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FUZZY IDEALS OF SKEW LATTICES

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Abstract. In this paper, the concept of fuzzy ideal of a skew lattice S is introduced, a fuzzy ideal of S is characterized in terms of a crisp ideal of S and equivalent conditions between a fuzzy subset of S and a fuzzy ideal of S are established. Also, it is proved that the set of all fuzzy ideals of a strongly distributive skew lattice forms a complete lattice. Lastly a mapping from the set $I(S)$ of all ideals of a skew lattice S into the set $F\alpha_I(S)$ of all $\alpha-$level fuzzy ideals (α_I) of S corresponding to the ideal I is defined and it is proved that this map is an isomorphism.

1. Introduction

The concept of fuzzy subsets of a nonempty set S as a function of S into the unit interval $[0, 1]$ introduced by L. Zadch [**12**] initiated several algebraists to take up the study of fuzzy subalgebras of various algebraic systems such as groups, rings, modules, lattices etc. The study of fuzzy algebraic structure has started by Rosenfeld [**11**] and since then this concept has been applied to a variety of algebraic structures such as the notion of a fuzzy subgroup of a group [**7**] and [**11**], fuzzy subrings [**8**], fuzzy ideals of rings [**7, 9, 10**], and fuzzy ideals of lattices [**1**].

As skew lattices are non-commutative generalization of lattices introduced by Leech [**6**], the order structure has an important role in the study of these algebras. Skew lattices can be seen as double regular bands where two different order concepts can be defined: the natural preorder, denoted by $\preceq$, and the natural partial order, denoted by $\leqslant$, one weaker than the other and both of them motivated by analogous order concepts defined for bands. They generalize the partial order of the correspondent lattice. Two kinds of ideals can be naturally derived from these preorders (ideal of a skew lattice defined using preorder and skew ideal defined using the natural partial order). The strong concept of ideal is naturally derived

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from the preorder and has been largely studied having an important role in the research centered on the congruences of skew Boolean algebras with intersections, a particular case of the Boolean version of a skew lattice [**3**] and [**4**]. These ideas motivates the researcher to develop the concept of fuzzy ideals of skew lattices.

The paper is organized as follows: Section two deals on preliminary concepts which can be used in proving lemmas, theorems and corollaries in the subsequent section. In section three the notion of fuzzy ideals of skew lattices are introduced, some basic arithmetical properties are presented and different characterizations of fuzzy ideals of skew lattices are given.

2. Preliminaries

First we give the necessary definitions and results on skew lattices, ideals of skew lattices, fuzzy lattices and fuzzy ideals which will be used in the next section.

Definition 2.1. ([**6**]) An algebra $(S, \vee, \wedge)$ of type (2, 2) is said to be a skew lattice if the operations $\vee$ and $\wedge$ are associative and the absorption laws hold:

(1) (1) $x \wedge (y \vee x) = x = x \wedge (x \vee y)$;
(2) $(x \wedge y) \vee x = x = (y \wedge x) \vee x$.

On any skew lattice S two binary relations are defined as follows: the natural order relation denoted $\leqslant$: $x \leqslant y$ if and only if $x \wedge y = y \wedge x = x$ and the natural preorder relation denoted $\preceq$: $x \preceq y$ if and only if $x \wedge y \wedge x = x$ (or dually, $y \vee x \vee y = y$) .

Definition 2.2. ([**4**]) A nonempty subset I of a skew lattice S closed under $\vee$ is an ideal of S if, for all $x \in S$ and $y \in I, x \preceq y$ implies $x \in I$.

Theorem 2.1 ([**4**]). *All ideals in a skew lattice are sub skew lattices.*

Definition 2.3. [12] Let X be a nonempty set. By a fuzzy subset μ of X, we mean a map from X to the interval $[0, 1]$, $\mu : X \longrightarrow [0, 1]$.

If μ is a fuzzy subset of X and $t \in [0, 1]$, then the level subset μ_t is defined as follows: $\mu_t = \{x \in X | \mu(x) \geqslant t\}$.

Definition 2.4. ([**9**]) A fuzzy subset μ of a ring R is called a fuzzy left (right) ideal of R if

i. $\mu(x - y) = min\{\mu(x), \mu(y)\}$,
ii. $\mu(xy) = \mu(y)(\mu(xy) = \mu(x))$ for all $x, y \in R$.

Theorem 2.2 ([**9**]). *Let μ be a fuzzy subset of a ring R. If $\forall t \in Im(\mu)$, μ_t is a left (right) ideal of R, then μ is a fuzzy left (right) ideal of R.*

Definition 2.5. ([**1**]) Let μ be a fuzzy set in a lattice L. Then μ is called a fuzzy sublattice of L if

i. $\mu(x + y) \geqslant min(\mu(x), \mu(y))$;
ii. $\mu(xy) \geqslant min(\mu(x), \mu(y))$. for all $x, y \in L$

Definition 2.6. ([**1**]) Let μ be a fuzzy sublattice of L. Then μ is called a fuzzy ideal if $x \leqslant y$ in L implies $\mu(x) \geqslant \mu(y)$.

3. Fuzzy Ideals of Skew Lattices

In this section, we introduce the concept of fuzzy ideal of skew lattices.

Definition 3.1. A nonempty subset I of a skew lattice S closed under $\vee$ is an ideal of S whenever $y \wedge x \wedge y \in I$ for all $x \in I, y \in S$.

Through out the rest part of this section S denote a skew lattice. Next we define fuzzy ideal of a skew Lattice.

Definition 3.2. A fuzzy subset μ of S is called a fuzzy subskew lattice of S, if $\mu(x \vee y) \wedge \mu(x \wedge y) \geqslant \mu(x) \wedge \mu(y),$ for all $x, y \in S$.

Definition 3.3. A fuzzy subset μ of S is called a fuzzy ideal of S if

(1) $\mu(x \vee y) = min(\mu(x), \mu(y))$ for all $x, y \in S$.
(2) $\mu(x \wedge y \wedge x) \geqslant \mu(y)$ for all $x, y \in S$.

Example 3.1. Consider the set $S = \{a, b, c\}$. Let $\vee$ and $\wedge$ are binary operations on S defined by the tables given below. Then $(S, \vee, \wedge)$ is a skew lattice. Now define a fuzzy subset $\mu : S \longrightarrow [0, 1]$, by $\mu(a) = 0.6 = \mu(b)$ and $\mu(c) = 0.9$. It is easy to check that μ is a fuzzy ideal of S.

$\vee$	a	b	c
a	a	a	a
b	b	b	b
c	a	b	c

$\wedge$	a	b	c
a	a	b	c
b	a	b	c
c	c	c	c

Let $\mathfrak{X}_I$ denote the characteristic function of any subset I of a skew lattice S. i.e, $\mathfrak{X}_I(x) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{if } x \notin I. \end{cases}$

Lemma 3.1. *A fuzzy subset μ is a fuzzy ideal of a skew lattice S if and only if the level subset μ_t of S is an ideal of a skew lattice S where $t \in Im\mu$. In particular $\mathfrak{X}_I$ is a fuzzy ideal of S if and only if I is an ideal of S.*

Proof. Let S be a skew lattice and assume μ be a fuzzy ideal of S. Consider a level subset μ_t. Then

(1) For $x, y \in \mu_t$. Clearly $\mu(x), \mu(y) \geqslant t$. Then

$$\mu(x \vee y) = min(\mu(x), \mu(y)) \geqslant t.$$

Hence $x \vee y \in \mu_t$.

(2) For $y \in \mu_t$ and $x \in S$. Then

$$\mu(x \wedge y \wedge x) \geqslant \mu(y) \geqslant t.$$

Hence $x \wedge y \wedge x \in \mu_t$. This proves that μ_t is an ideal of a skew lattice S.

Conversely, suppose μ_t be an ideal of S. Let $x, y \in S$. Then $\mu(x) = t_1$ and $\mu(y) = t_2$ for some $t_1, t_2 \in [0, 1]$. Thus $x \in \mu_{t_1}$ and $y \in \mu_{t_2}$. Which implies that $x \in \mu_t$ and $y \in \mu_t$ for some $t \in [0, 1]$ such that $t = min\{t_1, t_2\}$. Since μ_t is an ideal we have $x \vee y \in \mu_t$. Hence $\mu(x \vee y) \geqslant t = min\{t_1, t_2\}$. Therefore

$\mu(x \vee y) \geqslant min\{\mu(x), \mu(y)\}$. Also, let $x, y \in S$ and $\mu(x \vee y) = t$ for some $t \in [0,1]$ which implies that $x \vee y \in \mu_t$. Clearly $x, y \preceq x \vee y$ and since μ_t is an ideal $x, y \in \mu_t$. Thus $\mu(x), \mu(y) \geqslant t$ (i.e, $\mu(x) \wedge \mu(y) \geqslant t$). Hence $\mu(x \vee y) \leqslant \mu(x) \wedge \mu(y)$. Therefore $\mu(x \vee y) = min\{\mu(x), \mu(y)\}$.

On the other hand let $x, y \in S$. Then $\mu(x) = t_1$ and $\mu(y) = t_2$ for some $t_1, t_2 \in [0,1]$. By assumption μ_{t_1} and μ_{t_2} are ideals. Clearly, due to a skew lattice is regular we obtain that $(x \wedge y \wedge x) \wedge x \wedge (x \wedge y \wedge x) = x \wedge y \wedge x$ and $(x \wedge y \wedge x) \wedge y \wedge (x \wedge y \wedge x) = x \wedge y \wedge x$. Thus as $x \in \mu_{t_1}$ and $y \in \mu_{t_2}$, from Definition 3.1 we have $x \wedge y \wedge x \in \mu_{t_1}, \mu_{t_2}$. This implies that $\mu(x \wedge y \wedge x) \geqslant t_1, t_2$. Hence $\mu(x \wedge y \wedge x) \geqslant max\{\mu(x), \mu(y)\} \geqslant \mu(y)$. Therefore μ is a fuzzy ideal of S.

Suppose I is an ideal of S and let $x, y \in S$. We show that $\mathfrak{X}_I$ is a fuzzy ideal of S. For this we consider different cases:

Case 1: if $x, y \in I$, then $x \vee y \in I$. Which implies that $\mathfrak{X}_I(x \vee y) = 1 = min\{\mathfrak{X}_I(x), \mathfrak{X}_I(y)\}$. From Definition 3.1, whenever $x, y \in I$ implies that $x \in S, y \in I$ or vice versa we have $x \wedge y \wedge x \in I$. Thus $\mathfrak{X}_I(x \wedge y \wedge x) = 1 \geqslant \mathfrak{X}_I(y)$. Hence $\mathfrak{X}_I$ is a fuzzy ideal of S.

Case 2: let $x \in S$ and $y \in I$. By the same reason as above $x \wedge y \wedge x \in I$. Thus $\mathfrak{X}_I(x \wedge y \wedge x) = 1 \geqslant \mathfrak{X}_I(y)$. For any $x, y \in S$ if $x \vee y \in I$, then $x \wedge (x \vee y) \wedge x = x \in I$ and $y \wedge (x \vee y) \wedge y = y \in I$. Hence $\mathfrak{X}_I(x \vee y) = 1 = min\{\mathfrak{X}_I(x), \mathfrak{X}_I(y)\}$. But if $x \vee y \notin I$, then $x \notin I$. Hence $\mathfrak{X}_I(x \vee y) = 0 = min\{0, 1\} = min\{\mathfrak{X}_I(x), \mathfrak{X}_I(y)\}$.

Case 3: Let $x, y \notin I$. Then $\mathfrak{X}_I(x) = 0 = \mathfrak{X}_I(y)$. Since $x \vee y \notin I$ we obtained that $\mathfrak{X}_I(x \vee y) = 0$. Following this what ever is $x \vee y$ belongs to I or not we have that $\mathfrak{X}_I(x \vee y) = min\{\mathfrak{X}_I(x), \mathfrak{X}_I(y)\}$. Also $\mathfrak{X}_I(x \wedge y \wedge x) \geqslant 0 = \mathfrak{X}_I(y)$. Therefore in all the three cases it is shown that $\mathfrak{X}_I$ is a fuzzy ideal of a skew lattice S.

Conversely suppose $\mathfrak{X}_I$ is a fuzzy ideal of S. We claim that I is an ideal of S. Let $x, y \in I$. Then as $\mathfrak{X}_I$ is a fuzzy ideal $\mathfrak{X}_I(x \vee y) = min\{\mathfrak{X}_I(x), \mathfrak{X}_I(y)\} = 1$. Which implies that $\mathfrak{X}_I(x \vee y) = 1$. Thus $x \vee y \in I$. Once again since $\mathfrak{X}_I$ is a fuzzy ideal of S, if $x \in S, y \in I$, then we have $\mathfrak{X}_I(x \wedge y \wedge x) \geqslant \mathfrak{X}_I(y) = 1$. Hence $\mathfrak{X}_I(x \wedge y \wedge x) = 1$. In turn it gives $x \wedge y \wedge x \in I$. Therefore I is an ideal of a skew lattice S. □

THEOREM 3.1. *Let μ be a fuzzy subset of S such that $\mu(x \vee y) = \mu(x) \wedge \mu(y)$. Then the following statements equivalently define a fuzzy ideal μ.*

(i) $\mu(x \wedge y \wedge x) \geqslant \mu(y)$
(ii) *if $x \preceq y$ for all $x, y \in S$, then $\mu(x) \geqslant \mu(y)$*
(iii) $\mu(x \wedge y), \mu(y \wedge x) \geqslant \mu(y)$.

PROOF. Suppose S be a skew lattice and μ be a fuzzy subset of S such that $\mu(x \vee y) = \mu(x) \wedge \mu(y)$. Assume (i) holds and let $x \preceq y$. Then

$$\begin{aligned} x = x \wedge y \wedge x \quad &\Rightarrow \quad \mu(x) = \mu(x \wedge y \wedge x) \\ &\Rightarrow \quad \mu(x) \geqslant \mu(y). \end{aligned}$$

Assume (ii) holds. Clearly $x \wedge y \preceq y$ and $y \wedge x \preceq y$. Thus by (ii) we obtained that $\mu(x \wedge y), \mu(y \wedge x) \geqslant \mu(y)$. Finally assume that $\mu(x \wedge y), \mu(y \wedge x) \geqslant \mu(y)$. Then $\mu(x \wedge y \wedge x) = \mu((x \wedge y) \wedge (y \wedge x)) \geqslant \mu(x \wedge y) \geqslant \mu(y)$ for all $x, y \in S$. Therefore in all the three cases μ is a fuzzy ideal of S. □

THEOREM 3.2. *Let μ be a fuzzy subset of a skew lattice S such that $\mu(x \vee y) = \mu(x) \wedge \mu(y)$. Then μ is a fuzzy ideal of S if and only if $\mu(x \vee y \vee x) = \mu(x) \wedge \mu(y)$.*

PROOF. Suppose μ be a fuzzy ideal of S. Then

$$\begin{aligned} \mu(x \vee y \vee x) &= \mu((x \vee y) \vee x) \\ &= \mu(x \vee y) \wedge \mu(x) \\ &= \mu(x) \wedge \mu(y) .... \lceil \text{ since } \mu(x \vee y) = \mu(x) \wedge \mu(y) \rfloor. \end{aligned}$$

Conversely, suppose $\mu(x \vee y \vee x) = \mu(x) \wedge \mu(y)$. Clearly,

$$\begin{aligned} x \wedge y \wedge x \preceq x, y &\Rightarrow y \vee (x \wedge y \wedge x) \vee y = y \\ &\Rightarrow \mu(y) = \mu(y \vee (x \wedge y \wedge x) \vee y) = \mu(y) \wedge \mu(x \wedge y \wedge x) \\ &\Rightarrow \mu(x \wedge y \wedge x) \geqslant \mu(y). \end{aligned}$$

Therefore μ is a fuzzy ideal of S. □

COROLLARY 3.1. *If μ is a fuzzy ideal of a skew lattice S, then $\mu(x \vee y) = \mu(x \vee y \vee x)$.*

PROOF. By Definition of a fuzzy ideal we have $\mu(x \vee y) = \mu(x) \wedge \mu(y)$ and since μ is a fuzzy ideal we have $\mu(x \vee y \vee x) = \mu(x) \wedge \mu(y)$. Therefore $\mu(x \vee y) = \mu(x \vee y \vee x)$. □

COROLLARY 3.2. *A fuzzy ideal μ of a skew lattice S is a fuzzy subskew lattice of S.*

PROOF. Suppose μ be a fuzzy ideal of S. Then, the fact $\mu(x \vee y) = \mu(x) \wedge \mu(y)$ implies that$\mu(x \vee y) \geqslant \mu(x) \wedge \mu(y)$ and $\mu(x \wedge y \wedge x) \geqslant \mu(y) \geqslant \mu(x) \wedge \mu(y)$. Since $x \wedge y \preceq x \wedge y \wedge x, \mu(x \wedge y) \geqslant \mu(x \wedge y \wedge x) \geqslant \mu(x) \wedge \mu(y)$. Hence $\mu((x \vee y) \wedge (x \wedge y)) \geqslant \mu(x) \wedge \mu(y)$ and therefore μ is a fuzzy subskew lattice. □

THEOREM 3.3. *If μ is a fuzzy ideal of a skew lattice S and $a, x \in S$, then $\mu(a \wedge x \wedge a) = \mu(x \wedge a \wedge x)$.*

PROOF. Let $y = x \wedge a \wedge x$

$$\begin{aligned} &\Rightarrow a \wedge y \wedge a = a \wedge (x \wedge a \wedge x) \wedge a = a \wedge x \wedge a \\ &\Rightarrow \mu(a \wedge x \wedge a) = \mu(a \wedge y \wedge a) \geqslant \mu(a), \mu(y) \\ &\Rightarrow \mu(a \wedge x \wedge a) \geqslant \mu(y) = \mu(x \wedge a \wedge x) \\ &\Rightarrow \mu(a \wedge x \wedge a) \geqslant \mu(x \wedge a \wedge x) \end{aligned}$$

Also let $y = a \wedge x \wedge a$.

$$\begin{aligned} &\Rightarrow x \wedge y \wedge x = x \wedge (a \wedge x \wedge a) \wedge x = x \wedge a \wedge x \\ &\Rightarrow \mu(x \wedge a \wedge x) = \mu(x \wedge y \wedge x) \geqslant \mu(x), \mu(y) \\ &\Rightarrow \mu(x \wedge a \wedge x) \geqslant \mu(y) = \mu(a \wedge x \wedge a) \end{aligned}$$

Therefore $\mu(x \wedge a \wedge x) = \mu(a \wedge x \wedge a)$. □

LEMMA 3.2. *Let μ be a fuzzy ideal of a skew lattice S. If $x \mathfrak{D} y$, then $\mu(x) = \mu(y)$*

PROOF. Suppose $x\mathfrak{D}y$. Then $x \wedge y \wedge x = x$ and $y \wedge x \wedge y = y$. Since μ is a fuzzy ideal of S, $\mu(x \wedge y \wedge x) \geqslant \mu(y)$ and $\mu(y \wedge x \wedge y) \geqslant \mu(x)$. This in turn implies that $\mu(x) \geqslant \mu(y)$ and $\mu(y) \geqslant \mu(x)$ respectively. Hence $\mu(x) = \mu(y)$. □

Consider $FI(S)$ is the class of all fuzzy ideals of S.

THEOREM 3.4. *The set of all fuzzy ideals of a strongly distributive skew lattice S is a complete lattice.*

PROOF. Let $\{\mu_i : i \in \Delta\}$ be a class of fuzzy ideals of S. Let μ be the point-wise infimum of $\{\mu_i\}_{i\in\Delta}$, that is $\mu(x) = \bigwedge_{i\in\Delta} \mu_i(x)$. Let $x, y \in S$. Then

$$\begin{aligned} \mu(x \vee y) &= \bigwedge_{i\in\Delta} \mu_i(x \vee y) \\ &= \bigwedge_{i\in\Delta} (\mu_i(x) \wedge \mu_i(y)) \\ &= (\bigwedge_{i\in\Delta} \mu_i(x)) \wedge (\bigwedge_{i\in\Delta} \mu_i(x)) \\ &= \mu(x) \wedge \mu(y). \end{aligned}$$

Also for any $x, y \in S$,

$$\begin{aligned} \mu(x \wedge y \wedge x) &= \bigwedge_{i\in\Delta} \mu_i(x \wedge y \wedge x) \\ &\geqslant \bigwedge_{i\in\Delta} \mu_i(y) \\ &= \mu(y). \end{aligned}$$

So $\mu(x \wedge y \wedge x) \geqslant \mu(y)$. Thus μ is a fuzzy ideal of S. Therefore $\mu \in FI(S)$. This shows that $FI(S)$ is a complete lattice. □

DEFINITION 3.4. For any ideal I of a skew lattice S and for each $\alpha \in [0, 1]$ the mapping α_I, $\alpha_I : S \longrightarrow [0, 1]$ defined by

$$\alpha_I(y) = \begin{cases} 1 & \text{if } y \in I \\ \alpha & \text{if } y \notin I \end{cases}$$

is called the α-level fuzzy ideal corresponding to I.

LEMMA 3.3. *α_I is a fuzzy ideal of S.*

PROOF. From the above definition we have

$$\alpha_I(x \vee y) = \begin{cases} 1 & \text{if } x \vee y \in I \\ \alpha & \text{if } x \vee y \notin I. \end{cases}$$

Clearly for any $x, y \in S$, $x, y \preceq x \vee y$. Thus if $x \vee y \in I$,then $x, y \in I$. In turn this shows that $\alpha_I(x) = 1, \alpha_I(y) = 1$. Hence $\alpha_I(x \vee y) = 1 = 1 \wedge 1 = \alpha_I(x) \wedge \alpha_I(y)$. Consider $x \vee y \notin I$ and assume that both $x, y \in I$. Thus as I is closed under $\vee$, $x \vee y \in I$ which is a contradiction. Thus either $x \notin I$ or $y \notin I$. This implies $\alpha_I(x) = \alpha$ or $\alpha_I(y) = \alpha$. Hence $\alpha_I(x \vee y) = \alpha = \alpha \wedge \alpha = \alpha_I(x) \wedge \alpha_I(y)$. Clearly,

$$\alpha_I(x \wedge y \wedge x) = \begin{cases} 1 & \text{if } x \wedge y \wedge x \in I \\ \alpha & \text{if } x \wedge y \wedge x \notin I \end{cases}$$

Suppose $x \wedge y \wedge x \in I$, then $\alpha_I(x \wedge y \wedge x) = 1 \geqslant \alpha_I(x) \vee \alpha_I(y) \geqslant \alpha_I(y)$. Consider that $x \wedge y \wedge x \notin I$. Suppose either x or y belongs to I. With out loss of generality, let $x \in I$. Since $x \wedge y \wedge x \preceq x$ we obtain that $x \wedge y \wedge x \in I$ which is a contradiction. Hence both $x, y \notin I$. This implies that $\alpha_I(x) = \alpha = \alpha_I(y)$. Thus $\alpha_I(x \wedge y \wedge x) = \alpha = \alpha \vee \alpha = \alpha_I(x) \vee \alpha_I(y) \geqslant \alpha_I(y)$. Therefore α_I is a fuzzy ideal. □

THEOREM 3.5. *For each $x, y \in S$ and $\alpha \in [0,1]$, define $\alpha_x : S \longrightarrow [0,1]$ by*

$$\alpha_x(y) = \begin{cases} 1 & \textit{if } y \preceq x \\ \alpha & \textit{otherwise.} \end{cases}$$

Then α_x is a fuzzy ideal of S.

PROOF. Let $y, z \in S$. If $y \vee z \preceq x$, then $x = x \vee (y \vee z) \vee x$. Thus

$$\begin{aligned} x \vee y \vee x &= (x \vee (y \vee z) \vee x) \vee y \vee (x \vee (y \vee z) \vee x) \\ &= x \vee (y \vee z \vee x \vee y \vee x \vee y \vee z) \vee x \\ &= x \vee y \vee z \vee x \vee y \vee z \vee x \\ &= x \vee (y \vee z) \vee (y \vee z) \vee x \\ &= x \vee (y \vee z) \vee x \\ &= x. \end{aligned}$$

Similarly,

$$\begin{aligned} x \vee z \vee x &= (x \vee y \vee z \vee x) \vee z \vee (x \vee y \vee z \vee x) \\ &= x \vee y \vee z \vee x \vee y \vee z \vee x \\ &= x \vee y \vee z \vee x \\ &= x. \end{aligned}$$

This implies that $y \preceq x$ and $z \preceq x$. Hence $\alpha_x(y \vee z) = 1 = 1 \wedge 1 = \alpha_x(y) \wedge \alpha_x(z)$.

Now assume that $y \vee z \npreceq x$ in such a way that both $y \preceq x$ and $z \preceq x$.

$$\begin{aligned} &\Rightarrow \quad x = x \vee y \vee x \text{ and } x = x \vee z \vee x \\ &\Rightarrow \quad x = x \vee y \vee x \vee z \vee x = x \vee y \vee z \vee x \\ &\Rightarrow \quad y \vee z \preceq x \end{aligned}$$

which is a contradiction. This shows that either $y \npreceq x$ or $z \npreceq x$, and hence $\alpha_x(y) = \alpha$ or $\alpha_x(z) = \alpha$ so that $\alpha_x(y \vee z) = \alpha = \alpha \wedge \alpha = \alpha_x(y) \wedge \alpha_x(z)$. Also let $y, z \in S$. If $z \wedge y \wedge z \preceq x$, then $\alpha_x(z \wedge y \wedge z) = 1 \geqslant \alpha_x(y) \vee \alpha_x(z) \geqslant \alpha_x(y)$. Consider $z \wedge y \wedge z \npreceq x$. Suppose both $y \preceq x$ and $z \preceq x \Rightarrow y \wedge x \wedge y = y$ and $z \wedge x \wedge z = z$. Thus $z \wedge y \wedge z \wedge x \wedge z \wedge y \wedge z = z \wedge y \wedge z \wedge y \wedge z = z \wedge y \wedge z \Rightarrow z \wedge y \wedge z \preceq x$ which is a contradiction. This implies that either $y \npreceq x$ or $z \npreceq x$. Thus $\alpha_x(y) = \alpha$ or $\alpha_x(z) = \alpha$ so that $\alpha_x(z \wedge y \wedge z) = \alpha = \alpha \vee \alpha = \alpha_x(y) \vee \alpha_x(z) \geqslant \alpha_x(y)$. Therefore α_x is a fuzzy ideal of S. □

α_x defined in the above theorem is called the $\alpha-$level principal fuzzy ideal corresponding to x. If S has largest element 1, then $\alpha_1 = \mathfrak{X}_S$.

Remark 3.1. Let S be a skew lattice and

$$S^{\downarrow} = \{y \in S : y \preceq x \text{ for some } x \in S\}.$$

Then $\alpha_{S^{\downarrow}} = \alpha_S = \alpha_1 = \mathfrak{X}_S$.

Proof. $\alpha_S(y) = \begin{cases} 1 & \text{if } y \in S \\ \alpha & \text{if } y \notin S \end{cases} = 1$, since every y belongs to S.

Also,

$$\begin{aligned} \alpha_{S^{\downarrow}}(y) &= \begin{cases} 1 & \text{if } y \in S^{\downarrow} \\ \alpha & \text{if } y \notin S^{\downarrow} \end{cases} \\ &= \begin{cases} 1 & \text{if } y \preceq x \ \text{ for some } x \in S \\ \alpha & \text{if } y \npreceq x \ \text{ for all } x \in S \end{cases} \\ &= 1, \text{ since } y \preceq y \text{ for any } y \in S. \end{aligned}$$

Therefore $\alpha_S = \alpha_{S^{\downarrow}}$. Similarly,

$$\alpha_1(y) = \begin{cases} 1 & \text{if } y \preceq 1 \\ \alpha & \text{if } y \npreceq 1 \end{cases} = 1, \text{ since } y \preceq 1 \text{ for every } y \in S.$$

and

$$\mathfrak{X}_S(y) = \begin{cases} 1 & \text{if } y \in S \\ 0 & \text{if } y \notin S \end{cases} = 1,$$

since the underline set is S and for all $y, y \in S$. Hence $\alpha_{S^{\downarrow}} = \alpha_S = \alpha_1 = \mathfrak{X}_S$. □

Theorem 3.6. *I is an ideal of a skew lattice S if and only if α_I is a fuzzy ideal of S.*

Proof. Suppose I be a nonempty subset of S such that α_I is a fuzzy ideal of S. Let $x, y \in I$. Then $\alpha_I(x) = \alpha_I(y) = 1$ and $\alpha_I(x \vee y) = \alpha_I(x) \wedge \alpha_I(y) = 1 \wedge 1 = 1 \Rightarrow x \vee y \in I$. Let $x \in S$ and $y \in I$. Clearly $x \wedge y \wedge x \preceq y$ and $\alpha_I(y) = 1$. Thus we claim that $x \wedge y \wedge x \in I$. Since α_I is a fuzzy ideal, $\alpha_I(x \wedge y \wedge x) \geqslant \alpha_I(y) = 1 \Rightarrow \alpha_I(x \wedge y \wedge x) = 1 \Rightarrow x \wedge y \wedge x \in I$. Therefore I is an ideal of a skew lattice. The for ward proof is done on the previous lemma. □

Theorem 3.7. *Suppose S be a skew lattice and $0 \neq \alpha \in [0, 1]$. Then $f : I(S) \longrightarrow F\alpha_I(S)$ defined by $f(I) = \alpha_I$ is an isomorphism.*

Proof. Suppose I and J are ideals of S. Then

$$\begin{aligned}
\alpha_I(y) \wedge \alpha_J(y) &= \begin{cases} 1 & \text{if } y \in I \\ \alpha & \text{if } y \notin I \end{cases} \wedge \begin{cases} 1 & \text{if } y \in J \\ \alpha & \text{if } y \notin J \end{cases} \\
&= \begin{cases} 1 & \text{if } y \in I \text{ and } y \in J \\ \alpha & \text{if } y \notin I \text{ and } y \notin J \end{cases} \\
&= \begin{cases} 1 & \text{if } y \in I \cap J \\ \alpha & \text{if } y \notin I \cap J \end{cases} \\
&= \alpha_{I\cap J}(y)
\end{aligned}$$

Hence (i): $\alpha_{I\cap J} = \alpha_I \wedge \alpha_J$.

Suppose $\alpha_I \leqslant \alpha_J$ and let $x \in I$. Then as $\alpha_I = \alpha_I \wedge \alpha_J$ we have $\alpha_I(x) \wedge \alpha_J(x) = \alpha_I(x) \Rightarrow 1 \wedge \alpha_J(x) = 1 \Rightarrow \alpha_J(x) = 1 \Rightarrow x \in J$. Hence $I \subseteq J$. Suppose $I \subseteq J$. If $x \in I$, then $\alpha_I(x) \wedge \alpha_J(x) = 1 \wedge 1 = 1 \Rightarrow \alpha_I(x) = 1 = \alpha_J(x) \Rightarrow \alpha_I = \alpha_J$. If $x \in J$ and $x \notin I$, then $\alpha_I(x) = \alpha, \alpha_J(x) = 1$. Hence $\alpha_I(x) \wedge \alpha_J(x) = \alpha \wedge 1 = \alpha = \alpha_I(x) \Rightarrow \alpha_I \leqslant \alpha_J$. If $x \notin I$ and $x \notin J$, then $\alpha_I(x) = \alpha_J(x) = \alpha$. Then $\alpha_I = \alpha_J$. Thus from the above three cases we conclude that $\alpha_I \leqslant \alpha_J$.

Therefore, (ii) $\alpha_I \leqslant \alpha_J \Leftrightarrow I \subseteq J$.

From (i) we have (iii), $f(I \wedge J) = \alpha_{I\wedge J} = \alpha_I \wedge \alpha_J = f(I) \wedge f(J)$.

We recall that $I \vee J = \{x \in S : x \leqslant a \vee b, a \in I, b \in J\}$ and $(\alpha_I \vee \alpha_J)(x) = \bigvee\{\bigwedge_{i=1}^{n}(\alpha_I(a_i) \vee \alpha_J(a_i)) : x \leqslant \bigvee_{i=1}^{n} a_i; a_i \in S\}$.

Since $I \subseteq I \vee J$ from (**) we have $\alpha_I \leqslant \alpha_{I\vee J}$ and $\alpha_J \leqslant \alpha_I \vee \alpha_J$. Hence (iv): $\alpha_I \vee \alpha_J \leqslant \alpha_{I\vee J}$.

Also $\alpha_I(x) = 1$ or $\alpha_I(x) = \alpha$ and $\alpha_J(x) = 1$ or $\alpha_J(x) = \alpha$ for any $x \in S$. Hence $(\alpha_I \vee \alpha_J)(x) \geqslant \alpha = \alpha_{I\vee J}(x)$, if $x \notin I \vee J$. That is $\alpha_I \vee \alpha_J \geqslant \alpha_{I\vee J}$ if $x \notin I \vee J$. On the other hand, suppose $x \in I \vee J$. Then there exist $a \in I$ and $b \in J$ such that $x \leqslant a \vee b$,

$$\begin{aligned}
(\alpha_I \vee \alpha_J)(x) &\geqslant (\alpha_I(a) \vee \alpha_J(a)) \wedge (\alpha_I(b) \vee \alpha_J(b)) \\
&\geqslant \alpha_I(a) \wedge \alpha_J(b) \\
&= 1 \wedge 1 \\
&= 1 \\
&= \alpha_{I\vee J}(x)
\end{aligned}$$

Consequently, (v) : $(\alpha_I \vee \alpha_J)(x) \geqslant \alpha_{I\vee J}(x)$ for all $x \in S$, (i.e. $\alpha_I \vee \alpha_J \geqslant \alpha_{I\vee J}$).

Hence using (iv) and (v) we have $\alpha_I \vee \alpha_J = \alpha_{I\vee J}$. Now $f(I \vee J) = \alpha_{I\vee J} = \alpha_I \vee \alpha_J = f(I) \vee f(J)$. Hence f is homomorphism. Suppose $f(I) = f(J) \Rightarrow \alpha_I = \alpha_J \Rightarrow I = J$. Thus f is an embedding of $I(S)$ in to $F\alpha_I(S)$. Therefore f is an isomorphism. □

THEOREM 3.8. *Let S be a skew lattice with 0. Let $\{I_\alpha\}_{\alpha\in[0,1]}$ be a class of ideals of S such that $\bigcap_{\alpha\in M} I_\alpha = I_{\bigvee_{\alpha\in M}\alpha}$ for any $M \subseteq [0,1]$. For any $x \in S$ define $\mu(x) = \bigvee\{\alpha \in [0,1] : x \in I_\alpha\}$. Then μ is a fuzzy ideal of S such that I_α is precisely* *the $\alpha-$cut of μ for any $\alpha \in [0,1]$. Conversely, every fuzzy ideal of S can be obtained as above.*

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