetilde{\mu}_A(x) \bullet \alpha$,

(b) $\lambda_A^\alpha(x) = \lambda_A(x) \bullet \alpha$,

for all $x \in X$.

REMARK 3.1. Clearly, $\langle\widetilde{\mu}_A^1, \lambda_A^1\rangle = \langle\widetilde{\mu}_A, \lambda_A\rangle$.

LEMMA 3.1. *$A = \langle\widetilde{\mu}_A, \lambda_A\rangle$ is a cubic sub-algebra of a PU-algebra $X$ such that $A^\alpha$ is $\alpha$-dot cubic subset, where $\alpha \in [0,1]$, then $\widetilde{\mu}^\alpha_A(x*y) \geqslant r\,min\{\widetilde{\mu}^\alpha_A(x), \widetilde{\mu}^\alpha_A(y)\}$ and $\lambda^\alpha_A(x*y) \leqslant max\{\lambda^\alpha_A(x), \lambda^\alpha_A(y)\}$ for all $x, y \in X$.*

PROOF. Let $X$ be a PU-algebra, $\alpha\in[0,1]$. Then by Definition 3.2, we have that

$$\begin{aligned}\widetilde{\mu}^{\tilde{\alpha}}_A(x*y) = \widetilde{\mu}_A(x*y)\bullet\alpha &\geqslant r\,min\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\}\bullet\alpha\\ &= r\,min\{\widetilde{\mu}_A(x)\bullet\alpha, \widetilde{\mu}_A(y)\bullet\alpha\}\\ &= r\,min\{\widetilde{\mu}^\alpha_A(x), \widetilde{\mu}^\alpha_A(y)\},\end{aligned}$$

for all $x, y \in X$ and

$$\begin{aligned}\lambda^\alpha_A(x*y) = \lambda_A(x*y)\bullet\alpha &\leqslant max\{\lambda_A(x), \lambda_A(y)\}\bullet\alpha\\ &= max\{\lambda_A(x)\bullet\alpha, \lambda_A(y)\bullet\alpha\}\\ &= max\{\lambda^\alpha_A(x), \lambda^\alpha_A(y)\},\end{aligned}$$

for all $x, y \in X$. □

DEFINITION 3.3. Let $X$ be a PU-algebra. A cubic subset $A^\alpha=\langle\widetilde{\mu}^\alpha_A, \lambda^\alpha_A\rangle$ in $X$ is called $\alpha$-dot cubic sub-algebra of $X$ if

$(C_{\tilde{\alpha}})$ $\widetilde{\mu}^\alpha_A(x*y) \geqslant r\,min\{\widetilde{\mu}^\alpha_A(x), \widetilde{\mu}^\alpha_A(y)\}$,

$(C_\alpha)$ $\lambda^\alpha_A(x*y) \leqslant max\{\lambda^\alpha_A(x), \lambda^\alpha_A(y)\}$

for all $x, y \in X$.

It is clear that $\alpha$-dot cubic sub-algebra of a PU-algebra $X$ is a generalization of a cubic sub-algebra of $X$ and a cubic sub-algebra of $X$ is a special case when $\alpha = 1$.

DEFINITION 3.4. Let $(X, *, 0)$ be a PU-algebra, a cubic subset $A=\langle\widetilde{\mu}_A, \lambda_A\rangle$ in $X$ is called a cubic new-ideal of $X$ if it satisfies the following conditions:

$(A_1)$ $\widetilde{\mu}_A(0) \geqslant \widetilde{\mu}_A(x)$ and $\lambda_A(0) \leqslant \lambda_A(x)$,

$(A_2)$ $\widetilde{\mu}_A((x*(y*z))*z) \geqslant r\,min\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\}$ and

$\lambda_A((x*(y*z))*z) \leqslant max\{\lambda_A(x), \lambda_A(y)\}$,

for all $x, y, z \in X$.

LEMMA 3.2. *If $A = \langle\widetilde{\mu}_A, \lambda_A\rangle$ is a cubic new-ideal of a PU-algebra $X$ such that $A^\alpha$ is $\alpha$-dot cubic subset, where $\alpha \in [0,1]$, then*

(1) $\widetilde{\mu}^\alpha_A(0) \geqslant \widetilde{\mu}^\alpha_A(x)$ *and* $\lambda^\alpha_A(0) \leqslant \lambda^\alpha_A(x)$,

(2) $\widetilde{\mu}^\alpha_A((x*(y*z))*z) \geqslant r\,min\{\widetilde{\mu}^\alpha_A(x), \widetilde{\mu}^\alpha_A(y)\}$,

(3) $\lambda^\alpha_A((x*(y*z))*z) \leqslant max\{\lambda^\alpha_A(x), \lambda^\alpha_A(y)\}$,

*for all $x, y, z \in X$.*

PROOF. Let $X$ be a PU-algebra, $\alpha \in [0,1]$. Then we have that:

(1) Let $x\in X$, then

$$\widetilde{\mu}^\alpha_A(0) = \widetilde{\mu}_A(0)\bullet\alpha \geqslant \widetilde{\mu}_A(x)\bullet\alpha = \widetilde{\mu}^\alpha_A(x)$$

and

$$\lambda^\alpha_A(0) = \lambda_A(0)\bullet\alpha \leqslant \lambda_A(x)\bullet\alpha = \lambda^\alpha_A(x).$$

(2) $\widetilde{\mu}_A^\alpha((x*(y*z))*z) = \widetilde{\mu}_A((x*(y*z))*z)\bullet\alpha$
$\geqslant r\,min\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(y)\} \bullet \alpha$
$= r\,min\{\widetilde{\mu}_A(x) \bullet \alpha, \widetilde{\mu}_A(y) \bullet \alpha\}$
$= r\,min\{\widetilde{\mu}_A^\alpha(x), \widetilde{\mu}_A^\alpha(y)\}$.

(3) $\lambda_A^\alpha((x*(y*z))*z) = \lambda_A((x*(y*z))*z) \bullet \alpha$
$\leqslant max\{\lambda_A(x), \lambda_A(y)\} \bullet \alpha$
$= max\{\lambda_A(x) \bullet \alpha, \lambda_A(y) \bullet \alpha\}$
$= max\{\lambda_A^\alpha(x), \lambda_A^\alpha(y)\}$,

for all $x, y, z \in X$ □

DEFINITION 3.5. Let $(X, *, 0)$ be a PU-algebra. An $\alpha$-dot cubic subset $A^\alpha = \langle\widetilde{\mu}_A^\alpha, \lambda_A^\alpha\rangle$ is called $\alpha$-dot cubic new-ideal of $X$ if it satisfies the following conditions:

$(A_0^\alpha)$ $\widetilde{\mu}_A^\alpha(0) \geqslant \widetilde{\mu}_A^\alpha(x)$ and $\lambda_A^\alpha(0) \leqslant \lambda_A^\alpha(x)$,

$(A_1^\alpha)$ $\widetilde{\mu}_A^\alpha((x*(y*z))*z) \geqslant r\,min\{\widetilde{\mu}_A^\alpha(x), \widetilde{\mu}_A^\alpha(y)\}$,

$(A_2^\alpha)$ $\lambda_A^\alpha((x*(y*z))*z) \leqslant max\{\lambda_A^\alpha(x), \lambda_A^\alpha(y)\}$,

for all $x, y, z \in X$.

It is clear that $\alpha$-dot cubic new-ideal of a PU-algebra $X$ is a generalization of a cubic new-ideal of $X$ and a cubic new-ideal of $X$ is special case, when $\alpha = 1$.

EXAMPLE 3.1. Let $X = \{0, 1, 2, 3\}$ in which $*$ is defined by Table 3.1.

| * | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 |
| 1 | 1 | 0 | 3 | 2 |
| 2 | 2 | 3 | 0 | 1 |
| 3 | 3 | 2 | 1 | 0 |

Table 3.1.

Then $(X, *, 0)$ is a PU-algebra. Define $\alpha$-dot cubic subset $A^\alpha = \langle\widetilde{\mu}_A^\alpha, \lambda_A^\alpha\rangle$ in $X$ by

$$\widetilde{\mu}_A^\alpha(x) = \begin{cases} \alpha \bullet [0.3, 0.9] & \text{if } x \in \{0, 1\} \\ \alpha \bullet [0.1, 0.6] & otherwise \end{cases}$$

and

$$\lambda_A^\alpha(x) = \begin{cases} \alpha \bullet 0.3 & \text{if } x \in \{0, 1\} \\ \alpha \bullet 0.7 & otherwise \end{cases}$$

where $\alpha \in [0, 1]$. Routine calculation gives that $A^\alpha = \langle\widetilde{\mu}_A^\alpha, \lambda_A^\alpha\rangle$ is an $\alpha$-dot cubic new-ideal of $X$.

LEMMA 3.3. *Let $A^\alpha = \langle\widetilde{\mu}_A^\alpha, \lambda_A^\alpha\rangle$ be $\alpha$-dot cubic new-ideal of a PU-algebra $X$. If the inequality $x * y \leqslant z$ holds in $X$, then $\widetilde{\mu}_A^\alpha(y) \geqslant r\,min\{\widetilde{\mu}_A^\alpha(x), \widetilde{\mu}_A^\alpha(z)\}$ and $\lambda_A^\alpha(y) \leqslant max\{\lambda_A^\alpha(x), \lambda_A^\alpha(z)\}$.*

Proof. Assume that the inequality $x * y \leqslant z$ holds in $X$. Then $z * (x * y) = 0$ and by $(A_1^\alpha)$, we have $\widetilde{\mu}_A^\alpha((z * (x * y)) * y) \geqslant r\,min\{\widetilde{\mu}_A^\alpha(x), \widetilde{\mu}_A^\alpha(z)\}$. Since $\widetilde{\mu}_A^\alpha(y) = \widetilde{\mu}_A^\alpha(0*y)$, then we have $\widetilde{\mu}_A^\alpha(y) \geqslant r\,min\{\widetilde{\mu}_A^\alpha(x), \widetilde{\mu}_A^\alpha(z)\}$. By $(A_2^\alpha)$, we obtain $\lambda_A^\alpha((z * (x * y)) * y) \leqslant max\{\lambda_A^\alpha(x), \lambda_A^\alpha(z)\}$. Since $\lambda_A^\alpha(y) = \lambda_A^\alpha(0 * y)$, therefore we have that $\lambda_A^\alpha(y) \leqslant max\{\lambda_A^\alpha(x), \lambda_A^\alpha(z)\}$. □

Corollary 3.1. *Let $A = \langle \widetilde{\mu}_A, \lambda_A \rangle$ be a cubic new-ideal of a PU-algebra $X$. If the inequality $x * y \leqslant z$ holds in $X$, then $\widetilde{\mu}_A(y) \geqslant r\,min\{\widetilde{\mu}_A(x), \widetilde{\mu}_A(z)\}$ and $\lambda_A(y) \leqslant max\{\lambda_A(x), \lambda_A(z)\}$.*

Lemma 3.4. *If $A^\alpha = \langle \widetilde{\mu}_A^\alpha, \lambda_A^\alpha \rangle$ is an $\alpha$-dot cubic subset of a PU-algebra $X$ and $x \leqslant y$ then $\widetilde{\mu}_A^\alpha(x) = \widetilde{\mu}_A^\alpha(y)$ and $\lambda_A^\alpha(x) = \lambda_A^\alpha(y)$.*

Proof. If $x \leqslant y$ then $y * x = 0$. Hence by the definition of PU-algebra and its properties we have that

$$\widetilde{\mu}_A^\alpha(x) = \widetilde{\mu}_A(x) \bullet \alpha = \widetilde{\mu}_A(0 * x) \bullet \alpha = \widetilde{\mu}_A((y * x) * x) \bullet \alpha = \widetilde{\mu}_A(y) \bullet \alpha = \widetilde{\mu}_A^\alpha(y)$$

and

$$\lambda_A^\alpha(x) = \lambda_A(x) \bullet \alpha = \lambda_A(0 * x) \bullet \alpha = \lambda_A((y * x) * x) \bullet \alpha = \lambda_A(y) \bullet \alpha = \lambda_A^\alpha(y). \quad \square$$

Corollary 3.2. *If $A = \langle \widetilde{\mu}_A, \lambda_A \rangle$ is a cubic subset of a PU-algebra $X$ and $x \leqslant y$ then $\widetilde{\mu}_A(x) = \widetilde{\mu}_A(y)$ and $\lambda_A(x) = \lambda_A(y)$.*

Definition 3.6. Let $A^\alpha = \langle \widetilde{\mu}_A^\alpha, \lambda_A^\alpha \rangle$ and $B^\alpha = \langle \widetilde{\mu}_B^\alpha, \lambda_B^\alpha \rangle$ be two $\alpha$-dot cubic sets in a PU-algebra $X$. Then we define

$$\begin{aligned} A^\alpha \cap B^\alpha &= \{\langle x, r\,min\{\widetilde{\mu}_A^\alpha(x), \widetilde{\mu}_B^\alpha(x)\}, max\{\lambda_A^\alpha(x), \lambda_B^\alpha(x)\}\rangle : x \in X\} \\ &= \{\langle x, \widetilde{\mu}_A^\alpha(x) \cap \widetilde{\mu}_B^\alpha(x), \lambda_A^\alpha(x) \cup \lambda_B^\alpha(x)\rangle : x \in X\}. \end{aligned}$$

Proposition 3.1. *If $\{A_i^\alpha\}_{i\in\Delta}$ is a family of $\alpha$-dot cubic new-ideals of a PU-algebra $X$, then $\cap_{i\in\Delta} A_i^\alpha$ is $\alpha$-dot cubic new-ideal of $X$.*

Proof. Let $\{A_i^\alpha\}_{i\in\Delta}$ be a family of $\alpha$-dot cubic new-ideals of a PU-algebra $X$. Then for any $x, y, z \in X$, we have

$$\begin{aligned} (\cap_{i\in\Delta}\widetilde{\mu}_{A_i}^\alpha)(0) &= r\,inf(\widetilde{\mu}_{A_i}^\alpha(0))_{i\in\Delta} \\ &\geqslant r\,inf(\widetilde{\mu}_{A_i}^\alpha(x))_{i\in\Delta} = (\cap_{i\in\Delta}\widetilde{\mu}_{A_i}^\alpha)(x) \end{aligned}$$

and

$$\begin{aligned} (\cup_{i\in\Delta}\lambda_{A_i}^\alpha)(0) &= sup(\lambda_{A_i}^\alpha(0))_{i\in\Delta} \\ &\leqslant sup(\lambda_{A_i}^\alpha(x))_{i\in\Delta} = (\cup_{i\in\Delta}\lambda_{A_i}^\alpha)(x). \end{aligned}$$

Also,

$$\begin{aligned} (\cap_{i\in\Delta}\widetilde{\mu}_{A_i}^\alpha)((x * (y * z)) * z) &= r\,inf(\widetilde{\mu}_{A_i}^\alpha((x * (y * z)) * z)_{i\in\Delta} \\ &\geqslant r\,inf(r\,min\{\widetilde{\mu}_{A_i}^\alpha(x), \widetilde{\mu}_{A_i}^\alpha(y)\})_{i\in\Delta} \\ &= r\,min(r\,inf(\widetilde{\mu}_{A_i}^\alpha(x))_{i\in\Delta}, r\,inf(\widetilde{\mu}_{A_i}^\alpha(y))_{i\in\Delta}\} \\ &= r\,min\{(\cap_{i\in\Delta}\widetilde{\mu}_{A_i}^\alpha)(x), (\cap_{i\in\Delta}\widetilde{\mu}_{A_i}^\alpha)(y)\} \end{aligned}$$

and

$$\begin{aligned} (\cup_{i\in\Delta}\lambda_{A_i}^\alpha)((x * (y * z)) * z) &= sup(\lambda_{A_i}^\alpha((x * (y * z)) * z)_{i\in\Delta} \\ &\leqslant sup(max\{\lambda_{A_i}^\alpha(x), \lambda_{A_i}^\alpha(y)\})_{i\in\Delta} \end{aligned}$$

$$= max(sup(\lambda^{\alpha}_{A_i}(x))_{i\in\Delta}, sup(\lambda^{\alpha}_{A_i}(y))_{i\in\Delta}\}$$
$$= max\{(\cup_{i\in\Delta}\lambda^{\alpha}_{A_i})(x), (\cup_{i\in\Delta}\lambda^{\alpha}_{A_i})(y)\}.$$

This completes the proof. □

THEOREM 3.1. *If* $A^{\alpha}=\langle\widetilde{\mu}^{\alpha}_{A},\lambda^{\alpha}_{A}\rangle$ *is an* $\alpha$*-dot cubic subset of a PU-algebra* $X$*, then it is* $\alpha$*-dot cubic new-ideal of* $X$ *if and only if it satisfies:*

$(\forall\widetilde{\varepsilon}\in D[0,1])(U(\widetilde{\mu}^{\alpha}_{A},\widetilde{\varepsilon})\neq\emptyset\Longrightarrow U(\widetilde{\mu}^{\alpha}_{A},\widetilde{\varepsilon})$ *is a new-ideal of* $X)$,

*where* $U(\widetilde{\mu}^{\alpha}_{A},\widetilde{\varepsilon})=\{x\in X:\widetilde{\mu}^{\alpha}_{A}(x)\geqslant\widetilde{\varepsilon}\}$

*and*

$(\forall\varepsilon\in[0,1])(L(\lambda^{\alpha}_{A},\varepsilon)\neq\emptyset\Longrightarrow L(\lambda^{\alpha}_{A},\varepsilon)$ *is a new-ideal of* $X)$,

*where* $L(\lambda^{\alpha}_{A},\varepsilon)=\{x\in X:\lambda^{\alpha}_{A}(x)\leqslant\varepsilon\}$.

PROOF. Assume that $A^{\alpha}=\langle\widetilde{\mu}^{\alpha}_{A},\lambda^{\alpha}_{A}\rangle$ is an $\alpha$-dot cubic new-ideal of $X$. Let $\widetilde{\varepsilon}\in D[0,1])$ with $U(\widetilde{\mu}^{\alpha}_{A},\widetilde{\varepsilon})\neq\emptyset$. If $x\in U(\widetilde{\mu}^{\alpha}_{A},\widetilde{\varepsilon})$, then $\widetilde{\mu}^{\alpha}_{A}(x)\geqslant\widetilde{\varepsilon}$. Since $\widetilde{\mu}^{\alpha}_{A}(0)\geqslant\widetilde{\mu}^{\alpha}_{A}(x)$, for all $x\in X$, then $\widetilde{\mu}^{\alpha}_{A}(0)\geqslant\widetilde{\varepsilon}$. Thus $0\in U(\widetilde{\mu}^{\alpha}_{A},\widetilde{\varepsilon})$. Let $x\in X$ and $a,b\in U(\widetilde{\mu}^{\alpha}_{A},\widetilde{\varepsilon})$. Then $\widetilde{\mu}^{\alpha}_{A}(a)\geqslant\widetilde{\varepsilon}$ and $\widetilde{\mu}^{\alpha}_{A}(b)\geqslant\widetilde{\varepsilon}$. It follows by the definition of $\alpha$-dot cubic new-ideal that $\widetilde{\mu}^{\alpha}_{A}((a*(b*x))*x)\geqslant r\,min\{\widetilde{\mu}^{\alpha}_{A}(a),\widetilde{\mu}^{\alpha}_{A}(b)\}\geqslant\widetilde{\varepsilon}$, so that $(a*(b*x))*x)\in U(\widetilde{\mu}^{\alpha}_{A},\widetilde{\varepsilon})$. Hence $U(\widetilde{\mu}^{\alpha}_{A},\widetilde{\varepsilon})$ is a new-ideal of $X$. Let $\varepsilon\in[0,1])$ with $L(\lambda^{\alpha}_{A},\varepsilon)\neq\emptyset$. If $x\in L(\lambda^{\alpha}_{A},\varepsilon)$, then $\lambda^{\alpha}_{A}(x)\leqslant\varepsilon$. Since $\lambda^{\alpha}_{A}(0)\leqslant\lambda^{\alpha}_{A}(x)$, for all $x\in X$ then $\lambda^{\alpha}_{A}(0)\leqslant\varepsilon$. Thus $0\in L(\lambda^{\alpha}_{A},\varepsilon)$. Let $x\in X$ and $a,b\in L(\lambda^{\alpha}_{A},\varepsilon)$. Then $\lambda^{\alpha}_{A}(a)\leqslant\varepsilon$ and $\lambda^{\alpha}_{A}(b)\leqslant\varepsilon$. It follows by the definition of $\alpha$-dot cubic new-ideal that $\lambda^{\alpha}_{A}((a*(b*x))*x)\leqslant max\{\lambda^{\alpha}_{A}(a),\lambda^{\alpha}_{A}(b)\}\leqslant\varepsilon$, so that $(a*(b*x))*x)\in L(\lambda_{A},\varepsilon)$. Hence $L(\lambda^{\alpha}_{A},\varepsilon)$ is a new-ideal of $X$.

Conversely, suppose that

$$(\forall\widetilde{\varepsilon}\in D[0,1])(U(\widetilde{\mu}^{\alpha}_{A},\widetilde{\varepsilon})\neq\emptyset\Longrightarrow U(\widetilde{\mu}^{\alpha}_{A},\widetilde{\varepsilon})\text{ is a new-ideal of }X),$$

where $U(\widetilde{\mu}^{\alpha}_{A},\widetilde{\varepsilon})=\{x\in X:\widetilde{\mu}^{\alpha}_{A}(x)\geqslant\widetilde{\varepsilon}\}$ and

$$(\forall\varepsilon\in[0,1])(L(\lambda^{\alpha}_{A},\varepsilon)\neq\emptyset\Longrightarrow L(\lambda^{\alpha}_{A},\varepsilon)\text{ is a new-ideal of }X),$$

where $L(\lambda^{\alpha}_{A},\varepsilon)=\{x\in X:\lambda^{\alpha}_{A}(x)\leqslant\varepsilon\}$. If $\widetilde{\mu}^{\alpha}_{A}(0)\leqslant\widetilde{\mu}^{\alpha}_{A}(x)$ for some $x\in X$ then $\widetilde{\mu}^{\alpha}_{A}(0)<\widetilde{\varepsilon}_0<\widetilde{\mu}^{\alpha}_{A}(x)$ by taking $\widetilde{\varepsilon}_0=\frac{\widetilde{\mu}^{\alpha}_{A}(0)+\widetilde{\mu}^{\alpha}_{A}(x)}{2}$. Hence $0\notin U(\widetilde{\mu}^{\alpha}_{A},\widetilde{\varepsilon}_0)$ which is a contradiction. If $\lambda^{\alpha}_{A}(0)\geqslant\lambda^{\alpha}_{A}(x)$ for some $x\in X$ then $\lambda^{\alpha}_{A}(0)>\varepsilon_0>\lambda^{\alpha}_{A}(x)$ by taking $\varepsilon_0=\frac{\lambda^{\alpha}_{A}(0)+\lambda^{\alpha}_{A}(x)}{2}$. Hence $0\notin L(\lambda^{\alpha}_{A},\varepsilon_0)$ which is a contradiction. Let $a,b,c\in X$ and $\widetilde{\mu}^{\alpha}_{A}((a*(b*c))*c)<r\,min\{\widetilde{\mu}^{\alpha}_{A}(a),\widetilde{\mu}^{\alpha}_{A}(b)\}$. Taking $\widetilde{\varepsilon}_1=\frac{\widetilde{\mu}^{\alpha}_{A}(a*(b*c)*c)+r min\{\widetilde{\mu}^{\alpha}_{A}(a),\widetilde{\mu}^{\alpha}_{A}(b)\}}{2}$, we have $\widetilde{\varepsilon}_1\in D[0,1]$ and $\widetilde{\mu}^{\alpha}_{A}((a*(b*c))*c)<\widetilde{\varepsilon}_1<r\,min\{\widetilde{\mu}^{\alpha}_{A}(a),\widetilde{\mu}^{\alpha}_{A}(b)\}$. It follows that $a,b\in U(\widetilde{\mu}^{\alpha}_{A},\widetilde{\varepsilon}_1)$ and $(a*(b*c)*c)\notin U(\widetilde{\mu}^{\alpha}_{A},\widetilde{\varepsilon}_1)$ which is a contradiction. Let $a,b,c\in X$ and $\lambda^{\alpha}_{A}((a*(b*c))*c)>max\{\lambda^{\alpha}_{A}(a),\lambda^{\alpha}_{A}(b)\}$. Taking $\varepsilon_1=\frac{\lambda^{\alpha}_{A}(a*(b*c)*c)+max\{\lambda^{\alpha}_{A}(a),\lambda^{\alpha}_{A}(b)\}}{2}$ we have $\varepsilon_1\in[0,1]$ and $\lambda^{\alpha}_{A}((a*(b*c))*c)>\varepsilon_1>max\{\lambda^{\alpha}_{A}(a),\lambda^{\alpha}_{A}(b)\}$. It follows that $a,b\in L(\lambda^{\alpha}_{A},\varepsilon_1)$ and $(a*(b*c)*c)\notin L(\lambda^{\alpha}_{A},\varepsilon_1)$ which is a contradiction. Therefore $A^{\alpha}=\langle\widetilde{\mu}^{\alpha}_{A},\lambda^{\alpha}_{A}\rangle$ is an $\alpha$-dot cubic new-ideal of $X$. □

COROLLARY 3.3. *Let* $A=\langle\widetilde{\mu}_{A},\lambda_{A}\rangle$ *be a cubic subset of a PU-algebra* $X$*. Then it is a cubic new-ideal of* $X$ *if and only if it satisfies:*

$(\forall\widetilde{\varepsilon} \in D[0,1])(U(\widetilde{\mu}_A, \widetilde{\varepsilon}) \neq \emptyset \Longrightarrow U(\widetilde{\mu}_A, \widetilde{\varepsilon})$ *is a new-ideal of* $X$),
*where* $U(\widetilde{\mu}_A, \widetilde{\varepsilon}) = \{x \in X : \widetilde{\mu}_A(x) \geqslant \widetilde{\varepsilon}\}$ *and*
$(\forall\varepsilon \in [0,1])(L(\lambda_A, \varepsilon) \neq \emptyset \Longrightarrow L(\lambda_A, \varepsilon)$ *is a new-ideal of* $X$),
*where* $L(\lambda_A, \varepsilon) = \{x \in X : \lambda_A(x) \leqslant \varepsilon\}$.

Definition 3.7. Let $f$ be a mapping from $X$ to $Y$. If $\langle\widetilde{\mu}^\alpha, \lambda^\alpha\rangle$ is an $\alpha$-dot cubic subset of $X$, then the $\alpha$-dot cubic subset $\langle\widetilde{\beta}^\alpha, \gamma^\alpha\rangle$ of $Y$ defined by

$$f(\widetilde{\mu}^\alpha)(y) = \widetilde{\beta}^\alpha(y) = \begin{cases} rsup_{x\in f^{-1}(y)}\widetilde{\mu}^\alpha(x) & \text{if } f^{-1}(y) \neq \emptyset \\ \widetilde{0} & otherwise \end{cases}$$

and

$$f(\lambda^\alpha)(y) = \gamma^\alpha(y) = \begin{cases} inf_{x\in f^{-1}(y)}\lambda^\alpha(x) & \text{if } f^{-1}(y) \neq \emptyset \\ 1 & otherwise \end{cases}$$

is said to be the image of $\langle\widetilde{\mu}^\alpha, \lambda^\alpha\rangle$ under $f$.

Similarly if $\langle\widetilde{\beta}^\alpha, \gamma^\alpha\rangle$ is an $\alpha$-dot cubic subset of $Y$, then the $\alpha$-dot cubic subset $\langle\widetilde{\mu}^\alpha, \lambda^\alpha\rangle$=$\langle\widetilde{\beta}^\alpha \circ f, \gamma^\alpha \circ f\rangle$ of $X$, where $(\widetilde{\beta}^\alpha \circ f)(x) = \widetilde{\beta}^\alpha(f(x))$ and $(\gamma^\alpha \circ f)(x) = \gamma^\alpha(f(x))$ for all $x \in X$ is called the pre-image of $\langle\widetilde{\beta}^\alpha, \gamma^\alpha\rangle$ under $f$.

Theorem 3.2. *Let* $(X, *, 0)$, $(\grave{X}, \grave{*}, \grave{0})$ *be PU-algebras and* $f : X \longrightarrow \grave{X}$ *be a homomorphism. If* $\langle\widetilde{\beta}^\alpha, \gamma^\alpha\rangle$ *is an* $\alpha$*-dot cubic new-ideal of* $\grave{X}$ *and* $\langle\widetilde{\mu}^\alpha, \lambda^\alpha\rangle$ *is the pre-image of* $\langle\widetilde{\beta}^\alpha, \gamma^\alpha\rangle$ *under* $f$, *then* $\langle\widetilde{\mu}^\alpha, \lambda^\alpha\rangle$ *is an* $\alpha$*-dot cubic new-ideal of* $X$.

Proof. Since $\langle\widetilde{\mu}^\alpha, \lambda^\alpha\rangle$ is the pre-image of $\langle\widetilde{\beta}^\alpha, \gamma^\alpha\rangle$ under $f$, then for all $x, y, z \in X$ we have
$\widetilde{\mu}^\alpha(0) = \widetilde{\beta}^\alpha(f(0)) \geqslant \widetilde{\beta}^\alpha(f(x)) = \widetilde{\mu}^\alpha(x)$ and $\lambda^\alpha(0)$=$\gamma^\alpha(f(0)) \leqslant \gamma^\alpha(f(x)) = \lambda^\alpha(x)$. Now

$$\begin{aligned}
\widetilde{\mu}^\alpha((x*(y*z))*z) &= \widetilde{\beta}^\alpha(f((x*(y*z))*z)) \\
&= \widetilde{\beta}^\alpha(f(x*(y*z))\grave{*}f(z)) \\
&= \widetilde{\beta}^\alpha((f(x)\grave{*}f(y*z))\grave{*}f(z)) \\
&= \widetilde{\beta}^\alpha((f(x)\grave{*}(f(y)\grave{*}f(z)))\grave{*}f(z)) \\
&\geqslant r\,min\{\widetilde{\beta}^\alpha(f(x)), \widetilde{\beta}^\alpha(f(y))\} \\
&= r\,min\{\widetilde{\mu}^\alpha(x), \widetilde{\mu}^\alpha(y)\}
\end{aligned}$$

and

$$\begin{aligned}
\lambda^\alpha((x*(y*z))*z) &= \gamma^\alpha(f((x*(y*z))*z)) \\
&=\gamma^\alpha(f(x*(y*z))\grave{*}f(z)) \\
&= \gamma^\alpha((f(x)\grave{*}f(y*z))\grave{*}f(z)) \\
&= \gamma^\alpha((f(x)\grave{*}(f(y)\grave{*}f(z)))\grave{*}f(z)) \\
&\leqslant max\{\gamma^\alpha(f(x)), \gamma^\alpha(f(y))\} \\
&= max\{\lambda^\alpha(x), \lambda^\alpha(y)\}.
\end{aligned}$$

Therefore $\langle\widetilde{\mu}^\alpha, \lambda^\alpha\rangle$ is an $\alpha$-dot cubic new-ideal of $X$. □

THEOREM 3.3. *Let* $(X, *, 0)$, $(Y, \dot{*}, \dot{0})$ *be PU-algebras,* $f : X \longrightarrow Y$ *be a homomorphism,* $\langle \widetilde{\mu}^\alpha, \lambda^\alpha \rangle$ *be an* $\alpha$*-dot cubic subset of* $X$, $\langle \widetilde{\beta}^\alpha, \gamma^\alpha \rangle$ *be the image of* $\langle \widetilde{\mu}^\alpha, \lambda^\alpha \rangle$ *under* $f$, $\widetilde{\mu}^\alpha(x) = \widetilde{\beta}^\alpha(f(x))$ *and* $\lambda^\alpha(x) = \gamma^\alpha(f(x))$ *for all* $x \in X$. *If* $\langle \widetilde{\mu}^\alpha, \lambda^\alpha \rangle$ *is an* $\alpha$*-dot cubic new-ideal of* $X$, *then* $\langle \widetilde{\beta}^\alpha, \gamma^\alpha \rangle$ *is an* $\alpha$*-dot cubic new-ideal of* $Y$.

PROOF. Since $0 \in f^{-1}(\dot{0})$, then $f^{-1}(\dot{0}) \neq \emptyset$. It follows that

$$\widetilde{\beta}^\alpha(\dot{0}) = r\,sup_{t \in f^{-1}(\dot{0})} \widetilde{\mu}^\alpha(t) = \widetilde{\mu}^\alpha(0) \geqslant \widetilde{\mu}^\alpha(x)$$

for all $x \in X$. Thus $\widetilde{\beta}^\alpha(\dot{0}) = r\,sup_{t \in f^{-1}(\dot{x})} \widetilde{\mu}^\alpha(t)$ for all $\dot{x} \in Y$. Hence $\widetilde{\beta}^\alpha(\dot{0}) \geqslant \widetilde{\beta}^\alpha(\dot{x})$ for all $\dot{x} \in Y$. Since $0 \in f^{-1}(\dot{0})$, then $f^{-1}(\dot{0}) \neq \emptyset$. It follows that $\gamma^\alpha(\dot{0}) = inf_{t \in f^{-1}(\dot{0})} \lambda^\alpha(t) = \lambda^\alpha(0) \leqslant \lambda^\alpha(x)$ for all $x \in X$. Thus $\gamma^\alpha(\dot{0}) = inf_{t \in f^{-1}(\dot{x})} \lambda^\alpha(t)$ for all $\dot{x} \in Y$. Hence $\gamma^\alpha(\dot{0}) \leqslant \gamma^\alpha(\dot{x})$ for all $\dot{x} \in Y$. For any $\dot{x}, \dot{y}, \dot{z} \in Y$. If $f^{-1}(\dot{x}) = \emptyset$ or $f^{-1}(\dot{y}) = \emptyset$, then $\widetilde{\beta}^\alpha(\dot{x}) = \widetilde{0}$ or $\widetilde{\beta}^\alpha(\dot{y}) = \widetilde{0}$, it follows that r min$\{\widetilde{\beta}^\alpha(\dot{x}), \widetilde{\beta}^\alpha(\dot{y})\} = \widetilde{0}$ and hence $\widetilde{\beta}^\alpha((\dot{x}\dot{*}(\dot{y}\dot{*}\dot{z}))\dot{*}\dot{z}) \geqslant r\,min\{\widetilde{\beta}^\alpha(\dot{x}), \widetilde{\beta}^\alpha(\dot{y})\}$. If $f^{-1}(\dot{x}) \neq \emptyset$ and $f^{-1}(\dot{y}) \neq \emptyset$, let $x_0 \in f^{-1}(\dot{x})$, $y_0 \in f^{-1}(\dot{y})$ with $\widetilde{\mu}^\alpha(x_0) = r\,sup_{t \in f^{-1}(\dot{x})} \widetilde{\mu}^\alpha(t)$ and $\widetilde{\mu}^\alpha(y_0) = r\,sup_{t \in f^{-1}(\dot{y})} \lambda^\alpha(t)$. It follows by given and properties of PU-algebra that

$$\begin{aligned}
\widetilde{\beta}^\alpha((\dot{x}\dot{*}(\dot{y}\dot{*}\dot{z}))\dot{*}\dot{z}) &= \widetilde{\beta}^\alpha((\dot{z}\dot{*}(\dot{y}\dot{*}\dot{z}))\dot{*}\dot{x}) \\
&= \widetilde{\beta}^\alpha((\dot{y}\dot{*}(\dot{z}\dot{*}\dot{z}))\dot{*}\dot{x}) \\
&= \widetilde{\beta}^\alpha((\dot{y}\dot{*}\dot{0})\dot{*}\dot{x}) \\
&= \widetilde{\beta}^\alpha((f(y_0)\dot{*}(f(0))\dot{*}(f(x_0)) \\
&= \widetilde{\beta}^\alpha((f(y_0 * 0) * x_0)) \\
&= \widetilde{\mu}^\alpha((y_0 * 0) * x_0) \\
&= \widetilde{\mu}^\alpha((y_0 * (z_0 * z_0)) * x_0) \\
&= \widetilde{\mu}^\alpha((z_0 * (y_0 * z_0)) * x_0) \\
&= \widetilde{\mu}^\alpha((x_0 * (y_0 * z_0)) * z_0) \\
&\geqslant r\,min\{\widetilde{\mu}^\alpha(x_0), \widetilde{\mu}^\alpha(y_0)\} \\
&= r\,min\{r\,sup_{t \in f^{-1}(\dot{x})} \widetilde{\mu}^\alpha(t), r\,sup_{t \in f^{-1}(\dot{y})} \widetilde{\mu}^\alpha(t)\} \\
&= r\,min\{\widetilde{\beta}^\alpha(\dot{x}), \widetilde{\beta}^\alpha(\dot{y})\}.
\end{aligned}$$

for any $\dot{x}, \dot{y}, \dot{z} \in Y$.

If $f^{-1}(\dot{x}) = \emptyset$ or $f^{-1}(\dot{y}) = \emptyset$, then $\gamma^\alpha(\dot{x}) = 1$ or $\gamma^\alpha(\dot{y}) = 1$, it follows that $max\{\gamma^\alpha(\dot{x}), \gamma^\alpha(\dot{y})\} = 1$ and hence $\gamma^\alpha((\dot{x}\dot{*}(\dot{y}\dot{*}\dot{z}))\dot{*}\dot{z}) \leqslant max\{\gamma^\alpha(\dot{x}), \gamma^\alpha(\dot{y})\}$.

If $f^{-1}(\dot{x}) \neq \emptyset$ and $f^{-1}(\dot{y}) \neq \emptyset$, let $x_0 \in f^{-1}(\dot{x})$, $y_0 \in f^{-1}(\dot{y})$ with $\lambda^\alpha(x_0) = inf_{t \in f^{-1}(\dot{x})} \lambda^\alpha(t)$ and $\lambda^\alpha(y_0) = inf_{t \in f^{-1}(\dot{y})} \lambda^\alpha(t)$. It follows by given and properties of PU-algebra that

$$\begin{aligned}
\gamma^\alpha((\dot{x}\dot{*}(\dot{y}\dot{*}\dot{z}))\dot{*}\dot{z}) &= \gamma^\alpha((\dot{z}\dot{*}(\dot{y}\dot{*}\dot{z}))\dot{*}\dot{x}) \\
&= \gamma^\alpha((\dot{y}\dot{*}(\dot{z}\dot{*}\dot{z}))\dot{*}\dot{x}) \\
&= \gamma^\alpha((\dot{y}\dot{*}\dot{0})\dot{*}\dot{x}) \\
&= \gamma^\alpha((f(y_0)\dot{*}(f(0))\dot{*}(f(x_0)) \\
&= \gamma^\alpha((f(y_0 * 0) * x_0))
\end{aligned}$$

$$= \lambda^{\alpha}((y_0 * 0) * x_0)$$
$$= \lambda^{\alpha}((y_0 * (z_0 * z_0)) * x_0)$$
$$= \lambda^{\alpha}((z_0 * (y_0 * z_0)) * x_0)$$
$$= \lambda^{\alpha}((x_0 * (y_0 * z_0)) * z_0)$$
$$\leqslant max\{\lambda^{\alpha}(x_0), \lambda^{\alpha}(y_0)\}$$
$$= max\{inf_{t\in f^{-1}(\grave{x})}\lambda^{\alpha}(t), inf_{t\in f^{-1}(\grave{y})}\lambda^{\alpha}(t)\}$$
$$= max\{\gamma^{\alpha}(\grave{x}), \gamma^{\alpha}(\grave{y})\}$$

Therefore $\langle\widetilde{\beta}, \gamma\rangle$ is an $\alpha$-dot cubic new-ideal of $Y$. □

COROLLARY 3.4. *Let $(X, *, 0)$, $(Y, \grave{*}, \grave{0})$ be PU-algebras, $f : X \longrightarrow Y$ be a homomorphism, $\langle\widetilde{\mu}, \lambda\rangle$ be a cubic subset of $X$, $\langle\widetilde{\beta}, \gamma\rangle$ be the image of $\langle\widetilde{\mu}, \lambda\rangle$ under $f$, $\widetilde{\mu}(x) = \widetilde{\beta}(f(x))$ and $\lambda(x) = \gamma(f(x))$ for all $x \in X$. If $\langle\widetilde{\mu}, \lambda\rangle$ is a cubic new-ideal of $X$, then $\langle\widetilde{\beta}, \gamma\rangle$ is a cubic new-ideal of $Y$.*

## 4. Cartesian product of $\alpha$-dot cubic new-ideal of PU-algebra

In this section, we introduce the concept of Cartesian product of an $\alpha$-dot cubic new-ideal of a PU-algebra.

DEFINITION 4.1. Let $S$ be a non-empty set. The cubic set $\langle\widetilde{\mu}^{\alpha}, \lambda^{\alpha}\rangle$ is called an $\alpha$-dot cubic-relation on $S$ if $\widetilde{\mu}^{\alpha}, \lambda^{\alpha}$ are two functions defined by $\widetilde{\mu}^{\alpha} : S \times S \longrightarrow D[0,1]$, $\lambda^{\alpha} : S \times S \longrightarrow [0,1]$.

DEFINITION 4.2. If $\langle\widetilde{\mu}^{\alpha}, \lambda^{\alpha}\rangle$ is an $\alpha$-dot cubic relation on a set $S$ and $\langle\widetilde{\beta}^{\alpha}, \gamma^{\alpha}\rangle$ is an $\alpha$-dot cubic subset of $S$, then $\langle\widetilde{\mu}^{\alpha}, \lambda^{\alpha}\rangle$ is an $\alpha$-dot cubic relation on $\langle\widetilde{\beta}^{\alpha}, \gamma^{\alpha}\rangle$ if $\widetilde{\mu}^{\alpha}(x, y) \leqslant r\,min\{\widetilde{\beta}^{\alpha}(x), \widetilde{\beta}^{\alpha}(y)\}$ and $\lambda^{\alpha}(x, y) \geqslant max\{\gamma^{\alpha}(x), \gamma^{\alpha}(y)\}$ for all $x, y \in S$.

DEFINITION 4.3. If $\langle\widetilde{\beta}^{\alpha}, \gamma^{\alpha}\rangle$ is an $\alpha$-dot cubic subset of a set $S$, the strongest $\alpha$-dot cubic relation on $S$ that is an $\alpha$-dot cubic relation on $\langle\widetilde{\beta}^{\alpha}, \gamma^{\alpha}\rangle$ is $\langle\widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}, \lambda^{\alpha}_{\gamma^{\alpha}}\rangle$ given by $\widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}(x, y) = r\,min\{\widetilde{\beta}^{\alpha}(x), \widetilde{\beta}^{\alpha}(y)\}$ and $\lambda^{\alpha}_{\gamma^{\alpha}} = max\{\gamma^{\alpha}(x), \gamma^{\alpha}(y)\}$ for all $x, y \in S$.

DEFINITION 4.4. We define the binary operation $*$ on the cartesian product $X \times X$ as follows:

$$(x_1, x_2) * (y_1, y_2) = (x_1 * y_1, x_2 * y_2)$$

for all $(x_1, x_2), (y_1, y_2) \in X \times X$.

LEMMA 4.1. *If $(X, *, 0)$ is a PU-algebra, then $(X\times X, *, (0, 0))$ is a PU-algebra, where $(x_1, x_2) * (y_1, y_2) = (x_1 * y_1, x_2 * y_2)$ for all $(x_1, x_2), (y_1, y_2) \in X \times X$.*

PROOF. Clear. □

THEOREM 4.1. *Let $\langle\widetilde{\beta}^{\alpha}, \gamma^{\alpha}\rangle$ be an $\alpha$-dot cubic subset of a PU-algebra $X$ and $\langle\widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}, \lambda^{\alpha}_{\gamma^{\alpha}}\rangle$ be the strongest $\alpha$-dot cubic relation on $X$, then $\langle\widetilde{\beta}^{\alpha}, \gamma^{\alpha}\rangle$ is an $\alpha$-dot cubic new-ideal of $X$ if and only if $\langle\widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}, \lambda^{\alpha}_{\gamma^{\alpha}}\rangle$ is an $\alpha$-dot cubic new-ideal of $X\times X$.*

Proof. ($\Rightarrow$) Assume that $\langle\widetilde{\beta}^\alpha,\gamma^\alpha\rangle$ is an $\alpha$-dot cubic new-ideal of $X$, we note that:

$\widetilde{\mu}^\alpha_{\widetilde{\beta}^\alpha}(0,0)=r\,min\{\widetilde{\beta}^\alpha(0),\widetilde{\beta}^\alpha(0)\}\geqslant r\,min\{\widetilde{\beta}^\alpha(x),\widetilde{\beta}^\alpha(y)\}=\widetilde{\mu}^\alpha_{\widetilde{\beta}^\alpha}(x,y)$ and

$\lambda^\alpha_{\gamma^\alpha}(0,0)=max\{\gamma^\alpha(0),\gamma^\alpha(0)\}\leqslant max\{\gamma^\alpha(x),\gamma^\alpha(y)\}=\lambda^\alpha_{\gamma^\alpha}(x,y)$

for all $x,y\in X$. Now, for any $(x_1,x_2),(y_1,y_2),(z_1,z_2)\in X\times X$ we have:

$$\begin{aligned}
&\widetilde{\mu}^\alpha_{\widetilde{\beta}^\alpha}(((x_1,x_2)*((y_1,y_2)*(z_1,z_2)))*(z_1,z_2))\\
&\qquad=\widetilde{\mu}^\alpha_{\widetilde{\beta}^\alpha}(((x_1,x_2)*(y_1*z_1,y_2*z_2))*(z_1,z_2))\\
&\qquad=\widetilde{\mu}^\alpha_{\widetilde{\beta}^\alpha}((x_1*(y_1*z_1),x_2*(y_2*z_2))*(z_1,z_2))\\
&\qquad=\widetilde{\mu}^\alpha_{\widetilde{\beta}^\alpha}((x_1*(y_1*z_1))*z_1,(x_2*(y_2*z_2))*z_2)\\
&\qquad=r\,min\{\widetilde{\beta}^\alpha((x_1*(y_1*z_1))*z_1),\widetilde{\beta}^\alpha((x_2*(y_2*z_2))*z_2)\}\\
&\qquad\geqslant r\,min\{r\,min\{\widetilde{\beta}^\alpha(x_1),\widetilde{\beta}^\alpha(y_1)\},r\,min\{\widetilde{\beta}^\alpha(x_2),\widetilde{\beta}^\alpha(y_2)\}\}\\
&\qquad=r\,min\{\text{r min}\{\widetilde{\beta}^\alpha(x_1),\widetilde{\beta}^\alpha(x_2)\},r\,min\{\widetilde{\beta}^\alpha(y_1),\widetilde{\beta}^\alpha(y_2)\}\}\\
&\qquad=r\,min\{\widetilde{\mu}^\alpha_{\widetilde{\beta}^\alpha}(x_1,x_2),\widetilde{\mu}^\alpha_{\widetilde{\beta}^\alpha}(y_1,y_2)\}
\end{aligned}$$

and

$$\begin{aligned}
&\lambda^\alpha_{\gamma^\alpha}(((x_1,x_2)*((y_1,y_2)*(z_1,z_2)))*(z_1,z_2))\\
&\qquad=\lambda^\alpha_{\gamma^\alpha}(((x_1,x_2)*(y_1*z_1,y_2*z_2))*(z_1,z_2))\\
&\qquad=\lambda^\alpha_{\gamma^\alpha}((x_1*(y_1*z_1),x_2*(y_2*z_2))*(z_1,z_2))\\
&\qquad=\lambda^\alpha_{\gamma^\alpha}((x_1*(y_1*z_1))*z_1,(x_2*(y_2*z_2))*z_2)\\
&\qquad=max\{\gamma^\alpha((x_1*(y_1*z_1))*z_1),\gamma^\alpha((x_2*(y_2*z_2))*z_2)\}\\
&\qquad\leqslant max\{\max\{\gamma^\alpha(x_1),\gamma^\alpha(y_1)\},max\{\gamma^\alpha(x_2),\gamma^\alpha(y_2)\}\}\\
&\qquad=max\{max\{\gamma^\alpha(x_1),\gamma^\alpha(x_2)\},max\{\gamma^\alpha(y_1),\gamma^\alpha(y_2)\}\}\\
&\qquad=max\{\lambda^\alpha_{\gamma^\alpha}(x_1,x_2),\lambda^\alpha_{\gamma^\alpha}(y_1,y_2)\}.
\end{aligned}$$

Hence $\langle\widetilde{\mu}^\alpha_{\widetilde{\beta}^\alpha},\lambda^\alpha_{\gamma^\alpha}\rangle$ is an $\alpha$-dot cubic new-ideal of $X\times X$.

($\Leftarrow$) For all $(x,x)\in X\times X$ we have

$$\widetilde{\mu}^\alpha_{\widetilde{\beta}^\alpha}(0,0)=r\,min\{\widetilde{\beta}^\alpha(0),\widetilde{\beta}^\alpha(0)\}\geqslant\widetilde{\mu}^\alpha_{\widetilde{\beta}^\alpha}(x,x)$$

and

$$\lambda^\alpha_{\gamma^\alpha}(0,0)=max\{\gamma^\alpha(0),\gamma^\alpha(0)\}\leqslant\lambda^\alpha_{\gamma^\alpha}(x,x).$$

Then

$$\widetilde{\beta}^\alpha(0)=r\,min\{\widetilde{\beta}^\alpha(0),\widetilde{\beta}^\alpha(0)\}\geqslant r\,min\{\widetilde{\beta}^\alpha(x),\widetilde{\beta}^\alpha(x)\}=\widetilde{\beta}^\alpha(x)$$

and

$$\gamma^\alpha(0)=max\{\gamma^\alpha(0),\gamma^\alpha(0)\}\leqslant max\{\gamma^\alpha(x),\gamma^\alpha(x)\}=\gamma^\alpha(x)$$

for all $x\in X$. Now, for all $x,y,z\in X$. We have

$$\begin{aligned}
\widetilde{\beta}^\alpha((x*(y*z))*z)&=rmin\{\widetilde{\beta}^\alpha((x*(y*z))*z),\widetilde{\beta}^\alpha((x*(y*z))*z)\}\\
&=\widetilde{\mu}^\alpha_{\widetilde{\beta}^\alpha}((x*(y*z))*z,(x*(y*z))*z)\\
&=\widetilde{\mu}^\alpha_{\widetilde{\beta}^\alpha}((x*(y*z),x*(y*z))*(z,z))
\end{aligned}$$

$$
\begin{aligned}
&= \widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}(((x,x)*((y*z,y*z)))*(z,z))\\
&= \widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}(((x,x)*((y,y)*(z,z)))*(z,z))\\
&\geqslant r\,min\{\widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}(x,x),\widetilde{\mu}^{\alpha}_{\widetilde{\beta}^{\alpha}}(y,y)\}\\
&= r\,min\{r\,min\{\widetilde{\beta}^{\alpha}(x),\widetilde{\beta}^{\alpha}(x)\}, r\,min\{\widetilde{\beta}^{\alpha}(y),\widetilde{\beta}^{\alpha}(y)\}\}\\
&= r\,min\{\widetilde{\beta}^{\alpha}(x),\widetilde{\beta}^{\alpha}(y)\}
\end{aligned}
$$

and

$$
\begin{aligned}
\gamma^{\alpha}((x*(y*z))*z) &= max\{\gamma^{\alpha}((x*(y*z))*z),\gamma^{\alpha}((x*(y*z))*z)\}\\
&= \lambda^{\alpha}_{\gamma^{\alpha}}((x*(y*z))*z,(x*(y*z))*z)\\
&= \lambda^{\alpha}_{\gamma^{\alpha}}((x*(y*z),x*(y*z))*(z,z))\\
&= \lambda^{\alpha}_{\gamma^{\alpha}}(((x,x)*((y*z,y*z)))*(z,z))\\
&= \lambda^{\alpha}_{\gamma^{\alpha}}(((x,x)*((y,y)*(z,z)))*(z,z))\\
&\leqslant max\{\lambda^{\alpha}_{\gamma^{\alpha}}(x,x),\lambda^{\alpha}_{\gamma^{\alpha}}(y,y)\}\\
&= max\{max\{\gamma^{\alpha}(x),\gamma^{\alpha}(x)\}, max\{\gamma^{\alpha}(y),\gamma^{\alpha}(y)\}\}\\
&= max\{\gamma^{\alpha}(x),\gamma^{\alpha}(y)\}.
\end{aligned}
$$

Hence $\langle\widetilde{\beta}^{\alpha},\gamma^{\alpha}\rangle$ is an $\alpha$-dot cubic new-ideal of $X$. □

DEFINITION 4.5. Let $\langle\widetilde{\mu},\lambda\rangle$ and $\langle\widetilde{\delta},\Psi\rangle$ be cubic subsets in $X$. The cartesian product $\langle\widetilde{\mu},\lambda\rangle\times\langle\widetilde{\delta},\Psi\rangle$ is defined by $(\widetilde{\mu}\times\widetilde{\delta}) : X\times X\longrightarrow D[0,1]$ and $(\lambda\times\Psi) : X\times X\longrightarrow[0,1]$, where $(\widetilde{\mu}\times\widetilde{\delta})(x,y) = r\,min\{\widetilde{\mu}(x),\widetilde{\delta}(y)\}$ and $(\lambda\times\Psi)(x,y) = max\{\lambda(x),\Psi(y)\}$ for all $x,y\in X$.

DEFINITION 4.6. Let $\langle\widetilde{\mu}^{\alpha},\lambda^{\alpha}\rangle$ and $\langle\widetilde{\delta}^{\alpha},\Psi^{\alpha}\rangle$ be $\alpha$-dot cubic subsets in $X$. The cartesian product $\langle\widetilde{\mu}^{\alpha},\lambda^{\alpha}\rangle\times\langle\widetilde{\delta}^{\alpha},\Psi^{\alpha}\rangle$ is defined by $(\widetilde{\mu}^{\alpha}\times\widetilde{\delta}^{\alpha}) : X\times X\longrightarrow D[0,1]$ and $(\lambda^{\alpha}\times\Psi^{\alpha}) : X\times X\longrightarrow[0,1]$, where $(\widetilde{\mu}^{\alpha}\times\widetilde{\delta}^{\alpha})(x,y) = r\,min\{\widetilde{\mu}^{\alpha}(x),\widetilde{\delta}^{\alpha}(y)\}$ and $(\lambda^{\alpha}\times\Psi^{\alpha})(x,y) = max\{\lambda^{\alpha}(x),\Psi^{\alpha}(y)\}$ for all $x,y\in X$.

THEOREM 4.2. *If $\langle\widetilde{\mu}^{\alpha},\lambda^{\alpha}\rangle$ and $\langle\widetilde{\delta}^{\alpha},\Psi^{\alpha}\rangle$ are $\alpha$-dot cubic new-ideals in a PU-algebra $X$, then $\langle\widetilde{\mu}^{\alpha},\lambda^{\alpha}\rangle\times\langle\widetilde{\delta}^{\alpha},\Psi^{\alpha}\rangle$ is an $\alpha$-dot cubic new-ideal in $X\times X$.*

PROOF. We have

$$(\widetilde{\mu}^{\alpha}\times\widetilde{\delta}^{\alpha})(0,0) = r\,min\{\widetilde{\mu}^{\alpha}(0),\widetilde{\delta}^{\alpha}(0)\}\geqslant r\,min\{\widetilde{\mu}^{\alpha}(x_1),\widetilde{\delta}^{\alpha}(x_2)\} = (\widetilde{\mu}^{\alpha}\times\widetilde{\delta}^{\alpha})(x_1,x_2).$$

Also,

$$(\lambda^{\alpha}\times\Psi^{\alpha})(0,0) = max\{\lambda^{\alpha}(0),\Psi^{\alpha}(0)\}\leqslant max\{\lambda^{\alpha}(x_1),\Psi^{\alpha}(x_2)\} = (\lambda^{\alpha}\times\Psi^{\alpha})(x_1,x_2),$$

for all $(x_1,x_2)\in X\times X$. Let $(x_1,x_2),(y_1,y_2),(z_1,z_2)\in X\times X$. Then we have

$$
\begin{aligned}
&(\widetilde{\mu}^{\alpha}\times\widetilde{\delta}^{\alpha})(((x_1,x_2)*((y_1,y_2)*(z_1,z_2)))*(z_1,z_2))\\
&\quad= (\widetilde{\mu}^{\alpha}\times\widetilde{\delta}^{\alpha})(((x_1,x_2)*(y_1*z_1,y_2*z_2))*(z_1,z_2))\\
&\quad= (\widetilde{\mu}^{\alpha}\times\widetilde{\delta}^{\alpha})((x_1*(y_1*z_1),x_2*(y_2*z_2))*(z_1,z_2))\\
&\quad= (\widetilde{\mu}^{\alpha}\times\widetilde{\delta}^{\alpha})((x_1*(y_1*z_1))*z_1,(x_2*(y_2*z_2))*z_2)\\
&\quad= r\,min\{\widetilde{\mu}^{\alpha}(x_1*(y_1*z_1))*z_1),\widetilde{\delta}^{\alpha}(x_2*(y_2*z_2))*z_2)\}
\end{aligned}
$$

$$\geqslant r\,min\{rmin\{\widetilde{\mu}^{\alpha}(x_1), \widetilde{\mu}^{\alpha}(y_1)\}, r\,min\{\widetilde{\delta}^{\alpha}(x_2), \widetilde{\delta}^{\alpha}(y_2)\}\}$$
$$= r\,min\{(\widetilde{\mu}^{\alpha} \times \widetilde{\delta}^{\alpha})(x_1, x_2), (\widetilde{\mu}^{\alpha} \times \widetilde{\delta}^{\alpha})(y_1, y_2)\}.$$

and

$$(\lambda^{\alpha} \times \Psi^{\alpha})(((x_1, x_2) * ((y_1, y_2) * (z_1, z_2))) * (z_1, z_2))$$
$$= (\lambda^{\alpha} \times \Psi^{\alpha})(((x_1, x_2) * (y_1 * z_1, y_2 * z_2)) * (z_1, z_2))$$
$$= (\lambda^{\alpha} \times \Psi^{\alpha})((x_1 * (y_1 * z_1), x_2 * (y_2 * z_2)) * (z_1, z_2))$$
$$= (\lambda^{\alpha} \times \Psi^{\alpha})((x_1 * (y_1 * z_1)) * z_1, (x_2 * (y_2 * z_2)) * z_2)$$
$$= max\{\lambda^{\alpha}(x_1 * (y_1 * z_1)) * z_1), \Psi^{\alpha}(x_2 * (y_2 * z_2)) * z_2)\}$$
$$\leqslant max\{max\{\lambda^{\alpha}(x_1), \lambda^{\alpha}(y_1)\}, max\{\Psi^{\alpha}(x_2), \Psi^{\alpha}(y_2)\}\}$$
$$= max\{(\lambda^{\alpha} \times \Psi^{\alpha})(x_1, x_2), (\lambda^{\alpha} \times \Psi^{\alpha})(y_1, y_2)\}.$$

Therefore $\langle \widetilde{\mu}^{\alpha}, \lambda^{\alpha}\rangle \times \langle \widetilde{\delta}^{\alpha}, \Psi^{\alpha}\rangle$ is $\alpha$-dot cubic new-ideal in $XtimesX$. □

## 5. Conclusions

In the present paper, we have introduced the concept of $\alpha$-dot-fuzzy new-ideal of PU-algebras and investigated some of their useful properties. We believe that these results are very useful in developing algebraic structures also these definitions and main results can be similarly extended to some other algebraic structure such as PS-algebras, Q-algebras, SU-algebras, IS-algebras, $\beta$-algebras and semi rings. It is our hope that this work would other foundations for further study of the theory of BCI-algebras. In our future study of fuzzy structure of PU-algebras, may be the following topics should be considered:

(1) To establish the interval value, bipolar and intuitionistic $\alpha$-dot new-ideal in PU-algebras.

(2) To consider the structure of $(\widetilde{\tau}, \rho)$-interval valued $\alpha$-dot new-ideal of PU-algebras.

(3) To get more results in $\alpha$-dot new-ideal of PU -algebras and it's application (Cubic soft sets with applications in PU-algebras).

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