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REMARKS ON THE SUM OF POWERS OF LAPLACIAN EIGENVALUES OF GRAPHS

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Abstract. Let $G=(V,E)$, $V=\{v_1,v_2,\dots,v_n\}$, be a simple connected graph of order n, size m and vertex degree sequence $\Delta=d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n > 0$, $d_i=d(v_i)$. The Laplacian matrix of G is $\mathbf{L}=\mathbf{D}-\mathbf{A}$, where $\mathbf{D}$ is the diagonal matrix of vertex degrees and $\mathbf{A}$ the adjacency matrix of G. Eigenvalues of matrix $\mathbf{L}$, $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{n-1} > \mu_n = 0$, form the so-called Laplacian spectrum of G. With $S_\alpha(G)=\sum_{i=1}^{n-1}\mu_i^\alpha$, where α is an arbitrary real number, we denote the sum of powers of Laplacian eigenvalues of G. In this paper we establish a relationship between $S_{\alpha+\beta}(G)$ and $S_{\alpha+\beta-1}(G)$, where α and β are arbitrary real numbers and obtain new bounds for $S_\alpha(G)$. By the appropriate choice of parameters α and β, a number of new/old inequalities that reveal relationships between various topological indices are obtained.

1. Introduction

Let $G=(V,E)$, $V=\{v_1,v_2,\dots,v_n\}$, be a simple connected graph with n vertices, m edges and vertex degree sequence $\Delta=d_1 \geqslant d_2 \geqslant \cdots \geqslant d_n > 0$, $d_i = d(v_i)$. Further, let $\mathbf{A}$ be the adjacency matrix of G, and $\mathbf{D}=\operatorname{diag}(d_1,d_2,\dots,d_n)$ the diagonal matrix of its vertex degrees. Laplacian matrix of G is defined as $\mathbf{L}=\mathbf{D}-\mathbf{A}$. Eigenvalues of matrix $\mathbf{L}$, $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_{n-1} > \mu_n = 0$ form the so-called Laplacian spectrum of G.

In graph theory, an invariant is a property of graphs that depends only on their abstract structure, not on the labeling of vertices or edges. Such quantities are also referred to as topological indices. Here we list some vertex–degree–based and Laplacian–spectrum–based graph invariants that are of interest for this work.

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In 1972 in paper [**7**] Gutman and Trinajstić derived approximate formulas for the total π-electron energy. One of the terms occurring in these formulas was the sum of squares of vertex degrees of the molecular graph:

$$M_1(G) = \sum_{i=1}^{n} d_i^2 \,,$$

which was recognized to be a measure of the extent of branching of the carbon–atom skeleton of the underlying molecule. Ten years later it was named the *first Zagreb index* [**1**]. It became one of the most popular and most extensively studied graph-based molecular structure descriptors.

In the same paper [**7**] in the formulas for total π-electron energy, another quantity, the sum of cubes of vertex degrees

$$F(G) = \sum_{i=1}^{n} d_i^3 \,,$$

was encountered. This quantity is also a measure of branching. However, for the unknown reasons, it did not attracted any attention until 2015 when it was reinvented in [**6**] and named the *forgotten topological index*.

In [**9**] Klein and Randić introduced a graph invariant named the Kirchhoff index as

$$Kf(G) = \sum_{i<j} r_{ij},$$

where r_{ij} is the resistance between the vertices v_i and v_j, i.e. r_{ij} is equal to the resistance between equivalent points on an associated electrical network obtained by replacing each edge of G by a unit (1 ohm) resistor. The Kirchhoff index has a very nice purely mathematical interpretation. Namely, in [**8, 25**] it was demonstrated that the Kirchhoff index of a connected graph satisfies the relation:

$$Kf(G) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i}.$$

Another Laplacian-spectrum-based graph invariant was put forward by Liu and Liu [**11**]

$$LEL(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i}$$

and was named *Laplacian-energy-like invariant*.

In a similar way one can define reciprocal Laplacian–energy–like, $RLEL(G)$, as

$$RLEL(G) = \sum_{i=1}^{n-1} \frac{1}{\sqrt{\mu_i}}.$$

Laplacian–spectrum–based topological index called sum of powers of the Laplacian eigenvalues of graphs is defined as

$$S_\alpha(G) = \sum_{i=1}^{n-1} \mu_i^\alpha, \quad S_0(G) = n-1,$$

where α is an arbitrary real number [**2**, **11**, **12**, **14**, **19**–**21**, **23**, **24**]. It can be easily verified that the following identities are valid:

$$\begin{aligned}
&nS_{-1}(G) = Kf(G),\\
&S_{-1/2}(G) = RLEL(G),\\
&S_{1/2}(G) = LEL(G),\\
&S_1(G) = \sum_{i=1}^{n-1} \mu_i = tr(D-A) = \sum_{i=1}^{n} d_i = 2m,\\
&S_2(G) = \sum_{i=1}^{n-1} \mu_i^2 = tr(D-A)^2 = \sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i = M_1(G) + 2m,\\
&S_3(G) = \sum_{i=1}^{n-1} \mu_i^3 = tr(D-A)^3 = F(G) + 3M_1(G) - 6C_3(G),
\end{aligned}$$

where $C_3(G)$ is the number of triangles in G.

In [**23**] some properties for $S_\alpha(G)$, where $\alpha \neq 0, 1$, were established. In [**12**] new bounds for $S_\alpha(G)$ in terms of the vertex degrees of G, and a relation between $S_\alpha(G)$ and the first general Zagreb index were obtained. Upper and lower bounds on $S_\alpha(G)$ in terms of n, m, maximum degree, clique number and a number of spanning trees were obtained in [**2**].

In this paper we establish relationship between $S_{\alpha+\beta}(G)$ and $S_{\alpha+\beta-1}(G)$, where α and β are arbitrary real numbers and obtain new bounds for $S_\alpha(G)$. Also, by the appropriate choice of parameters α and β, a number of new/old inequalities that reveal relationships between aforementioned topological indices are obtained.

2. Preliminaries

In this section we recall some discrete analytical inequalities for real number sequences that will be used in the paper.

Let $p = (p_i)$, $i = 1, 2, \dots, n$, be nonnegative real number sequence and $a = (a_i)$, $i = 1, 2, \dots, n$, be positive real number sequence. Then for any real r, such that $r \geqslant 1$ or $r \leqslant 0$, holds

$$\left(\sum_{i=1}^{n} p_i\right)^{r-1} \sum_{i=1}^{n} p_i a_i^r \geqslant \left(\sum_{i=1}^{n} p_i a_i\right)^r. \tag{2.1}$$

If $0 \leqslant r \leqslant 1$, then the sense of (2.1) reverses. Equality holds if and only if either $r = 0$, or $r = 1$, or $p_1 = p_2 = \cdots = p_n$ and $a_1 = a_2 = \cdots = a_n$, or for some t, $1 \leqslant t \leqslant n-1$, holds $p_1 = p_2 = \cdots = p_t = 0$ and $p_{t+1} = p_{t+2} = \cdots = p_n$ and

$a_{t+1} = a_{t+2} = \cdots = a_n$. This inequality is referred to as Jensen's inequality in the literature (see e.g. [**17**]).

Let $p = (p_i)$ and $x = (x_i)$, $i = 1, 2, \ldots, n$, be two positive real number sequences. In [**18**] it was proved that for any $r \geqslant 0$ the following is valid

$$\sum_{i=1}^{n} \frac{x_i^{r+1}}{a_i^r} \geqslant \frac{\left(\sum_{i=1}^{n} x_i\right)^{r+1}}{\left(\sum_{i=1}^{n} a_i\right)^r}. \tag{2.2}$$

Equality occurs if and only if $r = 0$ or $\frac{x_1}{a_1} = \frac{x_2}{a_2} = \cdots = \frac{x_n}{a_n}$.

3. Main results

In the following theorem we establish relations between $S_{\alpha+\beta}(G)$ and $S_{\alpha+\beta-1}(G)$, where α and β are arbitrary real numbers.

THEOREM 3.1. *Let G be a simple connected graph, $G \ncong K_n$, with $n \geqslant 3$ vertices, β be an arbitrary real number and k real number such that $\mu_{n-1} \geqslant k > 0$. Then for any real number α, $\alpha \geqslant 1$ or $\alpha \leqslant 0$, holds*

$$kS_{\alpha+\beta-1}(G) + \frac{(S_{\beta+1}(G) - kS_\beta(G))^\alpha}{(S_\beta(G) - kS_{\beta-1}(G))^{\alpha-1}} \leqslant S_{\alpha+\beta}(G)$$

$$\leqslant nS_{\alpha+\beta-1}(G) - \frac{(nS_\beta(G) - S_{\beta+1}(G))^\alpha}{(nS_{\beta-1}(G) - S_\beta(G))^{\alpha-1}}. \tag{3.1}$$

If $0 \leqslant \alpha \leqslant 1$, then the opposite inequalities hold. Equality at the left side of (3.1) *holds if and only if either $\alpha = 0$, or $\alpha = 1$, or for some t, $1 \leqslant t \leqslant n-2$, holds $\mu_1 = \mu_2 = \cdots = \mu_t > \mu_{t+1} = \mu_{t+2} = \cdots = \mu_{n-1} = k$. Equality at the right side of* (3.1) *is attained if and only if either $\alpha = 0$, or $\alpha = 1$, or for some t, $1 \leqslant t \leqslant n-2$, holds $n = \mu_1 = \mu_2 = \cdots = \mu_t > \mu_{t+1} = \mu_{t+2} = \cdots = \mu_{n-1}$.*

PROOF. For real numbers α and β we have that

$$S_{\alpha+\beta}(G) - kS_{\alpha+\beta-1}(G) = \sum_{i=1}^{n-1} (\mu_i - k)\, \mu_i^{\alpha+\beta-1} \tag{3.2}$$

and

$$nS_{\alpha+\beta-1}(G) - S_{\alpha+\beta}(G) = \sum_{i=1}^{n-1} (n - \mu_i)\, \mu_i^{\alpha+\beta-1}. \tag{3.3}$$

From the conditions given in the statement of Theorem 3.1 we get that expression $\mu_i - k$ in (3.2) is always nonnegative. In [**10**] it is proven that $\mu_1 \leqslant n$, therefore $n - \mu_i$ is nonnegative.

For $r = \alpha$, $p_i = (\mu_i - k)\mu_i^{\beta-1}$, $a_i = \mu_i$, $i = 1, 2, \ldots, n-1$, where α is a real number such that $\alpha \geqslant 1$ or $\alpha \leqslant 0$, and β is an arbitrary real number, the inequality (2.1) becomes

$$\left(\sum_{i=1}^{n-1} (\mu_i - k)\mu_i^{\beta-1}\right)^{\alpha-1} \sum_{i=1}^{n-1} (\mu_i - k)\mu_i^{\alpha+\beta-1} \geqslant \left(\sum_{i=1}^{n-1} (\mu_i - k)\mu_i^{\beta}\right)^{\alpha}. \tag{3.4}$$

When $G \cong K_n$, $n = k$, in (3.4) equality is attained, therefore without loss of generality we can assume that $G \ncong K_n$. From (3.4) follows

$$\sum_{i=1}^{n-1}(\mu_i - k)\mu_i^{\alpha+\beta-1} \geqslant \frac{\left(\sum_{i=1}^{n-1}(\mu_i - k)\mu_i^{\beta}\right)^{\alpha}}{\left(\sum_{i=1}^{n-1}(\mu_i - k)\mu_i^{\beta-1}\right)^{\alpha-1}},$$

that is

$$\sum_{i=1}^{n-1}(\mu_i - k)\mu_i^{\alpha+\beta-1} \geqslant \frac{(S_{\beta+1}(G) - kS_\beta(G))^{\alpha}}{(S_\beta(G) - kS_{\beta-1}(G))^{\alpha-1}}. \tag{3.5}$$

From (3.5) and (3.2) we get left side of inequality (3.1).

For $r = \alpha$, $p_i = (n - \mu_i)\mu_i^{\beta-1}$, $a_i = \mu_i$, $i = 1, 2, \ldots, n-1$, $\alpha \geqslant 1$ or $\alpha \leqslant 0$, and an arbitrary real number β, the inequality (2.1) transforms into

$$\left(\sum_{i=1}^{n-1}(n - \mu_i)\mu_i^{\beta-1}\right)^{\alpha-1} \sum_{i=1}^{n-1}(n - \mu_i)\mu_i^{\alpha+\beta-1} \geqslant \left(\sum_{i=1}^{n-1}(n - \mu_i)\mu_i^{\beta}\right)^{\alpha}. \tag{3.6}$$

If $G \cong K_n$, then in (3.6) the equality holds, therefore without loss of generality we can assume that $G \ncong K_n$. From (3.6) we obtain

$$\sum_{i=1}^{n-1}(n - \mu_i)\mu_i^{\alpha+\beta-1} \geqslant \frac{\left(\sum_{i=1}^{n-1}(n - \mu_i)\mu_i^{\beta}\right)^{\alpha}}{\left(\sum_{i=1}^{n-1}(n - \mu_i)\mu_i^{\beta-1}\right)^{\alpha-1}},$$

i.e.

$$\sum_{i=1}^{n-1}(n - \mu_i)\mu_i^{\alpha+\beta-1} \geqslant \frac{(nS_\beta(G) - S_{\beta+1}(G))^{\alpha}}{(nS_{\beta-1}(G) - S_\beta(G))^{\alpha-1}}. \tag{3.7}$$

From (3.7) and (3.3) we arrive at right side of inequality (3.1).

By a similar procedure we prove that opposite inequalities hold in (3.1) when $0 \leqslant \alpha \leqslant 1$.

Since $G \ncong K_n$, the equality in (3.4), and therefore at the left side of inequality (3.1), holds if and only if $\alpha = 0$, or $\alpha = 1$, or for some t, $1 \leqslant t \leqslant n-2$, holds $\mu_1 = \mu_2 = \cdots = \mu_t > \mu_{t+1} = \mu_{t+2} = \cdots = \mu_{n-1} = k$. Similarly, the equality in (3.6), i.e. at the right side of inequality (3.1) is attained if and only if either $\alpha = 0$, or $\alpha = 1$, or for some t, $1 \leqslant t \leqslant n-2$, holds $n = \mu_1 = \mu_2 = \cdots = \mu_t > \mu_{t+1} = \mu_{t+2} = \cdots = \mu_{n-1}$. $\square$

THEOREM 3.2. *Let G be a simple connected graph, $G \ncong K_n$, with $n \geqslant 3$ vertices and β is an arbitrary real number. Then for any real α, $\alpha \geqslant 1$ or $\alpha \leqslant 0$, holds*

$$S_{\alpha+\beta}(G) \geqslant \frac{1}{n-k}\left(\frac{n(S_{\beta+1}(G) - kS_\beta(G))^{\alpha}}{(S_\beta(G) - kS_{\beta-1}(G))^{\alpha-1}} + \frac{k(nS_\beta(G) - S_{\beta+1}(G))^{\alpha}}{(nS_{\beta-1}(G) - S_\beta(G))^{\alpha-1}}\right). \tag{3.8}$$

If $0 \leqslant \alpha \leqslant 1$, then the opposite inequality holds. Equality holds if and only if either $\alpha = 0$, or $\alpha = 1$, or for some t, $1 \leqslant t \leqslant n-2$, holds

$$n = \mu_1 = \mu_2 = \cdots = \mu_t > \mu_{t+1} = \mu_{t+2} = \cdots = \mu_{n-1} = k.$$

PROOF. Let α be an arbitrary real number such that $\alpha \geqslant 1$ or $\alpha \leqslant 0$, and $G \ncong K_n$. According to (3.2) and (3.3) we have that

$$S_{\alpha+\beta}(G) - kS_{\alpha+\beta-1}(G) \geqslant \frac{(S_{\beta+1}(G) - kS_{\beta}(G))^{\alpha}}{(S_{\beta}(G) - kS_{\beta-1}(G))^{\alpha-1}}$$

and

$$nS_{\alpha+\beta-1}(G) - S_{\alpha+\beta}(G) \geqslant \frac{(nS_{\beta}(G) - S_{\beta+1}(G))^{\alpha}}{(nS_{\beta-1}(G) - S_{\beta}(G))^{\alpha-1}}.$$

From the above inequalities we have

$$(n-k)S_{\alpha+\beta}(G) \geqslant \frac{n(S_{\beta+1}(G) - kS_{\beta}(G))^{\alpha}}{(S_{\beta}(G) - kS_{\beta-1}(G))^{\alpha-1}} + \frac{k(nS_{\beta}(G) - S_{\beta+1}(G))^{\alpha}}{(nS_{\beta-1}(G) - S_{\beta}(G))^{\alpha-1}}.$$

Since $G \ncong K_n$, i.e. $n - k \neq 0$, according to the above we obtain (3.8).

In a similar way we prove that opposite inequality holds in (3.8) when $0 \leqslant \alpha \leqslant 1$. □

For the appropriately chosen values of parameters α and β, according to Theorems 3.1 and 3.2, a number of particular inequalities for $S_{\alpha}(G)$ as well as some other graph invariants can be obtained. In the following corollaries we list only some of them.

COROLLARY 3.1. *Let G be a simple connected graph, $G \ncong K_n$, with $n \geqslant 3$ vertices, m edges and k is real number such that $\mu_{n-1} \geqslant k > 0$. Then for any real number α, $\alpha \geqslant 1$ or $\alpha \leqslant 0$, holds*

$$kS_{\alpha-1}(G) + \frac{(2m - k(n-1))^{\alpha}}{(n-1-\frac{k}{n}Kf(G))^{\alpha-1}} \leqslant S_{\alpha}(G) \leqslant nS_{\alpha-1}(G) - \frac{(n(n-1)-2m)^{\alpha}}{(Kf(G)-n+1)^{\alpha-1}}.$$

If $0 \leqslant \alpha \leqslant 1$, then the opposite inequalities hold.

For $\alpha \geqslant 2$ or $\alpha \leqslant 1$ holds

$$\begin{aligned} kS_{\alpha-1}(G) + \frac{(M_1(G) - 2m(k-1))^{\alpha-1}}{(2m - k(n-1))^{\alpha-2}} &\leqslant S_{\alpha}(G) \leqslant \\ \leqslant \quad nS_{\alpha-1}(G) - \frac{(2m(n-1) - M_1(G))^{\alpha-1}}{(n(n-1)-2m)^{\alpha-2}}&. \end{aligned}$$

If $1 \leqslant \alpha \leqslant 2$, then the opposite inequalities hold.

For $\alpha \geqslant 3$ or $\alpha \leqslant 2$ holds

$$\begin{aligned} kS_{\alpha-1}(G) + \frac{(S_3(G) - k(M_1(G)+2m))^{\alpha-2}}{(M_1(G) - 2m(k-1))^{\alpha-3}} &\leqslant S_{\alpha}(G) \leqslant \\ \leqslant \quad nS_{\alpha-1}(G) - \frac{(n(2m + M_1(G) - S_3(G))^{\alpha-2}}{(2m(n-1) - M_1(G))^{\alpha-3}}& \end{aligned}$$

If $2 \leqslant \alpha \leqslant 3$, then the opposite inequalities hold.

COROLLARY 3.2. *Let G be a simple connected graph, $G \ncong K_n$, with $n \geqslant 3$ vertices, m edges and k is real number such that $\mu_{n-1} \geqslant k > 0$. Then for any real α, $\alpha \geqslant 1$ or $\alpha \leqslant 0$, holds*

$$S_\alpha(G) \geqslant \frac{1}{n-k}\left(\frac{k(n(n-1)-2m)^\alpha}{(Kf(G)-n+1)^{\alpha-1}} + \frac{n(2m-k(n-1))^\alpha}{(n-1-\frac{k}{n}Kf(G))^{\alpha-1}}\right).$$

If $0 \leqslant \alpha \leqslant 1$, then the opposite inequality holds.

For $\alpha \geqslant 2$ or $\alpha \leqslant 1$, holds

$$S_\alpha(G) \geqslant \frac{1}{n-k}\left(\frac{k(2m(n-1)-M_1(G))^{\alpha-1}}{(n(n-1)-2m)^{\alpha-2}} + \frac{n(M_1(G)-2m(k-1))^{\alpha-1}}{(2m-k(n-1))^{\alpha-2}}\right).$$

If $1 \leqslant \alpha \leqslant 2$, then the opposite inequality holds.

For $\alpha \geqslant 3$ or $\alpha \leqslant 2$ holds

$$S_\alpha(G) \geqslant \frac{1}{n-k}\left(\frac{n(S_3(G)-k(M_1(G)+2m))^{\alpha-2}}{(M_1(G)-2m(k-1))^{\alpha-3}} + \frac{k(n(M_1(G)+2m)-S_3(G))^{\alpha-2}}{(2m(n-1)-M_1(G))^{\alpha-3}}\right).$$

If $2 \leqslant \alpha \leqslant 3$, then the opposite inequality is valid.

COROLLARY 3.3. *Let G be a simple connected graph, $G \ncong K_n$, with $n \geqslant 3$ vertices and m edges and let p be a real number such that $\mu_{n-1} \geqslant p > 0$. Then*

$$M_1(G) \geqslant \frac{1}{n-p}\left(\frac{p(n(n-1)-2m)^2}{Kf(G)-n+1} + \frac{n^2(2m-p(n-1))^2}{n(n-1)-pKf(G)}\right) - 2m\,,$$

$$M_1(G) \geqslant 2m(p-1) + \frac{n(2m-p(n-1))^2}{n(n-1)-pKf(G)}\,,$$

$$M_1(G) \leqslant 2m(n-1) - \frac{(n(n-1)-2m)^2}{Kf(G)-n+1}\,,$$

$$LEL(G) \geqslant n\,RLEL(G) - (n(n-1)-2m)^{1/2}(Kf(G)-n+1)^{1/2}\,,$$

$$LEL(G) \geqslant \frac{1}{n-p}\left(\frac{p(n(n-1)-2m)^{3/2}}{(2m(n-1)-M_1(G))^{1/2}} + \frac{n(2m-p(n-1))^{3/2}}{(M_1(G)-2m(p-1))^{1/2}}\right),$$

$$LEL(G) \leqslant p\,RLEL(G) + (2m-p(n-1))^{1/2}\left(n-1-\frac{p}{n}Kf(G)\right)^{1/2}.$$

Based on Theorem 3.1 we can successively obtain a number of bounds for $S_\alpha(G)$ for $\alpha \geqslant 4$. This is illustrated in the next corollary for the case $\alpha = 4$.

COROLLARY 3.4. *Let G be a simple connected graph, $G \ncong K_n$, with $n \geqslant 3$ vertices and m edges and let p be a real number such that $\mu_{n-1} \geqslant p > 0$. Then*

$$
\begin{aligned}
S_4(G) &\geqslant p\, S_3(G) + \frac{(2m - p(n-1))^4}{\left(n - 1 - \frac{p}{n} Kf(G)\right)^3}\,, \\
S_4(G) &\geqslant p\, S_3(G) + \frac{(M_1(G) - 2m(p-1))^3}{(2m - p(n-1))^2}\,, \\
S_4(G) &\geqslant p\, S_3(G) + \frac{(S_3(G) - p(M_1(G) + 2m))^2}{M_1(G) - 2m(p-1)}\,, \\
S_4(G) &\leqslant n\, S_3(G) - \frac{(n(M_1(G) + 2m) - S_3(G))^2}{2m(n-1) - M_1(G)}\,, \\
S_4(G) &\leqslant n\, S_3(G) - \frac{(n(n-1) - 2m)^4}{(Kf(G) - n + 1)^3}\,, \\
S_4(G) &\leqslant n\, S_3(G) - \frac{(2m(n-1) - M_1(G))^3}{(n(n-1) - 2m)^2}\,.
\end{aligned}
$$

In the next theorem we establish a relation between $S_\alpha(G)$, $S_\beta(G)$ and $S_{2\alpha-\beta}(G)$, where α and β are arbitrary real numbers.

THEOREM 3.3. *Let G be a simple connected graph with $n \geqslant 3$ vertices and m edges. Then, for any real α and β, holds*

$$
\left(S_\beta(G) - \mu_1^\beta\right)\left(S_{2\alpha-\beta}(G) - \mu_1^{2\alpha-\beta}\right) \geqslant \left(S_\alpha(G) - \mu_1^\alpha\right)^2 . \tag{3.9}
$$

Equality holds if and only if either $\alpha = \beta$, or $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$ where n is even.

PROOF. For any real α and β holds

$$
\sum_{i=2}^{n-1} \mu_i^{2\alpha-\beta} = \sum_{i=2}^{n-1} \frac{(\mu_i^\alpha)^2}{\mu_i^\beta} .
$$

On the other hand, for $r = 1$, the inequality (2.2) can be considered in a form

$$
\sum_{i=2}^{n-1} \frac{x_i^2}{a_i} \geqslant \frac{\left(\sum_{i=2}^{n-1} x_i\right)^2}{\sum_{i=2}^{n-1} a_i} .
$$

For $x_i = \mu_i^\alpha$, $a_i = \mu_i^\beta$, $i = 2, 3, \ldots, n-1$, where α and β are arbitrary real numbers, the above inequality becomes

$$
\sum_{i=2}^{n-1} \frac{(\mu_i^\alpha)^2}{\mu_i^\beta} \geqslant \frac{\left(\sum_{i=2}^{n-1} \mu_i^\alpha\right)^2}{\sum_{i=2}^{n-1} \mu_i^\beta} ,
$$

that is

$$S_{2\alpha-\beta}(G) - \mu_1^{2\alpha-\beta} \geqslant \frac{(S_\alpha(G) - \mu_1^\alpha)^2}{S_\beta(G) - \mu_1^\beta}\,, \tag{3.10}$$

wherefrom (3.9) immediately follows.

Equality in (3.10) holds if and only if $\mu_2^{\alpha-\beta} = \mu_3^{\alpha-\beta} = \cdots = \mu_{n-1}^{\alpha-\beta}$, that is if and only if $\alpha = \beta$ or $\mu_2 = \mu_3 = \cdots = \mu_{n-1}$, which implies that equality in (3.9) holds if and only if either $\alpha = \beta$, or $G \cong K_n$, or $G \cong K_{1,n-1}$, or $G \cong K_{\frac{n}{2},\frac{n}{2}}$ where n is even (see [**4**]). □

By the appropriate choice of parameters α and β, from (3.9) a number of old/new inequalities that reveal relationship between various graph invariants can be obtained. We illustrate that in the next corollary of Theorem 3.3.

Corollary 3.5. *Let G be a simple connected graph with $n \geqslant 2$ vertices and m edges. Then*

$$\begin{aligned}
&\big(Kf(G) - 1\big)\big(M_1(G) + 2m - (1+\Delta)^2\big) \geqslant n(LEL(G) - \sqrt{n})^2\,, &(3.11)\\
&(Kf(G) - 1)(F(G) + 3M_1(G) - 6C_3(G) - (1+\Delta)^3) \geqslant n(2m-n)^2\,,\\
&(2m - \Delta - 1)(Kf(G) - 1) \geqslant n(n-2)^2\,.
\end{aligned}$$

Equalities occur if and only if $G \cong K_n$ or $G \cong K_{1,n-1}$.

Proof. The first inequality is obtained by substituting in (3.9) α with $\frac{1}{2}$ and β with -1, the second for $\alpha = 1$ and $\beta = -1$, and the third for $\alpha = 0$ and $\beta = 1$. Also, we used inequalities $\mu_1 \geqslant 1 + \Delta$, proved in [**13**], and $\mu_1 \leqslant n$, proved in [**10**].

The inequality (3.11) was proved in [**15**]. □

The proof of the next result is fully analogous to that of Theorem 3.3, and hence omitted.

Theorem 3.4. *Let G be a simple connected graph of order n and size m. Then for any real α and β*

$$S_\beta(G) S_{2\alpha-\beta}(G) \geqslant S_\alpha(G)^2\,.$$

Equality holds if and only if $\alpha = \beta$ or $G \cong K_n$.

Corollary 3.6. *Let G be a simple connected graph of order n and size m. Then*

$$\begin{aligned}
Kf(G) &\geqslant \frac{n(LEL(G))^2}{M_1(G) + 2m}\,, &(3.12)\\
Kf(G) &\geqslant \frac{4nm^2}{F(G) + 3M_1(G) - 6C_3(G)}\,,\\
Kf(G) &\geqslant \frac{n(n-1)^2}{2m}\,, &(3.13)
\end{aligned}$$

with equalities if and only if $G \cong K_n$.

The inequality (3.12) was proved in [**15**], and (3.13) in [**22**] (see also [**16**]).

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