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REDUCED AXIOMATIZATION OF LATTICE IMPLICATION ALGEBRAS

Mehmet Terziler, Berken Sertoğlu and Süleyman Polat

Abstract. In this study, we give an equivalent definition of a lattice implication algebra via a reduced number of axioms, and hereby we think we have significantly simplified the common accepted definition of a lattice implication algebra.

1. Introduction

Reasoning on the classical two-valued logic is based on certainty. Natural extensions of this logic deal with uncertainties, vagueness, and fuziness; this kind of logics are many-valued, i.e., non-classical logics. Among them, the lattice implication algebra is a logic system with truth value in a lattice, lattice-valued logic, based on an implication algebra.

So a lattice implication algebra is an alternative logic for knowledge repsesentation and reasoning; more precisely, it is a combination of an algebraic lattice and an implication algebra for which the first axiomatization is proposed in [**3**]. In this logic, the lattice is defined to describe uncertainties, and especially incomplarability whereas the operation $\longrightarrow$, of the implication algebra is intented to describe the way of humans reasoning. For more information, one can consult [**4**].

Different but equivalent definitions of lattice implication algebras can be formulated, for instance, see [**2**]. In this work, we give a new definition of lattice implication algebras with only four axioms equivalent to generally acceptable axiomatization as in [**1**].

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2. Preliminaries

The usual definiton of a lattice implication algebra is as follows:

DEFINITION 2.1. ([**3, 4**]) A lattice implication algebra, (briefly, an LIA), $(L, \vee, \wedge, 0, 1)$, is a bounded lattice with order-reversing involution "$'$" together with a binary relation on L, $(x, y) \longmapsto xy$ (meant to representation, $\longrightarrow$) satisfying the following axioms:
(I_1) $x(yz) = y(xz)$,
(I_2) $xx = 1$,
(I_3) $xy = y'x'$,
(I_4) $xy = yx = 1 \Longrightarrow x = y$
(I_5) $(xy)y = (yx)x$.
(L_1) $(x \vee y)z = (xz) \wedge (yz)$
(L_2) $(x \wedge y)z = (xz) \vee (yz)$.

Note that (L_1) and (L_2) are equivalent to (L_3) and (L_4), respectively, as we shall prove in Section 3:
(L_3) $z(x \wedge y) = (zx) \wedge (zy)$,
(L_4) $z(x \vee y) = (zx) \vee (zy)$.

Examples of lattice imlication algebras are abundant; we will copy out some of them here.

• Let $(L, \vee, \wedge, ', 0, 1)$ be a Boolean lattice. If $x \longrightarrow y$, xy, is defined to be $x' \vee y$, then $(L, \vee, \wedge, ', 0, 1)$ becomes an LIA.

• Luckasiewicz implication algebra on the unit interval $[0, 1]$ of reals is an LIA, if the operations on $[0, 1]$ are defined as follows:

$$x \vee y = max\{x, y\},\ x \wedge y = min\{x, y\},$$
$$x' = 1 - x,\ x \longrightarrow y = min\{1, 1 - x + y\}.$$

• Let $L = \{0, a, b, c, d, 1\}$ and $0 < d < a < 1$, $0 < c < b < 1$. Define the operations on L as follows:
$x \vee y = (xy)y$, $x \wedge y = ((x'y')y')'$, $0' = 1$, $a' = c$, $b' = d$, $c' = a$, $d' = b$, $1' = 0$

$\longrightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Then $(L, \vee, \wedge, ', 0, 1)$ is an LIA. This example is taken from [2].

However, the following is not an LIA: $L = \{0, a, b, 1\}$, $0 < a < b < 1$; $0' = 1$, $a' = b$, $b' = a$, $1' = 0$, $x \vee y = max\{x, y\}$, $x \wedge y = min\{x, y\}$,

$\longrightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then axioms $(I_1)(I_5)$ are satisfied in $(L,', \longrightarrow, 0, 1)$, but (L_1) fails to hold, because we have

$$(a \vee b)b = bb = 1 \text{ and } (ab) \wedge (bb) = b \wedge 1 = b,$$

hence $(L, \vee, \wedge,', 0, 1)$ is not an LIA.

2.1. Order Reversing Involution. We shall elaborate the concept of involution because some people seem to have missed some essentialities about this notation. Let $(E, \leqslant)$ be a partially ordered set. Attached to the partial order $\leqslant$, there is its dual, $\leqslant^*$ which is defined by

$$x \leqslant^* y \Leftrightarrow y \leqslant x.$$

Clearly enough, $\leqslant^*$ is also a partial order on E.

Definition 2.2. An order-reverving involution on $(E, \leqslant)$ is any isomorphism

$$f : (E, \leqslant) \longrightarrow (E, \leqslant^*)$$

such that $f(f(x)) = x$, for each $x \in E$. That is, f is an involutive bijection from E onto E such that

$$x \leqslant y \Leftrightarrow f(x) \leqslant^* f(y) \Leftrightarrow f(y) \leqslant f(x).$$

Example 2.1. (a) Consider the following lattices:

$$E = \{0, a, 1\},\ 0 < a < 1;$$
$$F = \{0, a, b, 1\},\ 0 < a < 1,\ 0 < b < 1;$$
$$G = \{0, a, b, c, 1\},\ 0 < a < c < 1,\ 0 < b < c < 1.$$

Then one can prove the following:

• There is one and only one order-reversing involution on E which is, simply, $0 \longmapsto 1$, $a \longmapsto a$, $1 \longmapsto 0$.

• There are two different order-reversing involutions on F,

$$0 \longmapsto 1,\ a \longmapsto a,\ b \longmapsto b,\ 1 \longmapsto 0;$$
$$0 \longmapsto 1,\ a \longmapsto b,\ b \longmapsto a,\ 1 \longmapsto 0.$$

• There is no order-reversing involution, whatsover, on G because the partially ordered sets $(G, \leqslant)$ and $(G, \leqslant^*)$ are simply not isomorphic. All that can be seen on drawings.

• The unit interval $[0, 1]$ of reals as on ordered set, has one and only one order-reversing isomorphism, namely $x \longmapsto 1 - x$.

Thus, not all lattices have order-reversing involutions. According to cases, they may have only one, many and not at all.

3. Two order relations on LIA

For any LIA, a binary relation is introduced, defined as follows:

$$x \leqslant y \Leftrightarrow xy = 1.$$

This is a partial order; indeed, reflexivity and antisymmetry of $\leqslant$ follow from (I_2) and (I_4), respectively. As to transitivity, let $x \leqslant y$ and $y \leqslant z$; then $xy = 1$ and $yz = 1$. Now, $1 = yz = y(xy)z = x(yy)z = xz$, so $x \leqslant z$.

The order is the partial order of the lattice itself, which we denote $\precsim$, defined by

$$x \precsim y \Leftrightarrow y = x \vee y.$$

That $\precsim$ is a partial order follows from lattices axioms.

It turns out that these two partial orders coincide. Before we prove this very important property, we need to prove some simple results.

Proposition 3.1. *Let L be a lattice with an order-reversing involution $'$. Then we have*

$$x' \wedge y' = (x \vee y)', \qquad (x' \wedge y')' = x \vee y.$$

Proof. Since L is a lattice, $x' \wedge y'$ is the greatest lower bound m for x' and y'; that is,

$$z \precsim m \quad \Leftrightarrow \quad z \precsim x' \;\; \textit{and} \;\; z \precsim y'.$$

The involution reserves orders, therefore

$$m' \precsim z' \quad \Leftrightarrow \quad x \precsim z' \;\; \textit{and} \;\; y \precsim z'.$$

This last statements amounts to say that m' is the least upper bound of x and y, i.e., $m' = x \vee y$, or equivalently, $m = (x \vee y)'$. Whence the following formulas are obtained:

$$\begin{array}{ll} x' \wedge y' = (x \vee y)', & x' \vee y' = (x \wedge y)' \\ (x' \wedge y')' = x \vee y, & (x' \vee y')' = x \wedge y. \end{array}$$

□

Of course, similarly, we also have the duals:

Proposition 3.2. *In an LIA, $(L_1) \Leftrightarrow (L_3)$ and $(L_2) \Leftrightarrow (L_4)$.*

Proof. We only prove $(L_1) \Leftrightarrow (L_3)$; a similar proof obtains $(L_2) \Leftrightarrow (L_4)$.

Starting with (L_1), $(x \vee y)z = (xz) \wedge (yz)$, for each x, y, z, we get

$$\begin{array}{rcll} z'(x \vee y)' & = & (z'x') \wedge (z'y') & (\textit{by } (I_3)) \\ z'(x' \wedge y') & = & (z'x') \wedge (z'y') & (\textit{by Proposition } 3.1) \end{array}$$

for each x', y', z', which yields

$$z(x \wedge y) = (zx) \wedge (zy).$$

This shows that we have $(L_1) \Rightarrow (L_3)$. On the orher hand, we can get $(L_1) \Leftarrow (L_3)$ similarly. □

Property. In an LIA, the partial order $\leqslant$ and the partial order $\precsim$ coincide.

Proof. We show that $x \leqslant y \Leftrightarrow x \precsim y$. Recall that $x \leqslant y \Leftrightarrow xy = 1$ and $x \precsim y \Leftrightarrow y = x \vee y$. We have

$$\begin{aligned} x \precsim y \Leftrightarrow y = x \vee y &\Rightarrow xy = x(x \vee y) \\ &\Rightarrow xy = (xx) \vee (xy) \\ &\Rightarrow xy = 1 \vee (xy) \\ &\Rightarrow xy = 1 \Leftrightarrow x \leqslant y. \end{aligned}$$

On the other hand, using (I_2) and hypothesis $(L_4), (I_4)$ respectively, we have

$$x \leqslant y \Leftrightarrow xy = 1 \Rightarrow (x \vee y)y = (xy) \wedge (yy) = 1$$

and

$$y(x \vee y) = (yx) \vee (yy) = 1$$

and

$$y(x \vee y) = 1 \Rightarrow y = x \vee y \Leftrightarrow x \precsim y.$$

That ends the proof. □

4. An equivalent definition of an LIA

We give an equivalent definition of a lattice implication algebra which (we find) is simpler and more natural.

Definition 4.1. A treillis implicationnal (briefly, a TI), is a bounded lattice $(L, \vee, \wedge, 0, \leqslant)$ with an order-reversing involution, $x \longmapsto x'$, and a multiplication $(x, y) \longmapsto xy$ which satisfies the following axioms:
(A) $x \leqslant y \Leftrightarrow xy = 1$,
(B) $xy = y'x'$,
(C) $x(yz) = y(xz)$,
(D) $(xy)y = (yx)x$.

We shall prove that this definition is equivalent to that given in [4]. To see this, we have to show that (I_2), (I_4), (L_1), and (L_2) hold in any TI. Observe that (I_2) and (I_4) are satisfied via (A). To prove that (L_1) and (L_2) also hold, we need an intermediate result.

Lemma 4.1. *In any TI, the following always hold:*
(1) $0x = 1$, $x1 = 1$,
(2) $1x = x$, $x0 = x'$,
(3) $((xy)y)y = xy$,
(4) $x \leqslant y \Rightarrow (\forall z)(yz \leqslant xz)$, *[i.e.,* $xy = 1 \Rightarrow (\forall z)((yz)(xz) = 1)$*],*
(5) $(x \vee y)z \leqslant (xz) \wedge (yz)$,
(6) $x \vee y = (xy)y$.

Proof. (1) follows from $0 \leqslant x \leqslant 1$, for each x.

(2) Start with $x(1x) = 1(xx) = 11 = 1$, which proves that we have $x \leqslant 1x$. Then take $y = 1$ in (D), and get $1 = 11 = (x1)1 = (1x)x$, which proves that we

have $1x \leqslant x$, whence the identity $1x = x$. The identity $x0 = x'$ follows from (B) (by involution), since $x0 = 0'x' = 1x' = x'$.

(3) Start with

$$\begin{aligned} ((xy)y)y &= (y(xy))(xy) && (by\ (D)) \\ &= (x(yy))(xy) && (by\ (C)) \\ &= (x1)(xy) \\ &= 1(xy) \\ &= xy. \end{aligned}$$

(4) We have

$$\begin{aligned} (yz)(xz) &= x((yz)z) && (by\ (C)) \\ &= x((zy)y) && (by\ (D)) \\ &= (zy)(xy). && (by\ (C)) \end{aligned}$$

If $xy = 1$, then $(yz)(xz) = (zy)(xy) = (zy)1 = 1$.

(5) We have $x \leqslant x \vee y$ and $y \leqslant x \vee y$, so $(x \vee y)z \leqslant xz$ and $(x \vee y)z \leqslant yz$, by (4), therefore, $(x \vee y)z \leqslant (xz) \wedge (yz)$.

(6) There are two parts in the proof.

Part I. We first prove that $x \vee y \leqslant (xy)y$. Indeed, $x((xy)y) = (xy)(xy) = 1$ and $y((xy)y) = (xy)(yy) = 1$, by (C), therefore, $x \leqslant (xy)y$ and $y \leqslant (xy)y$ so that $x \vee y \leqslant (xy)y$.

Part II. We then prove that $(xy)y \leqslant x \vee y$. Indeed, if $x \leqslant a$ and $y \leqslant a$, then $xa = 1$ and $ya = 1$, by (A). Then $((xy)y)a = ((xy)y)(1a) = ((xy)y)((ya)a)$, $((xy)y)((ay)y) = (ay)(((xy)y)y) = (ay)(xy)$ using (3) above. Using (4) above, we have $((xy)y)a = 1$, since $xa = 1$. Therefore, $(xy)y \leqslant a$, so that $(xy)y \leqslant x \vee y$. □

We shall now prove that (L_1) and (L_2) hold in every TI; accordingly, (L_3) and (L_4) also hold in any TI. Here is a proof that (L_1) holds; one can produce a similar proof to show that (L_2) also holds.

Proof. We shall use Lemma 4.1 (5) and (6), and its dual

$(6')$ $\quad x \wedge y = ((xy)x')'$.

Indeed, we have

$$\begin{aligned} ((xz) \wedge (yz))((x \vee y)z) &= ((xz)(yz)(xz)')'((x \vee y)z) && (by\ (6')) \\ &= (((y(xz)z))(xz)')'((x \vee y)z) && (by\ (C)) \\ &= ((y(x \vee z))(xz)')'((x \vee y)z) && (by\ (6)) \\ &= ((x \vee y)z)'((y(x \vee y))(xz)') && (by\ involution) \\ &= (y(x \vee y))(((x \vee y)z)'(xz)') && (by\ (C)) \\ &= (y(x \vee z))((xz)((x \vee y)z)) && (by\ involution) \\ &= (y(x \vee z))((x \vee y)(xz)z) && (by\ (C)) \end{aligned}$$

and

$$\begin{aligned} ((xz) \wedge (yz))((x \vee y)z) &= (y(x \vee z))((x \vee y)(x \vee z)) && (by\ (6)) \\ &= (x \vee y)((y(x \vee z))(x \vee z)). && (by\ (C)) \end{aligned}$$

We have

$$(y(x \vee z))(x \vee z) = ((x \vee z)y)y = (x \vee z) \vee y$$

by (D) and (6), so

$$(x \vee y)(y(x \vee z))(x \vee z) = (x \vee y)(y \vee x \vee z) = 1,$$

since $x \vee y \leqslant y \vee x \vee z$.

Summing up, we have

$$(xz) \wedge (yz) \leqslant (x \vee y)z.$$

We also have

$$(x \vee y)z \leqslant (xz) \wedge (yz)$$

by (5). This shows that (L_1) holds. □

PROPOSITION 4.1. *The lattice of a TI is always distributive.*

PROOF. We have from Lemma 4.1 (6), (D), that $x \vee y = (xy)y = (yx)x$. We want to show that $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. Start with

$$\begin{aligned} x \vee (y \wedge z) &= (y \wedge z) \vee x \\ &= ((y \wedge z)x)x && (by\ Lemma\ 4.1\ (6)) \\ &= (yx \vee zx)x && (by\ (L_2)) \\ &= ((yx)x \wedge ((zx)x)) && (by\ (L_1)) \\ &= ((xy)y) \wedge ((xz)z) && (by\ (D)) \\ &= (x \vee y) \wedge (x \vee z). \end{aligned}$$

□

5. Conclusion

We have significaantly simplified the commonly accepted definition of an LIA. At this stage, we could ask several questions:

- Questions 1. Can our definition be simplified further?
- Questions 2. Can each lattice with involution receive a TI structure?
- Questions 3. Are there lattices which can receive two different TI structures?

As to Question 2, this is possible for each Boolean lattice. Just set $xy = x' \vee y$. Then axioms $(A) - (D)$ are easily verified.

References

[1] Y. B. Jun. The prime filter theorem of lattice implication algebras. *Int. J. Math. Math. Sci.*, **25**(2)(2001), 115–118.

[2] L.-X. Song and K.-L. Zhang. The equivalent definitions of Lattice Implication algebra. In: *Proceedings of the Future Information Technology and Management Engineering, FITME08, 2008* (656–658). ISBN: 978-0-7695-3480-0

[3] Y. Xu. Lattice implication algebra. *J. Southwest Jiaotong Univ.*, **28**(1)(1993), 20–27.

[4] Y. Xu, D. Ruan, K. Qin and J. Liu. *Lattice-Valued Logic, an Alternative Approach to Treat Fuzziness and Incomparability*, Springer, 2003.

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Yaşar University, Faculty of Science, Department of Mathematics, Izmir, Turkey
E-mail address: mehmet.terziler@yasar.edu.tr
E-mail address: berkensertoglu@hotmail.com
E-mail address: polat36@hotmail.com