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DISTANCE BASED F-INDEX OF SOME GRAPH PRODUCTS

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Abstract. In this paper, distance based F index of various types of products of graphs have been found.

1. Introduction

In this paper, all graphs considered are simple, connected and finite. Let $G = (V(G), E(G))$ be a connected graph of order p_i. For a graph G, the degree of a vertex v is the number of edges incident to v and denoted by $d_G(v)$. The number of edges of G is denoted by q_i.

For any $u, v \in V(G)$, the distance between u and v in G, symbolized by $d_G(u, v)$, is the length of a shortest (u, v)-path in G.

A topological index is a numerical quantity related to a graph that is invariant under graph automorphism. H. Wiener [**10**] in 1947, introduced a topological index based on the distance $d_G(u, v)$ which is named as Wiener index and it is established as

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u, v) = \frac{1}{2} \sum_{u,v\in V(G)} d_G(u, v).$$

There are some degree based topological indices of a graph which are known as Zagreb indices, established by Gutman et al. in [**3**]. The first Zagreb index $M_1(G)$

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and the second Zagreb index $M_2(G)$ of a graph G are established respectively:

$$\begin{aligned} M_1(G) &= \sum_{uv\in E(G)} [d_G(u)+d_G(v)] = \sum_{v\in V(G)} d_G^2(v) \\ M_2(G) &= \sum_{uv\in E(G)} d_G(u)d_G(v) \end{aligned}$$

In [**6**], Khalifeh et. al derived the first and second Zagreb indices of some graph operations. The degree distance was proposed by Dobrynin and Kochetova [**1**] and Gutman [**4**] as a weighted version of the Wiener index. The degree distance of G, symbolized by $DD(G)$, is established as

$$DD(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v)[d_G(u)+d_G(v)] = \frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v)[d_G(u)+d_G(v)].$$

In [**4**], the sum of cubes of vertex degrees was involved in the investigation of the total $\pi-$ electron energy and it was again investigated by B.Furtula et. al. in [2] as "Forgotten Topological index" or "F-index" . It is established for a graph G as

$$F(G) = \sum_{v\in V(G)} d_G(v)^3 = \sum_{uv\in E(G)} [d_G(u)^2 + d_G(v)^2].$$

Nilanjan De et. al. in [8] derived the exact values for $F-$ index of cetain graph operations.

In [**7**] , Muruganandam et al. have introduced the concept of distance version of F-index which is symbolized by $DF(G)$ and it is established as

$$\begin{aligned} DF(G) &= \sum_{\{u,v\}\subseteq V(G)} d_G(u,v)[d_G(u)^2+d_G(v)^2] \\ &= \frac{1}{2}\sum_{u,v\in V(G)} d_G(u,v)[d_G(u)^2+d_G(v)^2]. \end{aligned}$$

The *strong product* of the graphs G_1 and G_2, symbolized by $G_1 \boxtimes G_2$, is the graph with vertex set $V(G_1)\times V(G_2)$ and two vertices (u_1,u_2) and (v_1,v_2) are adjacent whenever $(i)u_1 = v_1$ and $u_2v_2 \in E(G_2)$, or, $(ii)u_2 = v_2$ and $u_1v_1 \in E(G_1)$, or, $(iii)u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$.

The *tensor product* of the graphs G_1 and G_2, symbolized by $G_1 \times G_2$, is the graph with vertex set $V(G_1\times G_2)$ and $E(G_1\times G_2) = \big\{(u_1,v_1)(u_2,v_2)|u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)\big\}$.

The *corona product* of the graphs G_1 and G_2, symbolized by $G_1 \odot G_2$, is the graph attained by taking one copy of G_1 and $|V(G_1|$ disjoint copies of G_2, and then joining the i^{th} vertex of G_1 to every vertex in i^{th} copy of G_2.

In [**9**], degree distance and Gutman index of corona product of graphs are attained.

In this present work, we attain the absolute values of distance based $F-$ index of strong, corona and tensor products of graphs.

2. Distance Based F-index of Strong Product of Graphs

In this section, we find the exact value of the distance based F- index of strong product $G_1 \boxtimes G_2$.

Lemma 2.1 ([**5**]). *The degree of the vertex* (u_i, v_l) *of* $V(G_1 \boxtimes G_2)$ *is*

$$d_{G_1}(u_i) + d_{G_2}(v_l) + d_{G_1}(u_i)d_{G_2}(v_l).$$

That is

$$d_{G_1\boxtimes G_2}(u_i, v_l) = d_{G_1}(u_i) + d_{G_2}(v_l) + d_{G_1}(u_i)d_{G_2}(v_l).$$

Lemma 2.2 ([**5**]). *Let* $w_{il} = (u_i, v_l)$ *and* $w_{mn} = (u_m, v_n)$ *be in* $V(G_1 \boxtimes G_2)$. *Then the distance between* w_{il} *and* w_{mn} *is*

$$d_{G_1\boxtimes G_2}(w_{il}, w_{mn}) = \begin{cases} d_{G_2}(v_l, v_n), & i = m, l \neq n \\ d_{G_1}(u_i, u_m), & i \neq m, l = n \\ d_{G_2}(v_l, v_n), & i \neq m, l \neq n. \end{cases}$$

Theorem 2.1. *Let* $G_i, i = 1, 2$, *be a* (p_i, q_i)*- graph. Then*

$$\begin{aligned} &2 \times DF(G_1 \boxtimes G_2) = \\ &2p_2DF(G_1) + 4M_1(G_2)W(G_1) + 2DF(G_1)M_1(G_2) + 8DD(G_1)q_2 \\ &+ 4DD(G_1)M_1(G_2) + 8DF(G_1)q_2 + 2p_1DF(G_2) + 4M_1(G_1)W(G_2) \\ &+ 2DF(G_2)M_1(G_1) + 8DD(G_2)q_1 + 4DD(G_2)M_1(G_1) + 8DF(G_2)q_1 \\ &+ 2p_1(p_1 - 1)DF(G_2) + 4(p_1 - 1)M_1(G_1)W(G_2) + 8M_1(p_1 - 1)DD(G_2) \\ &+ 2(p_1 - 1)M_1(G_1)DF(G_2) + 8q_1(p_1 - 1)DF(G_2) \\ &+ 4(p_1 - 1)M_1(G_1)DD(G_2) \end{aligned}$$

Proof. Let $G = G_1 \boxtimes G_2$. Then,

$$\begin{aligned} 2 \times DF(G) &= \sum_{w_{il}, w_{mn} \in V(G)} d_G(w_{il}, w_{mn})[d_G^2(w_{il}) + d_G^2(w_{mn})] \\ &= \sum_{l=0}^{p_2-1} \sum_{i,m=0, i\neq m}^{p_1-1} d_G(w_{il}, w_{ml})[d_G^2(w_{il}) + d_G^2(w_{ml})] \\ &+ \sum_{i=0}^{p_1-1} \sum_{l,n=0, l\neq m}^{p_2-1} d_G(w_{il}, w_{in})[d_G^2(w_{il}) + d_G^2(w_{in})] \\ &+ \sum_{l,n=0, l\neq n}^{p_2-1} \sum_{i,m=0, i\neq m}^{p_1-1} d_G(w_{il}, w_{mn})[d_G^2(w_{il}) + d_G^2(w_{mn})] \\ &= S_1 + S_2 + S_3, \end{aligned}$$

where $S_1,\ S_2,\ S_3$ are the sums of the above terms in order. We calculate $S_1,\ S_2,$ and S_3 separately. First we calculate S_1.

$$\begin{aligned}
S_1 &= \sum_{l=0}^{p_2-1} \sum_{i,m=0,i\neq m}^{p_1-1} d_G(w_{il}, w_{ml})[d_G^2(w_{il}) + d_G^2(w_{ml})] \\
&= \sum_{l=0}^{p_2-1} \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i, u_m)\Big[d_{G_1}(u_i) + d_{G_2}(v_l) + d_{G_1}(u_i)d_{G_2}(v_l)\Big]^2 \\
&+ \Big[d_{G_1}(u_m) + d_{G_2}(v_l) + d_{G_1}(u_m)d_{G_2}(v_l)\Big]^2 \\
&= \sum_{l=0}^{p_2-1} \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i, u_m)\Big\{\Big[d_{G_1}^2(u_i) + d_{G_2}^2(v_l) + d_{G_1}^2(u_i)d_{G_2}^2(v_l) \\
&+ 2d_{G_1}(u_i)d_{G_2}(v_l) + 2d_{G_1}(u_i)d_{G_2}^2(v_l) + 2d_{G_1}^2(u_i)d_{G_2}(v_l)\Big] \\
&+ \Big[d_{G_1}^2(u_m) + d_{G_2}^2(v_l) + d_{G_1}^2(u_m)d_{G_2}^2(v_l) \\
&+ 2d_{G_1}(u_m)d_{G_2}(v_l) + 2d_{G_1}(u_m)d_{G_2}^2(v_l) + 2d_{G_1}^2(u_m)d_{G_2}(v_l)\Big]\Big\} \\
&= \sum_{l=0}^{p_2-1} \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i, u_m)\Big\{[d_{G_1}^2(u_i) + d_{G_1}^2(u_m)] \\
&+ 2d_{G_2}^2(v_l) + [d_{G_1}^2(u_i) + d_{G_1}^2(u_m)]d_{G_2}^2(v_l) \\
&+ 2[d_{G_1}(u_i) + d_{G_1}(u_m)]d_{G_2}(v_l) + 2[d_{G_1}(u_i) + d_{G_1}(u_m)]d_{G_2}^2(v_l) \\
&+ 2[d_{G_1}^2(u_i) + d_{G_1}^2(u_m)]d_{G_2}(v_l)\Big\} \\
&= \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i, u_m)[d_{G_1}^2(u_i) + d_{G_1}^2(u_m)] \sum_{l=0}^{p_2-1} 1 \\
&+ 2\sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i, u_m) \sum_{l=0}^{p_2-1} d_{G_2}^2(v_l) \\
&+ \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i, u_m)[d_{G_1}^2(u_i) + d_{G_1}^2(u_m)] \sum_{l=0}^{p_2-1} d_{G_2}^2(v_l) \\
&+ 2\sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i, u_m)[d_{G_1}(u_i) + d_{G_1}(u_m)] \sum_{l=0}^{p_2-1} d_{G_2}(v_l) \\
&+ 2\sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i, u_m)[d_{G_1}(u_i) + d_{G_1}(u_m)] \sum_{l=0}^{p_2-1} d_{G_2}^2(v_l) \\
&+ 2\sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i, u_m)[d_{G_1}^2(u_i) + d_{G_1}^2(u_m)] \sum_{l=0}^{p_2-1} d_{G_2}(v_l) \\
&= 2p_2DF(G_1) + 4M_1(G_2)W(G_1) + 2DF(G_1)M_1(G_2) + 8DD(G_1)q_2 \\
&+ 4DD(G_1)M_1(G_2) + 8DF(G_1)q_2
\end{aligned}$$

Next we compute S_2.

$$\begin{aligned}
S_2 &= \sum_{i=0}^{p_1-1} \sum_{l,n=0,l\neq n}^{p_2-1} d_G(w_{il}, w_{in})[d_G(w_{il})^2 + d_G(w_{in})^2] \\
&= \sum_{i=0}^{p_1-1} \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l, v_n)\Big\{\Big[d_{G_1}(u_i) + d_{G_2}(v_l) + d_{G_1}(u_i)d_{G_2}(v_l)\Big]^2 \\
&+ \Big[d_{G_1}(u_i) + d_{G_2}(v_n) + d_{G_1}(u_i)d_{G_2}(v_n)\Big]^2\Big\} \\
&= \sum_{i=0}^{p_1-1} \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l, v_n)\Big\{\Big[d^2_{G_1}(u_i) + d^2_{G_2}(v_l) + d^2_{G_1}(u_i)d^2_{G_2}(v_l) \\
&+ 2d_{G_1}(u_i)d_{G_2}(v_l) + 2d_{G_1}(u_i)d^2_{G_2}(v_l) + 2d^2_{G_1}(u_i)d_{G_2}(v_l)\Big] \\
&+ \Big[d^2_{G_1}(u_i) + d^2_{G_2}(v_n) + d^2_{G_1}(u_i)d^2_{G_2}(v_n) \\
&+ 2d_{G_1}(u_i)d_{G_2}(v_n) + 2d_{G_1}(u_i)d^2_{G_2}(v_n) + 2d^2_{G_1}(u_i)d_{G_2}(v_n)\Big]\Big\} \\
&= \sum_{i=0}^{p_1-1} \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l, v_n)\Big\{[d^2_{G_2}(v_l) + d^2_{G_2}(v_n)] + 2d^2_{G_1}(u_i) \\
&+ [d^2_{G_2}(v_l) + d^2_{G_2}(v_n)]d^2_{G_1}(u_i) \\
&+ 2[d_{G_2}(v_l) + d_{G_2}(v_n)]d_{G_1}(u_i) + 2[d_{G_2}(v_l) + d_{G_2}(v_n)]d^2_{G_1}(u_i) \\
&+ 2[d^2_{G_2}(v_l) + d^2_{G_2}(v_n)]d_{G_1}(u_i)\Big\} \\
&= \sum_{i=0}^{p_1-1} 1 \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l, v_n)[d^2_{G_2}(v_l) + d^2_{G_2}(v_n)] \\
&+ 2\sum_{i=0}^{p_1-1} d^2_{G_1}(u_i) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l, v_n) \\
&+ \sum_{i=0}^{p_1-1} d^2_{G_1}(u_i) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l, v_n)[d^2_{G_2}(v_l) + d^2_{G_2}(v_n)] \\
&+ 2\sum_{i=0}^{p_1-1} d_{G_1}(u_i) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l, v_n)[d_{G_2}(v_l) + d_{G_2}(v_n)] \\
&+ 2\sum_{i=0}^{p_1-1} d^2_{G_1}(u_i) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l, v_n)[d_{G_2}(v_l) + d_{G_2}(v_n)] \\
&+ \sum_{i=0}^{p_1-1} d_{G_1}(u_i) \sum_{l,n=0,l\neq n}^{p_2-1} d^2_{G_2}(v_l, v_n)[d^2_{G_2}(v_l) + d_{G_2}(v_n)] \\
&= 2p_1DF(G_2) + 4M_1(G_1)W(G_2) + 2DF(G_2)M_1(G_1) + 8DD(G_2)q_1 \\
&+ 4DD(G_2)M_1(G_1) + 8DF(G_2)q_1
\end{aligned}$$

Finally we compute S_3.

$$
\begin{aligned}
S_3 &= \sum_{l,n=0,l\neq n}^{p_2-1} \sum_{i,m=0,i\neq m}^{p_1-1} d_G(w_{il}, w_{mn})[d_G(w_{il})^2 + d_G(w_{mn})^2] \\
&= \sum_{l,n=0,l\neq n}^{p_2-1} \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_2}(v_l, v_n)\Big\{\Big[d_{G_1}(u_i) + d_{G_2}(v_l) + d_{G_1}(u_i)d_{G_2}(v_l)\Big]^2 \\
&+ \Big[d_{G_1}(u_m) + d_{G_2}(v_n) + d_{G_1}(u_m)d_{G_2}(v_n)\Big]^2\Big\} \\
&= \sum_{l,n=0,l\neq n}^{p_2-1} \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_2}(v_l, v_n)\Big\{d^2_{G_1}(u_i) + d^2_{G_2}(v_l) + d^2_{G_1}(u_i)d^2_{G_2}(v_l) \\
&+ 2d_{G_1}(u_i)d_{G_2}(v_l) + 2d^2_{G_1}(u_i)d_{G_2}(v_l) + 2d_{G_1}(u_i)d^2_{G_2}(v_l) \\
&+ d^2_{G_1}(u_m) + d^2_{G_2}(v_n) + d^2_{G_1}(u_m)d^2_{G_2}(v_n) \\
&+ 2d_{G_1}(u_m)d_{G_2}(v_n) + 2d^2_{G_1}(u_m)d_{G_2}(v_n) + 2d_{G_1}(u_m)d^2_{G_2}(v_n)\Big\} \\
&= \sum_{i,m=0,i\neq m}^{p_1-1} 1 \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l, v_n)[d^2_{G_2}(v_l) + d^2_{G_2}(v_n)] \\
&+ \sum_{i,m=0,i\neq m}^{p_1-1} [d^2_{G_1}(u_i) + d^2_{G_1}(u_m)] \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l, v_n) \\
&+ 2\sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l, v_n)d_{G_2}(v_l) \\
&+ 2\sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_m) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l, v_n)d_{G_2}(v_n) \\
&+ \sum_{i,m=0,i\neq m}^{p_1-1} d^2_{G_1}(u_i) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l, v_n)d^2_{G_2}(v_l) \\
&+ \sum_{i,m=0,i\neq m}^{p_1-1} d^2_{G_1}(u_m) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l, v_n)d^2_{G_2}(v_n) \\
&+ 2\sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l, v_n)d^2_{G_2}(v_l) \\
&+ 2\sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_m) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l, v_n)d^2_{G_2}(v_n) \\
&+ 2\sum_{i,m=0,i\neq m}^{p_1-1} d^2_{G_1}(u_i) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l, v_n)d_{G_2}(v_l) \\
&+ 2\sum_{i,m=0,i\neq m}^{p_1-1} d^2_{G_1}(u_m) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l, v_n)d_{G_2}(v_n)
\end{aligned}
$$

and finally

$$\begin{aligned} S_3 &= 2p_1(p_1-1)DF(G_2)+4(p_1-1)M_1(G_1)W(G_2)+8q_1(p_1-1)DD(G_2) \\ &+ 2(p_1-1)M_1(G_1)DF(G_2)+8q_1(p_1-1)DF(G_2) \\ &+ 4(p_1-1)M_1(G_1)DD(G_2) \end{aligned}$$

Adding S_1, S_2 and S_3 we get,

$2 \times DF(G_1 \boxtimes G_2) =$
$2p_2DF(G_1)+4M_1(G_2)W(G_1)+2DF(G_1)M_1(G_2)+8DD(G_1)q_2$
$+4DD(G_1)M_1(G_2)+8DF(G_1)q_2+2p_1DF(G_2)+4M_1(G_1)W(G_2)$
$+2DF(G_2)M_1(G_1)+8DD(G_2)q_1+4DD(G_2)M_1(G_1)+8DF(G_2)q_1$
$+2p_1(p_1-1)DF(G_2)+4(p_1-1)M_1(G_1)W(G_2)+8q_1(p_1-1)DD(G_2)$
$+2(p_1-1)M_1(G_1)DF(G_2)+8q_1(p_1-1)DF(G_2)+4(p_1-1)M_1(G_1)DD(G_2)$. □

3. Distance Based of F-index of Corona Product of Graphs

In this section, we find the exact value of the distance based F- index of corona product $G_1 \odot G_2$.

Let $V(G_1) = \{u_0, u_1, ..., u_{p_1-1}\}$ and $V(G_2) = \{v_0, v_1, ..., v_{p_2-1}\}$. For $0 \leqslant i \leqslant p_1 - 1$, denote by G_2^i the i^{th} copy of G_2 joined to the vertex u_i and $V(G_2^i) = \{v_{i0}, v_{i1}, ..., v_{i(p_2-1)}\}$.

Lemma 3.1 ([**9**]). *The degree of $w \in V(G_1 \odot G_2)$ is*

$$d_{G_1\odot G_2}(w) = \begin{cases} d_{G_1}(w)+p_2 & if\ w \in V(G_1) \\ d_{G_1}(w)+p_2 & if\ w \in V(G_1^i)\ for\ some\ 0 \leqslant i \leqslant p_1-1. \end{cases}$$

Lemma 3.2 ([**9**]). *Let G_1 be arbitrary graph.Let G_2^i be the i^{th} copy of G_2 in $d_{G_1\odot G_2}$ and let $V(G_2^i) = \{v_{i0}, v_{i1}, ..., v_{i(p_2-1)}\}$. Then*

$d_{(G_1\odot G_2)}(u_i, u_m) = d_{G_1}(u_i, u_m),\ if\ 0 \leqslant i, m \leqslant p_1 - 1,$

$d_{(G_1\odot G_2)}(u_i, v_{mn}) = d_{G_1}(u_i, u_m) + 1,\ if\ 0 \leqslant i, m \leqslant p_1 - 1, 0 \leqslant n \leqslant p_2 - 1,$

$$d_{G_1\odot G_2}(v_{il}, v_{mn}) = \begin{cases} d_{G_1}(u_i, u_m)+2, & if\ i \neq m, \\ 1, & if\ i = m,\ and\ v_lv_n \in E(G_2), \\ 2, & if\ i = m,\ and\ v_lv_n \notin E(G_2). \end{cases}$$

Theorem 3.1. *Let $G_i, i = 1, 2,$ be a (p_i, q_i)- graph. Then*

$2 \times DF(G_1 \odot G_2) =$
$2DF(G_1)+4p_2^2W(G_1)+4p_2DD(G_1)+2p_1\Big[2p_2M_1(G_2)+8p_2q_2$
$+2p_2(p_2-1)-F(G_2)-4M_1(G_2)-10q_2\Big]+2p_2DF(G_1)+4p_2^2DD(G_1)$
$+4W(G_1)[p_2(p_2^2+1)M_1(G_2)+4q_2]+2p_1p_2[M_1(G_1)+4q_1p_2]$
$+2p_1^2[M_1(G_2)+4q_2+p_2(p_2^2+1)]+4[W(G_1)+p_1(p_1-1)][p_2M_1(G_2)$
$+4p_2q_2+p_2^2]$.

PROOF. Let $G = G_1 \odot G_2$. Then,

$$\begin{aligned}2 \times DF(G) &= \sum_{u,v \in V(G)} d_G(u,v)[d_G(u)^2 + d_G(v)^2] \\ &= \sum_{i,m=0, i\neq m}^{p_1-1} d_G(u_i,u_m)[d_G(u_i)^2 + d_G(u_m)^2] \\ &+ \sum_{i=0}^{p_1-1} \sum_{l,n=0,l\neq m}^{p_2-1} d_G(v_{il},v_{in})[d_G(v_{il})^2 + d_G(v_{in})^2] \\ &+2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1}\sum_{n=0}^{p_2-1}[d_G(u_i,v_{mn})[d_G(u_i)^2 + d_G(v_{mn})^2] \\ &+ \sum_{i,m=0,i\neq m}^{p_1-1}\sum_{l,n=0}^{p_2-1} d_G(v_{il},v_{mn})[d_G(v_{il}) + d_G(v_{mn})] \\ &= S_1 + S_2 + S_3 + S_4\end{aligned}$$

where , S_1, S_2, S_3 S_4 are the sums of the above terms in order. We calculate S_1, S_2, S_3 and S_4 separately.

First we compute S_1

$$\begin{aligned}S_1 &= \sum_{i,m=0,i\neq m}^{p_1-1} d_G(u_i,u_m)[d_{G_1}(u_i)^2 + d_{G_1}(u_m)^2 \\ &= \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m)\Big[\big(d_{G_1}(u_i)+p_2\big)^2 + \big(d_{G_1}(u_m)+p_2\big)^2\Big] \\ &= \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m)\Big[d_{G_1}^2(u_i) + p_2^2 + 2p_2 d_{G_1}(u_i) + d_{G_1}^2(u_m) \\ &+ p_2^2 + 2p_2 d_{G_1}(u_m)\Big]\end{aligned}$$

$$\begin{aligned}S_1 &= \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m)\Big[d_{G_1}^2(u_i) + d_{G_1}^2(u_i)\Big] \\ &+ 2p_2^2 \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m) \\ &+ 2p_2 \sum_{i,m=0,i\neq m}^{p_1-1} d_{G_1}(u_i,u_m)\Big[d_{G_1}(u_i) + d_{G_1}(u_m)\Big] \\ &= 2DF(G_1) + 4p_2^2 W(G_1) + 4p_2 DD(G_1)\end{aligned}$$

Next we compute S_2

$$\begin{aligned}
S_2 &= \sum_{i=0}^{p_1-1} \sum_{l,n=0,l\neq n}^{p_2-1} d_G(v_{il},v_{in})\big[d_G^2(v_{il})^2 + d_G(v_{in})^2\big] \\
&= \sum_{i=0}^{p_1-1} \Big\{ \sum_{l,n=0,l\neq n,\ v_lv_n\in E(G_2)}^{p_2-1} d_G(v_{il},v_{in})\big[d_G(v_{il})^2 + d_G(v_{in})^2\big] \\
&+ \sum_{l,n=0,l\neq n,\ v_lv_n\notin E(G_2)}^{p_2-1} d_G(v_{il},v_{in})\big[d_G(v_{il})^2 + d_G(v_{in})^2\big]\Big\} \\
&= \sum_{i=0}^{p_1-1} \Big\{ \sum_{l,n=0,l\neq n,\ v_lv_n\in E(G_2)}^{p_2-1} \Big[\Big(d_{G_2}(v_l)+1\Big)^2 + \Big(d_{G_2}(v_n)+1\Big)^2\Big] \\
&+ \sum_{l,n=0,l\neq n,\ v_lv_n\notin E(G_2)}^{p_2-1} 2\Big[\Big(d_{G_2}(v_l)+1\Big)^2 + \Big(d_{G_2}(v_n)+1\Big)^2\Big]\Big\}
\end{aligned}$$

$$\begin{aligned}
S_2 = \sum_{i=0}^{p_1-1} \Big\{ & \sum_{l,n=0,l\neq n,\ v_lv_n\in E(G_2)}^{p_2-1} \Big[d_{G_2}^2(v_l) + d_{G_2}^2(v_n) + 2\Big(d_{G_2}(v_l)+d_{G_2}(v_n)\Big) + 2\Big] \\
&+ \sum_{l,n=0,l\neq n,\ v_lv_n\notin E(G_2)}^{p_2-1} 2\Big[d_{G_2}^2(v_l) + d_{G_2}^2(v_n) + 2\Big(d_{G_2}(v_l)+d_{G_2}(v_n)\Big) + 2\Big]\Big\} \\
= \sum_{i=0}^{p_1-1} \Big\{\Big[& \sum_{l,n=0,l\neq n,\ v_lv_n\in E(G_2)}^{p_2-1} \big[d_{G_2}^2(v_l) + d_{G_2}^2(v_n) + 2\Big(d_{G_2}(v_l)+d_{G_2}(v_n)\Big) + 2\big] \\
&+ \sum_{l,n=0,l\neq n,\ v_lv_n\notin E(G_2)}^{p_2-1} \big[d_{G_2}^2(v_l) + d_{G_2}^2(v_n) + 2\Big(d_{G_2}(v_l)+d_{G_2}(v_n)\Big) + 2\big]\Big] \\
&+ \sum_{l,n=0,l\neq n,\ v_lv_n\notin E(G_2)}^{p_2-1} \big[d_{G_2}^2(v_l) + d_{G_2}^2(v_n) + 2\Big(d_{G_2}(v_l)+d_{G_2}(v_n)\Big) + 2\big]\Big\}
\end{aligned}$$

$$
\begin{aligned}
S_2 &= \sum_{i=0}^{p_1-1}\Big\{\sum_{l,n=0,l\neq n,}^{p_2-1}\big[d^2_{G_2}(v_l)+d^2_{G_2}(v_n)+2\Big(d_{G_2}(v_l)+d_{G_2}(v_n)\Big)+2\big]\\
&+ \Big[\sum_{v_lv_n\in E(G_2)}^{p_2-1}\big[d^2_{G_2}(v_l)+d^2_{G_2}(v_n)+2\Big(d_{G_2}(v_l)+d_{G_2}(v_n)\Big)+2\big]\\
&+ \sum_{v_lv_n\in E(G_2)}^{p_2-1}\big[d^2_{G_2}(v_l)+d^2_{G_2}(v_n)+2\Big(d_{G_2}(v_l)+d_{G_2}(v_n)\Big)+2\big]\Big]\\
&- \sum_{l,n=0,l\neq n,\ v_lv_n\notin E(G_2)}^{p_2-1}\big[d^2_{G_2}(v_l)+d^2_{G_2}(v_n)+2\Big(d_{G_2}(v_l)+d_{G_2}(v_n)\Big)+2\big]\Big\}\\
&= \sum_{i=0}^{p_1-1}\Big\{2\sum_{l,n=0,l\neq n,}^{p_2-1}\Big[d^2_{G_2}(v_l)+d^2_{G_2}(v_n)+2\Big(d_{G_2}(v_l)+d_{G_2}(v_n)\Big)+2\Big]\\
&- 2\sum_{l,n=0,l\neq n,\ v_lv_n\notin E(G_2)}^{p_2-1}\Big[d^2_{G_2}(v_l)+d^2_{G_2}(v_n)+2\Big(d_{G_2}(v_l)+d_{G_2}(v_n)\Big)+2\Big]\Big\}\\
&= \sum_{i=0}^{p_1-1}\Big[4(p_2-1)q_1(G_2)+16(p_2-1)q_2+8p_2C_2-2F(G_2)-4q_1(G_2)-4q_2\Big]\\
&= \sum_{i=0}^{p_1-1}\Big[4p_2M_1(G_2)+16p_2q_2+4p_2(p_2-1)-2F(G_2)-8M_1(G_2)-20q_2\Big]\\
&= 2p_1\Big[2p_2M_1(G_2)+8p_2q_2+2p_2(p_2-1)-F(G_2)-4M_1(G_2)-10q_2\Big]
\end{aligned}
$$

Next we compute S_3

$$
S_3 = 2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1}\sum_{n=0}^{p_2-1} d_G(u_i,v_{mn})\Big[d_G(u_i)^2+d_G(v_{mn})^2\Big]
$$

$$
= 2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1}\sum_{n=0}^{p_2-1}\big[d_{G_1}(u_i,u_m)+1\big]\Big[\Big(d_{G_1}(u_i)+p_2\Big)^2+\Big(d_{G_2}(v_n)+1\Big)^2\Big]
$$

$$
\begin{aligned}
= 2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1}\sum_{n=0}^{p_2-1}\Big\{d_{G_1}(u_i,u_m)\Big[d^2_{G_1}(u_i)+2p_2d_{G_1}(u_i)+(p_2^2+1)+d^2_{G_2}(v_n)\\
+2d_{G_2}(v_n)\Big]\\
+\Big[d^2_{G}(u_i)+2p_2d_{G_1}(u_i)+(p_2^2+1)+d^2_{G_2}(v_n)+2d_{G_2}(v_n)\Big]\Big\}
\end{aligned}
$$

$$S_3 = 2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1}\Big[p_2 d_{G_1}(u_i,u_m)d^2_{G_1}(u_i) + 2p_2^2 d_{G_1}(u_i,u_m)d_{G_1}(u_i)$$
$$+d_{G_1}(u_i,u_m)p_2(p_2^2+1)$$
$$+d_{G_1}(u_i,u_m)q_1(G_2) + 4q_2 d_{G_1}(u_i,u_m) + p_2 d^2_{G_1}(u_i) + 2p_2^2 d_{G_1}(u_i) + p_2(p_2^2+1)$$
$$+q_1(G_2) + 4q_2\Big]$$
$$= 2\Big\{\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} p_2 d_{G_1}(u_i,u_m)d^2_{G_1}(u_i) + 2p_2^2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} d_{G_1}(u_i,u_m)d_{G_1}(u_i)$$
$$+p_2(p_2^2+1)\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} d_{G_1}(u_i,u_m) + M_1(G_2)\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} d_{G_1}(u_i,u_m)$$
$$+4q_2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} d_{G_1}(u_i,u_m) + p_2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} d^2_{G_1}(u_i)$$
$$+2p_2^2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} d_{G_1}(u_i) + p_2(p_2^2+1)\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} 1$$
$$+M_1(G_2)\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} 1 + 4q_2\sum_{m=0}^{p_1-1}\sum_{i=0}^{p_1-1} 1\Big\}$$
$$= 2p_2 DF(G_1) + 4p_2^2 DD(G_1) + 4W(G_1)\big[p_2(p_2^2+1)M_1(G_2) + 4q_2\big]$$
$$+2p_1p_2[M_1(G_1) + 4q_1p_2] + 2p_1^2\big[M_1(G_2) + 4q_2 + p_2(p_2^2+1)\big]$$

Finally, we calculate S_4

$$\begin{aligned}
S_4 &= \sum_{i,m=0,i\neq m}^{p_1-1}\sum_{l,n=0}^{p_2-1} d_G(v_{il},v_{mn})\Big(d^2_G(v_{il}) + d^2_G(v_{mn})\Big)\\
&= \sum_{i,m=0,i\neq m}^{p_1-1}\sum_{l,n=0}^{p_2-1}\Big(d_{G_1}(u_i,u_m)+2\Big)\Big[\Big(d_{G_2}(v_l)+1\Big)^2 + \Big(d_{G_2}(v_n)+1\Big)^2\Big]\\
&= \sum_{i,m=0,i\neq m}^{p_1-1}\Big(d_{G_1}(u_i,u_m)+2\Big)\sum_{l,n=0}^{p_2-1}\Big[d^2_{G_2}(v_l) + d^2_{G_2}(v_n)\\
&+ 2\Big(d_{G_2}(v_l)+d_{G_2}(v_n)\Big)+2\Big]\\
&= \sum_{i,m=0,i\neq m}^{p_1-1}\Big(d_{G_1}(u_i,u_m)+2\Big)\ 2\sum_{l,n=0}^{p_2-1}\Big[d^2_{G_2}(v_l) + 2d_{G_2}(v_l)+1\Big]\\
&= 4\Big[W(G_1)+p_1(p_1-1)\Big]\ \Big[p_2M_1(G_2)+4p_2q_2+p_2^2\Big]
\end{aligned}$$

Adding S_1, S_2, S_3 and S_4 we get,
$2\times DF(G_1\odot G_2)$

$$= 2DF(G_1) + 4p_2^2W(G_1) + 4p_2DD(G_1) + 2p_1\Big[2p_2M_1(G_2) + 8p_2q_2$$
$$+ 2p_2(p_2-1) - F(G_2) - 4M_1(G_2) - 10q_2\Big] + 2p_2DF(G_1) + 4p_2^2DD(G_1)$$
$$+ 4W(G_1)[p_2(p_2^2+1)M_1(G_2) + 4q_2] + 2p_1p_2[M_1(G_1) + 4q_1p_2]$$
$$+ 2p_1^2[M_1(G_2) + 4q_2 + p_2(p_2^2+1)] + 4[W(G_1) + p_1(p_1-1)][p_2M_1(G_2)$$
$$+ 4p_2q_2 + p_2^2].$$ □

4. Distance Based F-index of Tensor Product of Graphs

In this section, we find the exact value of the distance based F- index of tensor product $G_1 \times G_2$.

LEMMA 4.1 ([**5**]). *The following hold*

(i) $|V(G_1 \times G_2)| = |V(G_1)||V(G_2)|$

(ii) $|E(G_1 \times G_2)| = 2|E(G_1)||E(G_2)|$

(iii) $d_{G_1\times G_2}(u_i, v_l) = d_{G_1}(u_i)d_{G_2}(v_l)$.

LEMMA 4.2. *Let $w_{il} = (u_i, v_l)$ and $w_{mn} = (u_m, v_n)$ be in $V(G_1 \times G_2)$. Then the distance between w_{il} and w_{mn} is*

$$d_{G_1\times G_2}(w_{il}, w_{mn}) = \begin{cases} d_{G_2}(v_l, v_n), & \text{if, } i \neq m, l \neq n \\ 0, & \text{otherwise.} \end{cases}$$

PROOF. It can be easily derived from the definition of the tensor product of G_1 and G_2. □

THEOREM 4.1. *Let $G_i, i = 1, 2,$ be a (p_i, q_i)- graph. Then*

$$DF(G) = (p_1 - 1)M_1(G_1)DF(G_2).$$

PROOF. Let $G = G_1 \times G_2$. Then,

$$2 \times DF(G) = \sum_{w_{il},w_{mn}\in V(G)} d_G(w_{il}, w_{mn})\Big[d_G^2(w_{il}) + d_G^2(w_{mn})\Big]$$
$$= \sum_{i,m=0,i\neq m}^{p_1-1} \sum_{l,n=0,l\neq n}^{p_2-1} d_G(w_{il}, w_{mn})\Big[d_G^2(w_{il}) + d_G^2(w_{mn})\Big]$$
$$= \sum_{i,m=0,i\neq m}^{p_1-1} \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l, v_n)\Big[d_{G_1}^2(u_i)d_{G_2}^2(v_l) + d_{G_1}^2(u_m)d_{G_2}^2(v_n)\Big]$$

and

$$\begin{aligned}
2 \times DF(G) &= \sum_{i,m=0,i\neq m}^{p_1-1} d^2_{G_1}(u_i) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n) d^2_{G_2}(v_l) \\
&+ \sum_{i,m=0,i\neq m}^{p_1-1} d^2_{G_1}(u_m) \sum_{l,n=0,l\neq n}^{p_2-1} d_{G_2}(v_l,v_n) d^2_{G_2}(v_n)\Big] \\
&= (p_1-1)q_1(G_1)DF(G_2) + (p_1-1)M_1(G_1)DF(G_2) \\
2 \times DF(G) &= 2(p_1-1)M_1(G_1)DF(G_2) \\
DF(G) &= (p_1-1)M_1(G_1)DF(G_2)
\end{aligned}$$

□

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