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DECOMPOSITION THEOREMS IN PRINCIPAL RESIDUATED ALMOST DISTRIBUTIVE LATTICES

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Abstract. In this paper, we introduce the concept of meet representation of an element in an ADL L with ascending chain condition (a.c.c.). We prove decomposition theorems in a P-ADL and introduce the concepts of normal primary decomposition and isolated component of an element a in a complete residuated ADL L. We also prove the fundamental theorem on primary decompositions.

1. Introduction

Swamy, U.M. and Rao, G.C. [**8**] introduced the concept of an Almost Distributive Lattice as a common abstraction of almost all the existing ring theoretic generalizations of a Boolean algebra (like regular rings, $p-$rings, biregular rings, associate rings, P_1-rings etc.) on one hand and distributive lattices on the other.

In [**1**], Dilworth, R.P., has introduced the concept of a residuation in lattices and in [**9, 10**], Ward, M. and Dilworth, R.P., have studied residuated lattices. In [**11**], Ward, M. has introduced the concept of a principal residuated lattice (or simply a P-Lattice) and studied its properties. We introduced the concepts of a residuation and a multiplication in an ADL and the concept of a residuated ADL in our earlier paper [**5**]. We have proved some important properties of residuation ':' and multiplication '·' in a residuated ADL L in [**6**]. In [**3**], we introduced the concept of Noether ADL. In [**4**], we introduced the concepts of principal element in a residuated ADL and a principal residuated almost distributive lattice (or P-ADL). In this paper, we introduce the concept of meet representation of an element in an ADL L with ascending chain condition (a.c.c.). We prove decomposition theorems

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in a P-ADL and introduce the concepts of normal primary decomposition and isolated component of an element a in a complete residuated ADL L. We also prove the fundamental theorem on primary decompositions.

In Section 2, we recall the definition of an Almost Distributive Lattice (ADL) and certain elementary properties of an ADL from [**2, 8**] and some important results on a residuated almost distributive lattice from our earlier paper [**6**].

In Section 3, we define meet representation of an element in an ADL L with a.c.c. If L is a residuated ADL with a.c.c. and L has a meet representation, then we prove that the elements of L have primary decomposition if and only if every meet irreducible element of L is primary. In a principal residuated ADL L with a maximal element m, we prove that for each $a \in L$, there exist distinct primes p_1, p_2, ..., p_l such that

$$a \wedge m = p_1^{\alpha_1} \wedge p_2^{\alpha_2} \wedge ... \wedge p_l^{\alpha_l} \wedge m = (p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot ... \cdot p_l^{\alpha_l}) \wedge m.$$

We introduce the concepts of normal decomposition and isolated component of an element a in a complete residuated ADL L and prove that in a complete residuated ADL L, any two isolated components of an element a with the same set of corresponding primes are assosiates to each other.

2. Preliminaries

In this section we collect a few important definitions and results which are already known and which will be used more frequently in the paper.

We begin with the definition of an ADL.

Definition 2.1. ([**2**]). An *Almost Distributive Lattice(ADL)* is an algebra $(L, \vee, \wedge)$ of type $(2, 2)$ satisfying

(1) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
(2) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(3) $(a \vee b) \wedge b = b$
(4) $(a \vee b) \wedge a = a$
(5) $a \vee (a \wedge b) = a$, for all $a, b, c \in L$.

It can be seen directly that every distributive lattice is an ADL. If there is an element $0 \in L$ such that $0 \wedge a = 0$ for all $a \in L$, then $(L, \vee, \wedge, 0)$ is called an ADL with 0.

Example 2.1. ([**2**]). Let X be a non-empty set. Fix $x_0 \in X$. For any $x, y \in L$, define

$$x \wedge y = \begin{cases} x_0, & \text{if } x = x_0 \\ y, & \text{if } x \neq x_0 \end{cases} \quad x \vee y = \begin{cases} y, & \text{if } x = x_0 \\ x, & \text{if } x \neq x_0. \end{cases}$$

Then $(X, \vee, \wedge, x_0)$ is an ADL, with x_0 as its zero element. This ADL is called a *discrete* ADL.

For any $a, b \in L$, we say that a is less than or equals to b and write $a \leqslant b$, if $a \wedge b = a$. Then "$\leqslant$" is a partial ordering on L.

THEOREM 2.1 ([**2**]). *Let* $(L, \vee, \wedge, 0)$ *be an ADL with '0'. Then, for any* $a, b \in L$, *we have*

(1) $a \wedge 0 = 0$ *and* $0 \vee a = a$
(2) $a \wedge a = a = a \vee a$
(3) $(a \wedge b) \vee b = b,\ a \vee (b \wedge a) = a$ *and* $a \wedge (a \vee b) = a$
(4) $a \wedge b = a \Longleftrightarrow a \vee b = b$ *and* $a \wedge b = b \Longleftrightarrow a \vee b = a$
(5) $a \wedge b = b \wedge a$ *and* $a \vee b = b \vee a$ *whenever* $a \leqslant b$
(6) $a \wedge b \leqslant b$ *and* $a \leqslant a \vee b$
(7) $\wedge$ *is associative in* L
(8) $a \wedge b \wedge c = b \wedge a \wedge c$
(9) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(10) $a \wedge b = 0 \Longleftrightarrow b \wedge a = 0$
(11) $a \vee (b \vee a) = a \vee b$.

It can be observed that an ADL L satisfies almost all the properties of a distributive lattice except, possible the right distributivity of $\vee$ over $\wedge$, the commutativity of $\vee$, the commutativity of $\wedge$ and the absorption law $(a \wedge b) \vee a = a$. Any one of these properties convert L into a distributive lattice.

THEOREM 2.2 ([**2**]). *Let* $(L, \vee, \wedge, 0)$ *be an ADL with* 0.
Then the following are equivalent:

(1) $(L, \vee, \wedge, 0)$ *is a distributive lattice*
(2) $a \vee b = b \vee a$, *for all* $a, b \in L$
(3) $a \wedge b = b \wedge a$, *for all* $a, b \in L$
(4) $(a \wedge b) \vee c = (a \vee c) \wedge (b \vee c)$, *for all* $a, b, c \in L$.

PROPOSITION 2.1 ([**2**]). *Let* $(L, \vee, \wedge)$ *be an ADL. Then for any* $a, b, c \in L$ *with* $a \leqslant b$, *we have*

(1) $a \wedge c \leqslant b \wedge c$
(2) $c \wedge a \leqslant c \wedge b$
(3) $c \vee a \leqslant c \vee b$.

DEFINITION 2.2. ([**2**]) An element $m \in L$ is called maximal if it is maximal as in the partially ordered set $(L, \leqslant)$. That is, for any $a \in L,\ m \leqslant a$ implies $m = a$.

THEOREM 2.3 ([**2**]). *Let* L *be an ADL and* $m \in L$. *Then the following are equivalent:*

(1) m *is maximal with respect to* $\leqslant$
(2) $m \vee a = m$, *for all* $a \in L$ (3) $m \wedge a = a$, *for all* $a \in L$.

LEMMA 2.1 ([**2**]). *Let* L *be an ADL with a maximal element* m *and* $x, y \in L$. *If* $x \wedge y = y$ *and* $y \wedge x = x$ *then* x *is maximal if and only if* y *is maximal.Also the following conditions are equivalent:*

(i) $x \wedge y = y$ *and* $y \wedge x = x$
(ii) $x \wedge m = y \wedge m$.

DEFINITION 2.3. ([**7**]) If $(L, \vee, \wedge, 0, m)$ is an ADL with 0 and with a maximal element m, then the set $I(L)$ of all ideals of L is a complete lattice under set inclusion. In this lattice, for any $I, J \in I(L)$, the l.u.b. and g.l.b. of I, J are given

by $I \vee J = \{(x \vee y) \wedge m \mid x \in I, y \in J\}$ and $I \wedge J = I \cap J$.
The set $PI(L) = \{(a] \mid a \in L\}$ of all principal ideals of L forms a sublattice of $I(L)$. (Since $(a] \vee (b] = (a \vee b]$ *and* $(a] \cap (b] = (a \wedge b]$.)

Definition 2.4. ([**7**]) An ADL $L = (L, \vee, \wedge, 0, m)$ with a maximal element m is said to be a *complete* ADL , if $PI(L)$ is a complete sub lattice of the lattice $I(L)$.

Theorem 2.4 ([**7**]). *Let $L = (L, \vee, \wedge, 0, m)$ be an ADL with a maximal element m. Then L is a complete ADL if and only if the lattice $([0, m], \vee, \wedge)$ is a complete lattice.*

In the following, we give the concepts of residuation and multiplication in an almost distributive lattice (ADL) L and the definition of a residuated almost distributive lattice taken from our earlier paper [**5**].

Definition 2.5. ([**5**]) Let L be an ADL with a maximal element m. A binary operation : on an ADL L is called a *residuation* over L if, for $a, b, c \in L$ the following conditions are satisfied.

$(R1)$ $a \wedge b = b$ if and only if $a : b$ is maximal
$(R2)$ $a \wedge b = b \implies$ (i) $(a : c) \wedge (b : c) = b : c$ and (ii) $(c : b) \wedge (c : a) = c : a$
$(R3)$ $[(a : b) : c] \wedge m = [(a : c) : b] \wedge m$
$(R4)$ $[(a \wedge b) : c] \wedge m = (a : c) \wedge (b : c) \wedge m$
$(R5)$ $[c : (a \vee b)] \wedge m = (c : a) \wedge (c : b) \wedge m.$

Definition 2.6. ([**5**]) Let L be an ADL with a maximal element m. A binary operation . on an ADL L is called a *multiplication* over L if, for $a, b, c \in L$ the following conditions are satisfied.

$(M1)$ $(a \cdot b) \wedge m = (b \cdot a) \wedge m$
$(M2)$ $[(a \cdot b) \cdot c] \wedge m = [a \cdot (b \cdot c)] \wedge m$
$(M3)$ $(a \cdot m) \wedge m = a \wedge m$
$(M4)$ $[a \cdot (b \vee c)] \wedge m = [(a \cdot b) \vee (a \cdot c)] \wedge m.$

Definition 2.7. ([**5**]) An ADL L with a maximal element m is said to be a *residuated almost distributive lattice* (*residuated* ADL), if there exists two binary operations ':' and '·' on L satisfying conditions R1 to R5, M1 to M4 and the following condition (A).

(A) $(x : a) \wedge b = b$ if and only if $x \wedge (a \cdot b) = a \cdot b$, for any $x, a, b \in L$.

We use the following properties frequently later in the results.

Lemma 2.2 ([**5**]). *Let L be an ADL with a maximal element m and . a binary operation on L satisfying the conditions $M1 - M4$. Then for any $a, b, c, d \in L$,*

(i) $a \wedge (a \cdot b) = a \cdot b$ *and* $b \wedge (a \cdot b) = a \cdot b$
(ii) $a \wedge b = b \implies (c \cdot a) \wedge (c \cdot b) = c \cdot b$ *and* $(a \cdot c) \wedge (b \cdot c) = b \cdot c$
(iii) $d \wedge [(a \cdot b) \cdot c] = (a \cdot b) \cdot c$ *if and only if* $d \wedge [a \cdot (b \cdot c)] = a \cdot (b \cdot c)$
(iv) $(a \cdot c) \wedge (b \cdot c) \wedge [(a \wedge b) \cdot c] = (a \wedge b) \cdot c$
(v) $d \wedge (a \cdot c) \wedge (b \cdot c) = (a \cdot c) \wedge (b \cdot c) \implies d \wedge [(a \wedge b) \cdot c] = (a \wedge b) \cdot c$
(vi) $d \wedge [(a \cdot c) \vee (b \cdot c)] = (a \cdot c) \vee (b \cdot c) \Leftrightarrow d \wedge [(a \vee b) \cdot c] = (a \vee b) \cdot c$

The following result is a direct consequence of M1 of Definition 2.5.

Lemma 2.3 ([**5**]). *Let L be an ADL with a maximal element m and $\cdot$ a binary operation on L satisfying the condition $M1$. For $a,b,x \in L$, $a \wedge (x.b) = x.b$ if and only if $a \wedge (b \cdot x) = b \cdot x$*

In the following, we give some important properties of residuation ':' and multiplication '$\cdot$' in a residuated ADL L. These are taken from our earlier paper [**6**].

Lemma 2.4 ([**6**]). *Let L be a residuated ADL with a maximal element m. For $a,b,c,d \in L$, the following hold in L.*

(1) $(a:b) \wedge a = a$
(2) $[a:(a:b)] \wedge (a \vee b) = a \vee b$
(3) $[(a:b):c] \wedge [a:(b \cdot c)] = a:(b \cdot c)$
(4) $[a:(b \cdot c)] \wedge [(a:b):c] = (a:b):c$
(5) $[(a \wedge b):b] \wedge (a:b) = a:b$
(6) $(a:b) \wedge [(a \wedge b):b] = (a \wedge b):b$
(7) $[a:(a \vee b)] \wedge m = (a:b) \wedge m$
(8) $[c:(a \wedge b)] \wedge [(c:a) \vee (c:b)] = (c:a) \vee (c:b)$
(9) *If* $a:b = a$ *then* $a \wedge (b \cdot d) = b \cdot d \Longrightarrow a \wedge d = d$
(10) $\{a:[a:(a:b)]\} \wedge (a:b) = a:b$
(11) $[(a \vee b):c] \wedge [(a:c) \vee (b:c)] = (a:c) \vee (b:c)$
(12) $a \wedge m \geqslant b \wedge m \Longrightarrow (a:c) \wedge m \geqslant (b:c) \wedge m$
(13) $(a:b) \wedge \{a:[a:(a:b)]\} = a:[a:(a:b)]$
(14) $a \wedge b = b \Longrightarrow (a \cdot c) \wedge (b \cdot c) = b \cdot c$
(15) $a \wedge b \wedge (a \cdot b) = a \cdot b$
(16) $[(a \cdot b):a] \wedge b = b$
(17) $(a \cdot b) \wedge [(a \wedge b) \cdot (a \vee b)] = (a \wedge b) \cdot (a \vee b)$
(18) $a \vee b$ *is maximal* $\Longrightarrow (a \cdot b) \wedge a \wedge b = a \wedge b$

We recall the following concepts on a residuated ADL L from our earlier paper [**3**].

Definition 2.8. ([**3**]) An element p of a residuated ADL L is called
(i) *irreducible*, if for any $f,g \in L$, $f \wedge g = p \Longrightarrow$ either $f = p$ or $g = p$.
(ii) *prime*, if for any $a,b \in L$, $p \wedge (a.b) = a.b \Longrightarrow$ either $p \wedge a = a$ or $p \wedge b = b$.
(iii) *primary*, if for any $a,b \in L$, $p \wedge (a \cdot b) = a \cdot b$ and $p \wedge a \neq a \Longrightarrow p \wedge b^s = b^s$, for some $s \in Z^+$.

Definition 2.9. ([**3**]) An ADL L is said to satisfy the *ascending chain condition(a.c.c.)*, if for every increasing sequence $x_1 \leqslant x_2 \leqslant x_3 \leqslant ...$ in L, there exists a positive integer n such that $x_n = x_{n+1} = x_{n+2} = ...$

Definition 2.10. ([**3**]) Let L be a residuated ADL. An element a of L is called *principal*, if $b \in L$ and $a \wedge b = b$, then $a \cdot c = b$, for some $c \in L$.

Definition 2.11. ([**3**]) A residuated ADL L is said to be a *Noether ADL*, if
(N1) the ascending chain condition(a.c.c.) holds in L and
(N2) every irreducible element of L is primary.

Now, we have taken the following definitions from our earlier paper [**4**].

DEFINITION 2.12. ([**4**]) Let L be an ADL and $x, y \in L$.

(i) y is called a *divisor* of x if $y \wedge x = x$.

Observe that every maximal element m is a divisor of x, for any $x \in L$ and every associate of x is a divisor of x.

(ii) A divisor y of x other than maximal elements and associates of x is called a *proper divisor* of x.

DEFINITION 2.13. ([**4**]) Let L be an ADL with a maximal element m. An element x of L is called an *associate* of y if $x \wedge m = y \wedge m$ (or x is equivalent to y).

DEFINITION 2.14. ([**4**]) Let L be a residuated ADL with a.c.c. If every element of L is principal then L is called a *Principal Residuated Almost Distributive Lattice (or P-ADL).*

The following Lemma was proved in our earlier paper [**3**] and is used frequently later in the results.

LEMMA 2.5 ([**3**]). *Let L be a residuated ADL with a maximal element m. If $a, b \in L$ such that a is principal and $a \wedge b = b$ then $[(b : a) \cdot a] \wedge m = b \wedge m$.*

3. Decomposition Theorems in a P-ADL

In this section, we define meet representation of an element in an ADL L with a.c.c. If L is a residuated ADL with a.c.c. and if L has a meet representation, then we prove that the elements of L have primary decomposition if and only if every meet irreducible element of L is primary. In a P-ADL L, with a maximal element m, we prove that for each $a \in L$, there exist distinct primes p_1, p_2, ..., p_l such that

$$a \wedge m = p_1^{\alpha_1} \wedge p_2^{\alpha_2} \wedge ... \wedge p_l^{\alpha_l} \wedge m = (p_1^{\alpha_1} \cdot p_2^{\alpha_2} ... p_l^{\alpha_l}) \wedge m.$$

We introduce the concepts of normal decomposition and isolated component of an element a in a complete residuated ADL L and also prove the fundamental theorem on primary decompositions.

First we prove the following result in an ADL L.

THEOREM 3.1. *If an ADL L satisfies the ascending chain condition then every non empty subset of L has a maximal element.*

PROOF. Suppose L is an ADL satisfying the ascending chain condition. Let S be a non empty subset of L. Assume that S has no maximal element. If $x_1 \in S$, then x_1 is not a maximal element. Therefore, there exists an element $x_2 \in S$ such that $x_1 < x_2$. Again since x_2 is not a maximal element, there exists an element $x_3 \in S$ such that $x_1 < x_2 < x_3$. Proceeding like this, we get a strictly increasing chain $x_1 < x_2 < x_3 < x_4 < ...$ of elements of S in L. This contradicts the fact that every increasing sequence in L is stationary. Hence every non empty subset S of L has a maximal element. □

DEFINITION 3.1. Let L be an ADL and L satisfying the ascending chain condition. An element a of L is said to have a *meet reprsentation* if there exists a finite number of irreducible elements $s_1, s_2, ..., s_m$ in L such that $a = s_1 \wedge s_2 \wedge ... \wedge s_m$.

If every element of L has a meet representation, then L is said to have a meet representation.

Let us recall the following definition from [**3**].

Definition 3.2. ([**3**]) An element a of a residuated ADL L is said to have a *primary decomposition*, if there exists primary elements q_1, q_2, ..., q_l in L such that $a = q_1 \wedge q_2 \wedge ... \wedge q_l$.

Now, we prove the following result.

Theorem 3.2. *Let L be a residuated ADL with a.c.c. Suppose L has a meet representation. Then every element of L has a primary decomposition if and only if every meet irreducible element of L is primary.*

Proof. Suppose every element of L has a primary decompositin and p is a meet irreducible element of L. Then there exists primary elements $q_1, q_2, ..., q_l$ in L such that $p = q_1 \wedge q_2 \wedge ... \wedge q_l$. Since p is meet irreducible, we get $p = q_1$ or $p = q_2$ or ... or $p = q_l$ and hence p is primary.

Now, suppose that every irreducible element of L is primary and $a \in L$. Since L has a meet representation, we can write $a = s_1 \wedge s_2 \wedge ... \wedge s_m$, for irreducible elements $s_1, s_2, ..., s_m$ of L. Thus a has a primary decomposition. (Since each s_i is a primary element of L) □

We have taken the following definition and results from our earlier paper [**4**].

Definition 3.3. ([**4**]) Let L be a residuated ADL with a.c.c. and q a primary element of L. A prime element p of L is called *the prime corresponding to* q if $p \wedge q = q$, $q \wedge p^k = p^k$ and $q \wedge p^{k-1} \neq p^{k-1}$, for some $k \in Z^+$.

Lemma 3.1 ([**4**]). *Let L be a residuated ADL with a maximal element m and $a, b \in L$ such that $a \wedge m = b \wedge m$. Then $(p \cdot a) \wedge m = (p \cdot b) \wedge m$, for any $p \in L$.*

Lemma 3.2 ([**4**]). *Let L be a residuated ADL with a maximal element m and L satisfies the a.c.c. If q is a primary element of L and p is the prime corresponding to q. Then, for any $a \in L$, $(q : a) \wedge m = q \wedge m$ if and only if $p \wedge a \neq a$.*

Theorem 3.3 ([**4**]). *Let L be a P-ADL with a maximal element m. If q is a primary element of L and p is the prime corresponding to q then $q \wedge m = p^r \wedge m$, for some $r \in Z^+$.*

Now, we prove the following in a P-ADL.

Theorem 3.4. *Let L be a P-ADL with a maximal element m. Suppose L has a meet representation. Then, for each $a \in L$, there exist distinct primes $p_1, p_2, ..., p_l$ in L such that $a \wedge m = p_1^{\alpha_1} \wedge p_2^{\alpha_2} \wedge ... \wedge p_l^{\alpha_l} \wedge m$.*

Proof. Suppose L is a P-ADL with a maximal element m and $a \in L$. Since L has a meet representation, we can write $a = q_1 \wedge q_2 \wedge ... \wedge q_l$, where q_i's are irreducible elements of L. Since L is a P-ADL, it is a Noether ADL. Since every irreducible element of a Noether ADL is primary, we get that $q_1, q_2, ..., q_l$ are primary elements of L. Suppose $p_1, p_2, ..., p_l$ be the primes corresponding to $q_1, q_2, ..., q_l$, respectively. By Theorem 3.3, we get that

$$q_1 \wedge m = p_1^{\alpha_1} \wedge m,\ q_2 \wedge m = p_2^{\alpha_2} \wedge m, ..., q_l \wedge m = p_l^{\alpha_l} \wedge m,$$

for some natural numbers $\alpha_1, \alpha_2, ..., \alpha_l$.

Now,
$a \wedge m = q_1 \wedge q_2 \wedge ... \wedge q_l \wedge m.$
$= q_1 \wedge m \wedge q_2 \wedge m \wedge ... \wedge q_l \wedge m.$
$= p_1^{\alpha_1} \wedge m \wedge p_2^{\alpha_2} \wedge m \wedge ... \wedge p_l^{\alpha_l} \wedge m.$
$= p_1^{\alpha_1} \wedge p_2^{\alpha_2} \wedge ... \wedge p_l^{\alpha_l} \wedge m.$ □

Theorem 3.5. *Let L be a P-ADL with a maximal element m and L has a meet representation. Then, for each $a \in L$, there exist distinct primes $p_1, p_2, ..., p_l$ in L such that $a \wedge m = (p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot ... \cdot p_l^{\alpha_l}) \wedge m$.*

Proof. Suppose L is a P-ADL with a maximal element m and L has a meet representation and let $a \in L$. Then, by Theorem 3.4, there exist distinct prime elements $p_1, p_2, ..., p_l$ in L such that $a \wedge m = p_1^{\alpha_1} \wedge p_2^{\alpha_2} \wedge ... \wedge p_l^{\alpha_l} \wedge m$. We prove that $p_1^{\alpha_1} \wedge p_2^{\alpha_2} \wedge ... \wedge p_l^{\alpha_l} \wedge m = (p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot ... \cdot p_l^{\alpha_l}) \wedge m, \quad \longrightarrow (1)$ by using induction on l.

Clearly (1) is true for $l = 1$. Assume that (1) is true for any $l - 1$ distinct primes of L. We select from $p_1, p_2, ..., p_l$ a prime not divisible by any other prime, let it be p_1. Then for $2 \leqslant j \leqslant l$, $p_j \wedge p_1 \neq p_1$ and hence $p_j \wedge p_1^{\alpha_1} \neq p_1^{\alpha_1}$. Hence, by Lemma 3.2, $(p_j^{\alpha_j} : p_1^{\alpha_1}) \wedge m = p_j^{\alpha_j} \wedge m$. Since $p_1^{\alpha_1} \wedge m \geqslant a \wedge m$, we get $p_1^{\alpha_1} \wedge a = a$. Now, $(a \wedge m) : p_1^{\alpha_1} = (p_1^{\alpha_1} \wedge p_2^{\alpha_2} \wedge ...... \wedge p_l^{\alpha_l} \wedge m) : p_1^{\alpha_1}$. Thus
$[(a \wedge m) : p_1^{\alpha_1}] \wedge m = [(p_1^{\alpha_1} \wedge p_2^{\alpha_2} \wedge ...... \wedge p_l^{\alpha_l} \wedge m) : p_1^{\alpha_1}] \wedge m$
$= (p_1^{\alpha_1} : p_1^{\alpha_1}) \wedge (p_2^{\alpha_2} : p_1^{\alpha_1}) \wedge ...... \wedge (p_l^{\alpha_l} : p_1^{\alpha_1}) \wedge (m : p_1^{\alpha_1}) \wedge m$
$= p_2^{\alpha_2} \wedge p_3^{\alpha_3} \wedge ...... \wedge p_l^{\alpha_l} \wedge m.$
$= (p_2^{\alpha_2}.p_3^{\alpha_3}......p_l^{\alpha_l}) \wedge m.$ (By induction hypothesis)
$\Longrightarrow (a : p_1^{\alpha_1}) \wedge (m : p_1^{\alpha_1}) \wedge m = (p_2^{\alpha_2}.p_3^{\alpha_3}......p_l^{\alpha_l}) \wedge m.$
$\Longrightarrow (a : p_1^{\alpha_1}) \wedge m = (p_2^{\alpha_2}.p_3^{\alpha_3}......p_l^{\alpha_l}) \wedge m.$ (Since $(m : p_1^{\alpha_1}) \wedge m = m$)
$\Longrightarrow [(a : p_1^{\alpha_1}).p_1^{\alpha_1}] \wedge m = [(p_2^{\alpha_2}.p_3^{\alpha_3}......p_l^{\alpha_l}).p_1^{\alpha_1}] \wedge m = (p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}......p_l^{\alpha_l}) \wedge m.$
(By Lemma 3.1 and by condition M1 of definition 2.6)
$\Longrightarrow a \wedge m = (p_1^{\alpha_1}.p_2^{\alpha_2}.p_3^{\alpha_3}......p_l^{\alpha_l}) \wedge m.$ (Since $p_1^{\alpha_1}$ is principal)
$\Longrightarrow p_1^{\alpha_1} \wedge p_2^{\alpha_2} \wedge ...... \wedge p_l^{\alpha_l} \wedge m = (p_1^{\alpha_1}.p_2^{\alpha_2}......p_l^{\alpha_l}) \wedge m.$ □

Definition 3.4. Let L be a complete residuated ADL with a maximal element m and $a \in L$. A primary decomposition of a, $a = q_1 \wedge q_2 \wedge ...... \wedge q_l$ is said to be *reduced* if for each $q_i \in L$, $q_1 \wedge q_2 \wedge ...... \wedge q_{i-1} \wedge q_{i+1} \wedge ...... \wedge q_l \neq a$.

Definition 3.5. Let L be a complete residuated ADL with a maximal element m and $a \in L$. Suppose $a = q_1 \wedge q_2 \wedge ...... \wedge q_l$ be a primary decomposition of a. If superfluous q_i are removed and the primaries with same corresponding primes are combined, we obtain a reduced primary decomposition in which distinct primes corresponding to distinct primaries. Such a primary decomposition is called a *normal primary decomposition (or a normal decomposition).*

Definition 3.6. Let L be a complete residuated ADL with a maximal element m and $a \in L$. Suppose $a = q_1 \wedge q_2 \wedge ...... \wedge q_l$ be a normal decomposition of a and $p_1, p_2, ..., p_l$ denote distinct primes corresponding to primary elements $q_1, q_2, ..., q_l$. A subset S of $\{p_1, p_2, ..., p_l\}$ is said to be *isolated* if

$$p_i \in S \Longrightarrow p_j \in S \text{ when ever } p_i \wedge p_j = p_j.$$

In this case, the element $a_s = \wedge\{q_i \wedge m \mid p_i \in S\}$ is called the *isolated component* of a corresponding to S.

In the following, we prove the *fundamental theorem* on primary decompositions.

THEOREM 3.6. *Let L be a complete residuated ADL with a maximal element m and satisfying the a.c.c. Suppose $a \in L$. Then any two isolated components of a with the same set of corresponding primes are associates to each other.*

PROOF. Let S be an isolated subset of $\{p_1, p_2, ..., p_l\}$ and $a_s = \wedge\{q_i \wedge m \mid p_i \in S\}$ be an isolated component of an element a. Now, let $a = {q_1}^1 \wedge {q_2}^1 \wedge .. \wedge {q_l}^1$ be another normal decomposition of a and ${a_s}^1 = \wedge\{{q_i}^1 \wedge m \mid p_i \in S\}$. Take $b^1 = \wedge\{{q_j}^1 \wedge m \mid p_j \notin S\}$. For $1 \leqslant i \leqslant l$, we have $q_i \wedge m \geqslant a \wedge m = {a_s}^1 \wedge b^1 \wedge m \geqslant ({a_s}^1 \cdot b^1) \wedge m$. (By property (15) of Lemma 2.4)

$\Longrightarrow q_i \wedge ({a_s}^1 \cdot b^1) = {a_s}^1.b^1$

$\Longrightarrow q_i \wedge {a_s}^1 = {a_s}^1$ or $q_i \wedge {b^1}^k = {b^1}^k$, for some $k \in Z^+$. (Since q_i is primary)

$\Longrightarrow q_i \wedge {a_s}^1 = {a_s}^1$ or ${b^1}^k = q_i \wedge {b^1}^k = p_i \wedge q_i \wedge {b^1}^k = p_i \wedge {b^1}^k$, for some $k \in Z^+$.

$\Longrightarrow q_i \wedge {a_s}^1 = {a_s}^1$ or $p_i \wedge b^1 = b^1$. (Since p_i is prime)

If $p_i \wedge b^1 = b^1 = \wedge\{{q_j}^1 \wedge m \mid p_j \notin S\}$, then

$$p_i \wedge {q_j}^1 \wedge m = {q_j}^1 \wedge m \text{ and hence } p_i \wedge {q_j}^1 = {q_j}^1, \text{ for all } j \text{ such that } p_j \notin S.$$

We have ${q_j}^1 \wedge {p_j}^{k_j} = {p_j}^{k_j}$, for some $k_j \in Z^+$. Now, ${p_j}^{k_j} = {q_j}^1 \wedge {p_j}^{k_j} = p_i \wedge {q_j}^1 \wedge {p_j}^{k_j} = p_i \wedge {p_j}^{k_j}$. Since p_i is prime, we get $p_i \wedge p_j = p_j$. Hence $p_j \in S$ if $p_i \in S$. This is a contradiction to $p_j \notin S$. Therefore, $q_i \wedge {a_s}^1 = {a_s}^1$, for all i such that $p_i \in S$. Therefore, $[\wedge\{q_i \wedge m \mid p_i \in S\}] \wedge {a_s}^1 = {a_s}^1$. (Since $q_i \wedge {a_s}^1 = {a_s}^1$). Hence $a_s \wedge {a_s}^1 = {a_s}^1$. Similarly, we get ${a_s}^1 \wedge a_s = a_s$. Hence $a_s \wedge m = {a_s}^1 \wedge m$. Thus any two isolated components of an element a are associates to each other. □

References

[1] R. P. Dilworth. Abstract residuation over lattices. *Bull. Amer. Math. Soc.* **44**(4) (1938), 262–268.

[2] G. C. Rao. *Almost Distributive Lattices*. Doctoral Thesis. Deptartment of Mathematics, Andhra University, Visakhapatnam, 1980.

[3] G. C. Rao and S. S. Raju. Noether Almost Distributive Lattices. *Bull. Int. Math. Virtual Inst.*, **9**(1)(2019), 39–46.

[4] G. C. Rao and S. S. Raju. Principal Residuated Almost Distributive Lattices. *Bull. Int. Math. Virtual Inst.*, **9**(2)(2019), 219–230.

[5] G. C. Rao and S. S. Raju. Residuated Almost Distributive Lattices - I. *International Journal of Mathematical Archive,* **8**(10)(2017), 143–152.

[6] G. C. Rao and S. S. Raju. Residuated Almost Distributive Lattices - II. *International Journal of Mathematical Archive,* **8**(10)(2017), 190–198.

[7] G. C. Rao and Venugopalam Undurthi. Complete almost distributive lattices. *Asian-Eur. J. Math.* **7**(3)(2014), ID: 1450052 (8 pages).

[8] U. M. Swamy and G. C. Rao. Almost Distributive Lattices. *J. Aust. Math. Soc. (Series A)*, **31**(1)(1981), 77–91.

[9] M. Ward and R. P. Dilworth. Residuated lattices. *Proc. Natl. Acad. Sci. USA*, **24**(3)(1938), 162–164.
[10] M. Ward and R. P. Dilworth. Residuated lattices. *Trans. Am. Math. Soc.*, **45**(3)(1939), 335–354.
[11] M. Ward. Residuated distributive lattices. *Duke Math. J.*, **6**(3)(1940), 641–651.

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