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COMMON FIXED POINT THEOREMS IN COMPLETE FUZZY SPACE

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Abstract. The purpose of this article is to prove some common fixed point theorems in complete fuzzy metric space by using rational type contractive conditions. Our aim of this article is to generalized the results of various fixed point theorems in literature.

1. Introduction

In 1965, Zadeh introduced the concept of Fuzzy set as a new way to represent vagueness in our everyday life. Kramosil et al. (1975) have introduced the concept of fuzzy metric spaces in different ways. The purpose of this article is to prove some common fixed point theorems in complete fuzzy metric space by using rational type contractive conditions. Our aim of this article is to generalized the results of various fixed point theorems in literature.

2. Preliminaries

The concept of triangular norms (t-norms) is originally introduced by Menger in study of statistical metric spaces.

Definition 2.1. ([**9**]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is continuous t-norm if $*$ satisfies the following conditions:
(i) $*$ is commutative and associative;
(ii) $*$ is continuous;
(iii) $a * 1 = a$ for all $a \in [0,1]$;
(iv) $a * b \leqslant c * d$ whenever $a \leqslant c$ and $b \leqslant d$ for all $a, b, c, d \in [0,1]$.

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Examples of t-norms are:

$a * b = min\{a, b\}, a * b = ab$ and $a * b = max\{a + b - 1, 0\}$.

Kramosil et al. (1975) introduced the concept of fuzzy metric spaces as follows:

Definition 2.2. ([**3**]) A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if X is an arbitrary set, $*$ is a continuous t-norm, and M is fuzzy sets on $X^2 \times [0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$,

(i) $M(x, y, 0) = 0$;
(ii) $M(x, y, t) = 1$ for all $t > 0$ if and only if $x = y$;
(iii) $M(x, y, t) = M(y, x, t)$;
(iv) $M(x, y, t) * M(y, z, s) \leqslant M(x, z, t + s)$;
(v) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous.

Then $(X, M, *)$ is called a fuzzy metric space on X. The function $M(x, y, t)$ denote the degree of nearness between x and y w.r.t. t respectively.

Remark 2.1. ([**3**]) In fuzzy metric space $(X, M, *)$, $M(x, y, .)$ is non-decreasing for all $x, y \in X$.

Definition 2.3. ([**3**]) Let $(X, M, *)$ be a fuzzy metric space. Then a sequence $\{x_n\}$ in X is said to be

(a) convergent to a point $x \in X$ if, for all $t > 0$, $lim_n M(x_n, x, t) = 1$.
(b) Cauchy sequence if, for all $t > 0$ and $p > 0$, $lim_n M(x_{n+p}, x_n, t) = 1$.

Definition 2.4. ([**3**]) A fuzzy metric space $(X, M, *)$ is said to be complete if and only if every Cauchy sequence in X is convergent.

Definition 2.5. ([**10**]) A pair of self mappings (A, S) of a fuzzy metric space $(X, M, *)$ is said to be commuting if $M(ASx, SAx, t) = 1$ for all $x \in X$.

Definition 2.6. ([**10**]) A pair of self mappings (A, S) of a fuzzy metric space $(X, M, *)$ is said to be weakly commuting if $M(ASx, SAx, t) \geqslant M(Ax, Sx, t)$ for all $x \in X$ and $t > 0$.

Definition 2.7. ([**2**]) A pair of self mappings (A, S) of a fuzzy metric space $(X, M, *)$ is said to be compatible if $lim_n M(ASx_n, SAx_n, t) = 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $lim_n Ax_n = lim_n Sx_n = u$ for some $u \in X$.

Definition 2.8. ([**2**]) Let $(X, M, *)$ be a fuzzy metric space. A and S be self maps on X. A point $x \in X$ is called a coincidence point of A and S iff $Ax = Sx$. In this case, $w = Ax = Sx$ is called a point of coincidence of A and S.

Definition 2.9. ([**2**]) A pair of self mappings (A, S) of a fuzzy metric space $(X, M, *)$ is said to be weakly compatible if they commute at the coincidence points i.e., if $Au = Su$ for some $u \in X$, then $ASu = SAu$.

Definition 2.10. ([**10**]) A pair of self mappings (A, S) of a fuzzy metric space $(X, M, *)$ is said to be pointwise R-weakly commuting if given $x \in X$, there exist $R > 0$ such that $M(ASx, SAx, t) \geqslant M(Ax, Sx, t/R)$ for all $t > 0$. Clearly, every pair of weakly commuting mappings is pointwise R-weakly commuting with $R = 1$.

DEFINITION 2.11. ([7]) Two mappings A and S of a fuzzy metric space $(X, M,*)$ is said to be reciprocally continuous if $ASx_n \to Az, SAx_n \to Sz$, whenever $\{x_n\}$ is a sequence such that $Ax_n \to z, Sx_n \to z$ for some $z \in X$.

3. Main Results

THEOREM 3.1. *Let S and T be self- mappings of a fuzzy metric space $(X, M, *)$ satisfying the condition:*

(I) *for all $u, v, w, a \in X$,*

$$\begin{aligned} M(Su, Tv, kt) &\geqslant \alpha_1\left[\frac{M^2(u, Sw, \frac{k}{t}) + M^2(u, v, \frac{k}{t})}{1 + M(u, Sw, \frac{k}{t}) + M(u, v, \frac{k}{t})}\right] \\ &+ \alpha_2\left[\frac{M^2(v, Ta, \frac{k}{t}) + M^2(Sw, Ta, \frac{k}{t})}{1 + M(v, Ta, \frac{k}{t}) + M(Sw, Ta, \frac{k}{t})}\right] \\ &+ \alpha_3\sqrt{M(v, Sw, \frac{k}{t})M(u, Ta, \frac{k}{t})} + \alpha_4[M(u, v, \frac{k}{t})] \end{aligned}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are non-negative reals such that $2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 < 1$ then S, T have a unique common fixed point.

PROOF. Let x_0 be an arbitrary element of X and we construct a sequence $\{x_n\}$ defined as follows: $Sx_{n-1} = x_n, Tx_n = x_{n+1}, Sx_{n+1} = x_{n+2}, Tx_{n+2} = x_{n+3}...$ and $TSx_{n-1} = x_{n+1}, STx_n = x_{n+2}, TSx_{n+1} = x_{n+3}, STx_{n+2} = x_{n+4}...$ where n = 1, 2, 3, . . .

Now putting $u = Ty, v = Sx, w = x$ and $a = y$ in (I) then we have

(II)

$$\begin{aligned} M(STy, TSx, kt) &\geqslant \alpha_1\left[\frac{M^2(Ty, Sx, \frac{k}{t}) + M^2(Ty, Sx, \frac{k}{t})}{1 + M(Ty, Sx, \frac{k}{t}) + M(Ty, Sx, \frac{k}{t})}\right] \\ &+ \alpha_2\left[\frac{M^2(Sx, Ty, \frac{k}{t}) + M^2(Sx, Ty, \frac{k}{t})}{1 + M(Sx, Ty, \frac{k}{t}) + M(Sx, Ty, \frac{k}{t})}\right] \\ &+ \alpha_3\sqrt{M(Sx, Sx, \frac{k}{t})M(Ty, Ty, \frac{k}{t})} + \alpha_4[M(Ty, Sx, \frac{k}{t})]. \end{aligned}$$

$$M(STy, TSx, kt) \geqslant 2\alpha_1 M(Sx, Ty, \frac{k}{t}) + 2\alpha_2 M(Sx, Ty, \frac{k}{t}) + \alpha_4 M(Sx, Ty, \frac{k}{t}).$$

Now putting $x = x_{n-1}$ and $y = x_n$ in (II) then we have

$$\begin{gathered} M(STx_n, TSx_{n-1}, kt) \geqslant 2\alpha_1 M(Sx_{n-1}, Tx_n, \tfrac{k}{t}) + \\ 2\alpha_2 M(Sx_{n-1}, Tx_n, \tfrac{k}{t}) + \alpha_4 M(Sx_{n-1}, Tx_n, \tfrac{k}{t}) \end{gathered}$$

and

$$M(x_{n+2}, x_{n+1}, kt) \geqslant 2\alpha_1 M(x_n, x_{n+1}, \tfrac{k}{t}) + \\ 2\alpha_2 M(x_n, x_{n+1}, \tfrac{k}{t}) + \alpha_4 M(x_n, x_{n+1}, \tfrac{k}{t}). \text{ (III)}$$

From (III) we conclude that $M(x_{n-1}, x_n, \frac{k}{t})$ decreases with n. i.e.,

$$M(x_{n-1}, x_n, \frac{k}{t}) \to M(x_0, x_1, \frac{k}{t})$$

as $n \to \infty$.
If possible let $M(x_0, x_1, \frac{k}{t}) > 0$ taking limit $n \to \infty$ on (III) then we have

$$M(x_0, x_1, kt) \geqslant 2\alpha_1 M(x_0, x_1, \tfrac{k}{t}) + 2\alpha_2 M(x_0, x_1, \tfrac{k}{t}) + \alpha_4 M(x_0, x_1, \tfrac{k}{t})$$
$$= (2\alpha_1 + 2\alpha_2 + \alpha_4) M(x_0, x_1, \tfrac{k}{t}) > M(x_0, x_1, \tfrac{k}{t}).$$

Since $2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 < 1$, which gives contraction and hence

$$M(x_0, x_1, \frac{k}{t}) = 0.$$

Next we shall show that $\{x_n\}$ is a Cauchy sequence.
$M(x_m, x_n, kt) \geqslant M(x_m, x_{m+1}, \frac{k}{t}) + M(x_{m+1}, x_{n+1}, \frac{k}{t}) + M(x_{n+1}, x_n, \frac{k}{t})$.

Now,

$$M(x_m, x_n, kt) \geqslant M(x_m, x_{m+1}, \tfrac{k}{t}) + M(x_n, x_{n+1}, \tfrac{k}{t}) + M(Sx_n, Tx_m, \tfrac{k}{t}). \text{ (IV)}$$

By putting $u = x_n, v = x_m, w = x_{m-1}$ and $a = x_{n-1}$ in (I) then we have

$$\begin{aligned} M(Sx_n, Tx_m, kt) \quad &\geqslant \quad \alpha_1 \Big[\frac{M^2(x_n, Sx_{m-1}, \frac{k}{t}) + M^2(x_n, x_m, \frac{k}{t})}{1 + M(x_n, Sx_{m-1}, \frac{k}{t}) + M(x_n, x_m, \frac{k}{t})} \Big] \\ &+ \quad \alpha_2 \Big[\frac{M^2(x_m, Tx_{n-1}, \frac{k}{t}) + M^2(Sx_{m-1}, Tx_{n-1}, \frac{k}{t})}{1 + M(x_m, Tx_{n-1}, \frac{k}{t}) + M(Sx_{m-1}, Tx_{n-1}, \frac{k}{t})} \Big] \\ &+ \quad \alpha_3 \sqrt{M(x_m, Sx_{m-1}, \frac{k}{t}) M(x_n, Tx_{n-1}, \frac{k}{t})} \\ &+ \quad \alpha_4 [M(x_n, x_m, \frac{k}{t})] \end{aligned}$$

$$\begin{aligned} M(Sx_n, Tx_m, kt) \quad &\geqslant \quad \alpha_1 \Big[\frac{M^2(x_n, x_m, \frac{k}{t}) + M^2(x_n, x_m, \frac{k}{t})}{1 + M(x_n, x_m, \frac{k}{t}) + M(x_n, x_m, \frac{k}{t})} \Big] \\ &+ \quad \alpha_2 \Big[\frac{M^2(x_m, x_n, \frac{k}{t}) + M^2(x_m, x_n, \frac{k}{t})}{1 + M(x_m, x_n, \frac{k}{t}) + M(x_m, x_n, \frac{k}{t})} \Big] \\ &+ \quad \alpha_3 \sqrt{M(x_m, x_m, \frac{k}{t}) M(x_n, x_n, \frac{k}{t})} + \alpha_4 [M(x_n, x_m, \frac{k}{t})] \end{aligned}$$

$M(Sx_n, Tx_m, kt) \geqslant (2\alpha_1 + 2\alpha_2 + \alpha_4) M(x_n, x_m, \frac{k}{t})$ (V)

From (IV) and (V) we have
$M(x_m, x_n, kt) \geqslant M(x_m, x_{m+1}, \frac{k}{t}) + M(x_n, x_{n+1}, \frac{k}{t}) + (2\alpha_1 + 2\alpha_2 + \alpha_4) M(x_n, x_m, \frac{k}{t})$.

Letting $m, n \to \infty$ then $M(x_n, x_m, \frac{k}{t}) \to \infty$ as $2\alpha_1 + 2\alpha_2 + \alpha_4 < 1$. Hence $\{x_n\}$ is a Cauchy sequence in X. Now we prove z is a common fixed point of S and T. By putting $u = z, v = x_{n-1}, w = z$ and $a = x_{n-2}$ in (I) we have

$$\begin{aligned} M(Sz, Tx_{n-1}, kt) &\geqslant \alpha_1 [\frac{M^2(z, Sz, \frac{k}{t}) + M^2(z, x_{n-1}, \frac{k}{t})}{1 + M(z, Sz, \frac{k}{t}) + M(z, x_{n-1}, \frac{k}{t})}] \\ &+ \alpha_2 [\frac{M^2(x_{n-1}, Tx_{n-2}, \frac{k}{t}) + M^2(Sz, Tx_{n-2}, \frac{k}{t})}{1 + M(x_{n-1}, Tx_{n-2}, \frac{k}{t}) + M(Sz, Tx_{n-2}, \frac{k}{t})}] \\ &+ \alpha_3 \sqrt{M(x_{n-1}, Sz, \frac{k}{t}) M(z, Tx_{n-2}, \frac{k}{t})} \\ &+ \alpha_4 [M(z, x_{n-1}, \frac{k}{t})] \end{aligned}$$

$$\begin{aligned} M(Sz, x_n, kt) &\geqslant \alpha_1 [\frac{M^2(z, Sz, \frac{k}{t}) + M^2(z, x_{n-1}, \frac{k}{t})}{1 + M(z, Sz, \frac{k}{t}) + M(z, x_{n-1}, \frac{k}{t})}] \\ &+ \alpha_2 [\frac{M^2(x_{n-1}, x_{n-1}, \frac{k}{t}) + M^2(Sz, x_{n-1}, \frac{k}{t})}{1 + M(x_{n-1}, x_{n-1}, \frac{k}{t}) + M(Sz, x_{n-1}, \frac{k}{t})}] \\ &+ \alpha_3 \sqrt{M(x_{n-1}, Sz, \frac{k}{t}) M(z, x_{n-1}, \frac{k}{t})} + \alpha_4 [M(z, x_{n-1}, \frac{k}{t})]. \end{aligned}$$

Letting $n \to \infty$ then we have

$$\begin{aligned} M(Sz, z, kt) &\geqslant \alpha_1 [\frac{M^2(z, Sz, \frac{k}{t}) + M^2(z, z, \frac{k}{t})}{1 + M(z, Sz, \frac{k}{t}) + M(z, z, \frac{k}{t})}] \\ &+ \alpha_2 [\frac{M^2(z, z, \frac{k}{t}) + M^2(Sz, z, \frac{k}{t})}{1 + M(z, z, \frac{k}{t}) + M(Sz, z, \frac{k}{t})}] \\ &+ \alpha_3 \sqrt{M(z, Sz, \frac{k}{t}) M(z, z, \frac{k}{t})} + \alpha_4 [M(z, z, \frac{k}{t})]. \end{aligned}$$

$M(Sz, z, kt) \geqslant (\alpha_1 + \alpha_2) M(Sz, z, \frac{k}{t})$ and $M(Sz, z, kt) \geqslant M(Sz, z, \frac{k}{t})$ which gives $Sz = z$. Thus z is a fixed point of S.

Similarly by using (I), we can easily show that z is a fixed point of T. Hence z is a common fixed point of S and T.

For uniqueness, let $q \neq z$ be such that $Sq = q = Tq$. By putting $u = z, v = q, w = q$ and $a = z$ in (I) we have

$$\begin{aligned} M(Sz,Tq,kt) &\geqslant \alpha_1[\frac{M^2(z,Sq,\frac{k}{t})+M^2(z,q,\frac{k}{t})}{1+M(z,Sq,\frac{k}{t})+M(z,q,\frac{k}{t})}] \\ &+ \alpha_2[\frac{M^2(q,Tz,\frac{k}{t})+M^2(Sq,Tz,\frac{k}{t})}{1+M(q,Tz,\frac{k}{t})+M(Sq,Tz,\frac{k}{t})}] \\ &+ \alpha_3\sqrt{M(q,Sq,\frac{k}{t})M(z,Tz,\frac{k}{t})}+\alpha_4[M(z,q,\frac{k}{t})] \end{aligned}$$

$$\begin{aligned} M(z,q,kt) &\geqslant \alpha_1[\frac{M^2(z,q,\frac{k}{t})+M^2(z,q,\frac{k}{t})}{1+M(z,q,\frac{k}{t})+M(z,q,\frac{k}{t})}] \\ &+ \alpha_2[\frac{M^2(q,z,\frac{k}{t})+M^2(q,z,\frac{k}{t})}{1+M(q,z,\frac{k}{t})+M(q,z,\frac{k}{t})}] \\ &+ \alpha_3\sqrt{M(q,q,kt)M(z,z,\frac{k}{t})}+\alpha_4[M(z,q,\frac{k}{t})] \end{aligned}$$

$M(z,q,kt) \geqslant (2\alpha_1 + 2\alpha_2 + \alpha_4)M(z,q,\frac{k}{t})$ and $M(z,q,kt) \geqslant M(z,q,\frac{k}{t})$ which gives contradiction. Hence z is unique common fixed point of S and T. Result follows. □

Corollary 3.1. *Let T be self- mapping of a fuzzy complete metric space $(X, M, *)$ satisfying the condition: for all $u, v, w, a \in X$,*

$$\begin{aligned} M(Tu,Tv,kt) &\geqslant \alpha_1[\frac{M^2(u,Tw,\frac{k}{t})+M^2(u,v,\frac{k}{t})}{1+M(u,Tw,\frac{k}{t})+M(u,v,\frac{k}{t})}] \\ &+ \alpha_2[\frac{M^2(v,Ta,\frac{k}{t})+M^2(Tw,Ta,\frac{k}{t})}{1+M(v,Ta,\frac{k}{t})+M(Tw,Ta,\frac{k}{t})}] \\ &+ \alpha_3\sqrt{M(v,Tw,\frac{k}{t})M(u,Ta,\frac{k}{t})}+\alpha_4[M(u,v,\frac{k}{t})] \end{aligned}$$

where $\alpha 1, \alpha_2, \alpha_3, \alpha_4$ are non-negative reals such that $2\alpha 1 + 2\alpha_2 + \alpha_3 + \alpha_4 < 1$ then T has a unique common fixed point.

Proof. It is sufficient if we take $S = T$ in Theorem 3.1. □

THEOREM 3.2. *Let S,T,R be any three self mappings of a fuzzy complete metric space $(X,M,*)$ satisfying the condition: for all $u,v,w,a \in X$,* (VI)

$$\begin{aligned}
M(SRu,TRv,kt) &\geqslant \alpha_1\Big[\frac{M^2(u,SRw,\frac{k}{t})+M^2(u,TRa,\frac{k}{t})+M^2(u,SRw,\frac{k}{t})}{1+M(u,SRw,\frac{k}{t})+M(u,TRa,\frac{k}{t})+M(u,SRw,\frac{k}{t})}\Big]\\
&+ \alpha_2\Big[\frac{M^2(v,SRw,\frac{k}{t})+M^2(u,TRa,\frac{k}{t})+M^2(v,TRa,\frac{k}{t})}{1+M(v,SRw,\frac{k}{t})+M(u,TRa,\frac{k}{t})+M(v,TRa,\frac{k}{t})}\Big]\\
&+ \alpha_3\sqrt{M(v,SRw,\frac{k}{t})M(u,TRa,\frac{k}{t})}+\alpha_4[M(u,v,\frac{k}{t})]\\
&+ \alpha_5[M(u,v,\frac{k}{t})]
\end{aligned}$$

where $\alpha 1,\alpha_2,\alpha_3,\alpha_4,\alpha_5$ are non-negative reals such that $2\alpha_1+2\alpha_2+\alpha_3+\alpha_4+\alpha_5<1$ then SR,TR have a unique common fixed point.

PROOF. Let x_0 be an arbitrary element of X and we construct a sequence $\{x_n\}$ defined as follows

$$Sx_{n-1}=x_n, Tx_n=x_{n+1}, SRx_{n+1}=x_{n+2}, TRx_{n+2}=x_{n+3}...$$

and

$$TRSRx_{n-1}=x_{n+1}, SRTRx_n=x_{n+2}, TRSRx_{n+1}=x_{n+3}, SRTRx_{n+2}=x_{n+4}...$$

where n = 1, 2, 3, . . .

Now putting $u=TRx_n, v=SRx_{n-1}, w=x_{n-1}$ and $a=x_n$ in (VI) then we have

$$\begin{aligned}
&M(SRTRx_n,TRSRx_{n-1},kt)\geqslant\\
&\alpha_1\Big[\frac{M^2(TRx_n,SRx_{n-1},\frac{k}{t})+M^2(TRx_n,TRx_n,\frac{k}{t})+M^2(TRx_n,SRx_{n-1},\frac{k}{t})}{1+M(TRx_n,SRx_{n-1},\frac{k}{t})+M(TRx_n,TRx_n,\frac{k}{t})+M(TRx_n,SRx_{n-1},\frac{k}{t})}\Big]\\
&+\alpha_2\Big[\frac{M^2(SRx_{n-1},SRx_{n-1},\frac{k}{t})+M^2(TRx_n,TRx_n,\frac{k}{t})+M^2(SRx_{n-1},TRa,\frac{k}{t})}{1+M(SRx_{n-1},SRw,\frac{k}{t})+M(TRx_n,TRa,\frac{k}{t})+M(SRx_{n-1},TRx_n,\frac{k}{t})}\Big]\\
&+\alpha_3\sqrt{M(SRx_{n-1},SRx_{n-1},\frac{k}{t})M(TRx_n,TRx_n,\frac{k}{t})}+\alpha_4[M(TRx_n,SRx_{n-1},\frac{k}{t})]\\
&+\alpha_5[M(TRx_n,SRx_{n-1},\frac{k}{t})]
\end{aligned}$$

$$M(SRTRx_n,TRSRx_{n-1},kt)\geqslant(\alpha_1+\alpha_2+\alpha_4+\alpha_5)M(SRx_{n-1},TRx_n,\frac{k}{t})$$

$$M(x_{n+2},x_{n+1},kt)\geqslant(\alpha_1+\alpha_2+\alpha_4+\alpha_5)M(x_n,x_{n+1},\frac{k}{t}) \qquad \text{(VII)}$$

this concludes that $M(x_{n-1},x_n,t)$ decreases with n. Therefore,

$$M(x_{n-1},x_n,\frac{k}{t})\to M(x_0,x_1,\frac{k}{t}) \text{ as } n\to\infty.$$

If possible let $M(x_0, x_1, \frac{k}{t}) > 0$ taking limit $n \to \infty$ on (VII) then we have

$$M(x_0, x_1, kt) \geqslant (\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5) M(x_0, x_1, \frac{k}{t}) > M(x_0, x_1, \frac{k}{t})$$

which gives contraction and hence $M(x_0, x_1, \frac{k}{t}) = 0$.

Next we shall show that $\{x_n\}$ is a Cauchy sequence.

$$M(x_m, x_n, kt) \geqslant M(x_m, x_{m+1}, \frac{k}{t}) + M(x_{m+1}, x_{n+1}, \frac{k}{t}) + M(x_{n+1}, x_n, \frac{k}{t}).$$

Now,

$$M(x_m, x_n, kt) \geqslant M(x_m, x_{m+1}, \frac{k}{t}) + M(x_n, x_{n+1}, \frac{k}{t}) + M(SRx_n, TRx_m, \frac{k}{t}). \ (VIII)$$

By putting $u = x_n, v = x_m, w = x_{m-1}$ and $a = x_{n-1}$ in (VI) then we have

$$\begin{aligned} &M(SRx_n, TRx_m, kt) \geqslant \\ &\alpha_1 [\frac{M^2(x_n, SRx_{m-1}, \frac{k}{t}) + M^2(x_n, TRx_{n-1}, \frac{k}{t}) + M^2(x_n, SRx_{m-1}, \frac{k}{t})}{1 + M(x_n, SRx_{m-1}, \frac{k}{t}) + M(x_n, TRx_{n-1}, \frac{k}{t}) + M(x_n, SRx_{m-1}, \frac{k}{t})}] \\ &+ \alpha_2 [\frac{M^2(x_m, SRx_{m-1}, \frac{k}{t}) + M^2(x_n, TRx_{n-1}, \frac{k}{t}) + M^2(x_m, TRx_{n-1}, \frac{k}{t})}{1 + M(x_m, SRx_{m-1}, \frac{k}{t}) + M(x_n, TRa, \frac{k}{t}) + M(x_m, TRx_{n-1}, \frac{k}{t})}] \\ &+ \alpha_3 \sqrt{M(x_n, SRw, \frac{k}{t}) M(x_n, TRx_{n-1}, \frac{k}{t})} + \alpha_4 [M(x_n, x_m, \frac{k}{t})] \\ &+ \alpha_5 [M(x_n, x_m, \frac{k}{t})] \end{aligned}$$

$$M(Sx_n, Tx_m, kt) \geqslant (\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5) M(x_n, x_m, \frac{k}{t}) \ (IX)$$

From (VIII) and (IX) we have

$$\begin{gathered} M(x_m, x_n, kt) \geqslant \\ M(x_m, x_{m+1}, \tfrac{k}{t}) + M(x_n, x_{n+1}, \tfrac{k}{t}) + (\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5) M(x_n, x_m, \tfrac{k}{t}). \end{gathered}$$

Letting $m, n \to \infty$ then $M(x_n, x_m, \frac{k}{t}) \to \infty$ as $\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 < 1$. Hence $\{x_n\}$ is a Cauchy sequence in X.

Now we prove z is a common fixed point of SR and TR. By putting $u = z, v = x_{n-1}, w = z$ and $a = x_{n-2}$ in (VI) we have

$$\begin{aligned}
&M(SRz, TRx_{n-1}, kt) \geqslant \\
&\alpha_1[\frac{M^2(z, SRz, \frac{k}{t}) + M^2(z, TRa, \frac{k}{t}) + M^2(z, SRz, \frac{k}{t})}{1 + M(z, SRz, \frac{k}{t}) + M(z, TRa, \frac{k}{t}) + M(z, SRz, \frac{k}{t})}] \\
&+\alpha_2[\frac{M^2(x_{n-1}, SRz, \frac{k}{t}) + M^2(z, TRa, \frac{k}{t}) + M^2(x_{n-1}, TRa, \frac{k}{t})}{1 + M(x_{n-1}, SRz, \frac{k}{t}) + M(u, TRa, \frac{k}{t}) + M(x_{n-1}, TRa, \frac{k}{t})}] \\
&+\alpha_3\sqrt{M(x_{n-1}, SRz, \frac{k}{t})M(z, TRa, \frac{k}{t})} + \alpha_4[M(z, x_{n-1}, \frac{k}{t})] \\
&+\alpha_5[M(z, x_{n-1}, \frac{k}{t})].
\end{aligned}$$

Letting $n \to \infty$ then we have

$$M(SRz, z, kt) \geqslant (\alpha_1 + \alpha_3 + \alpha_4)M(SRz, z, \frac{k}{t}) \text{ and } M(SRz, z, kt) \geqslant M(SRz, z, \frac{k}{t})$$

which gives $SRz = z$. Thus z is a fixed point of SR.

Similarly by using (VI), we can easily show that z is a fixed point of TR. Hence z is a common fixed point of SR and TR.

For uniqueness, let $q \neq z$ be such that $SRq = q = TRq$. By putting $u = z, v = q, w = q$ and $a = z$ in (VI) we have

$$\begin{aligned}
M(Sz, Tq, kt) &\geqslant \alpha_1[\frac{M^2(z, Sq, \frac{k}{t}) + M^2(z, q, \frac{k}{t})}{1 + M(z, Sq, \frac{k}{t}) + M(z, q, \frac{k}{t})}] \\
&+ \alpha_2[\frac{M^2(q, Tz, \frac{k}{t}) + M^2(Sq, Tz, \frac{k}{t})}{1 + M(q, Tz, \frac{k}{t}) + M(Sq, Tz, \frac{k}{t}}] \\
&+ \alpha_3\sqrt{M(q, Sw, \frac{k}{t})M(z, Tz, \frac{k}{t})} + \alpha_4[M(z, vq, \frac{k}{t})]
\end{aligned}$$

$$M(z, q, kt) \geqslant (\alpha_1 + 2\alpha_2 + \alpha_4 + \alpha_5)M(z, q, \frac{k}{t}) \text{ and } M(z, q, kt) \geqslant M(z, q, \frac{k}{t})$$

which gives contradiction. Hence z is unique common fixed point of SR and TR. This completes proof of the theorem. □

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