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A NEW PROOF OF A REFINED YOUNG INEQUALITY

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Abstract. Y. Al-Mansrah and F. Kittaneh proved in [*A generalization of two refined inequalities*, Positivity, 19 (2015) 757-768] the following inequality:

$$(a^{\nu}b^{1-\nu})^m + r_0^m(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \leqslant (\nu a + (1-\nu)b)^m \qquad (M-K)$$

where $r_0 = \min\{\nu, 1-\nu\}$, for all positive integer m and all nonnegative numbers a, b and all parameter $\nu \in [0,1]$.

The inequality $(M-K)$ above is a considerable refinement to the well-known Young inequality and has many nice applications.

The aim of this note is to give a new proof of the inequality $(M-K)$. Our proof is based on the arithmetic-geometric mean inequality and will shorten considerably the lines of proof given by Y. Al-Mansrah and F. Kittaneh.

1. Introduction

The well-known weighted arithmetic-geometric mean (AM-GM) inequality says the following:

Theorem 1.1. *Let n be a positive integer. For all $i = 1, 2, \ldots, n$, let $a_i > 0$, and let $\nu_i \geqslant 0$ satisfy $\sum_{i=1}^n \nu_i = 1$. Then, we have*

$$\prod_{i=1}^{n} a_i^{\nu_i} \leqslant \sum_{i=1}^{n} \nu_i a_i. \tag{1.1}$$

The special case of the weighted AM-GM inequality ($n = 2$) is the classsical Young's inequality:

$$a^{\nu}b^{1-\nu} \leqslant \alpha a + (1-\nu)b. \tag{1.2}$$

valid for any positive real numbers a, b and $0 \leqslant \nu \leqslant 1$.

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Many mathematicians were interested by refining Young's inequality. The interested reader is invited to consult the references of this note and the references therein.

In these lines of investigations, we point out that Hirzallah and Kittaneh [**2**] refined Young's inequality (1.2) so that

$$(a^{\nu}b^{1-\nu})^2 + r_0^2(a-b)^2 \leqslant (\nu a + (1-\nu)b)^2 \tag{1.3}$$

where $r_0 = \min\{\nu, 1-\nu\}$.

Kittaneh and Manasrah [**3**] refined Young's inequality so that

$$a^{\nu}b^{1-\nu} + r_0(\sqrt{a}-\sqrt{b})^2 \leqslant \nu a + (1-\nu)b. \tag{1.4}$$

where $r_0 = \min\{\nu, 1-\nu\}$.

In 2015, Manasrah and Kittaneh [**4**] generalized the inequality (1.3) to

$$(a^{\nu}b^{1-\nu})^m + r_0^m(a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \leqslant (\nu a + (1-\nu)b)^m, \tag{$M-K$}$$

where m is any arbitrary positive integer.

Clearly, $(M-K)$ is a common generalization of the inequalities (1.3) and (1.4).

Recently, S. S. Dragomir (see [**1**]) has given some important applications of the inequality $(M-K)$ for isotonic functionals including Callebaut's inequality, Holder's inequality, certain integral inequalities and some discrete inequalities.

We notice that the proof given by Manasrah and Kittaneh in [**4**] is made through three lemmas a mathematical induction and several intermidiate results.

The purpose of this note is to give a new and short proof of the inequality $(M-K)$.

2. A new proof of the inequality $(M-K)$

We need to recall the following lemma.

LEMMA 2.1. *Let m be a positive integer. Let ν a positive number, such that $0 \leqslant \nu \leqslant 1$. Then we have*

$$\sum_{k=1}^{m} \binom{m}{k} k\nu^k(1-\nu)^{m-k} = m\nu, \tag{2.1}$$

and

$$\sum_{k=0}^{m-1} \binom{m}{k} (m-k)\nu^k(1-\nu)^{m-k} = m(1-\nu), \tag{2.2}$$

where $\binom{m}{k}$ is the usual binomial coefficient.

PROOF. It is sufficient to prove the identity (2.1). To this end, for any non-negative real numbers x_1 and x_2, we know that:

$$(x_1+x_2)^m=\sum_{k=0}^{m}\binom{m}{k}x_1^k x_2^{m-k}. \tag{2.3}$$

By derivation of (2.3) with respect x_1, we find that

$$m(x_1+x_2)^{m-1}=\sum_{k=1}^{m}\binom{m}{k}k x_1^{k-1}x_2^{m-k}. \tag{2.4}$$

By multiplying (2.4) by x_1,

$$m\,x_1(x_1+x_2)^{m-1}=\sum_{k=1}^{m}\binom{m}{k}k\,x_1^k x_2^{m-k}. \tag{2.5}$$

By setting $x_1=\nu$ and $x_2=1-\nu$ in (2.5), we obtain the identity (2.1).

The other identity (2.2) is obtained by a similar manner.

This ends the proof. □

Next we present our proof of the inequality $(M-K)$.

PROOF. Let m be any positive integer and let $a,b>0$ and $0\leqslant\nu\leqslant 1$.

Without loss of generality, we may suppose that $r_0=\nu$. That is, $\nu\leqslant 1-\nu$.

In this case, we claim that

$$(\nu a+(1-\nu)b)^m-\nu^m(a^{\frac{m}{2}}-b^{\frac{m}{2}})^2\geqslant(a^\nu b^{1-\nu})^m.$$

Indeed, we have the following identities

$$\begin{aligned}
&(\nu a+(1-\nu)b)^m-\nu^m(a^{\frac{m}{2}}-b^{\frac{m}{2}})^2\\
=\;&\sum_{k=0}^{m}\binom{m}{k}\nu^k(1-\nu)^{m-k}a^k b^{m-k}-\nu^m(a^m+b^m-2(ab)^{\frac{m}{2}})\\
=\;&\Big((1-\nu)^m-\nu^m\Big)b^m+\sum_{k=1}^{m-1}\binom{m}{k}\nu^k(1-\nu)^{m-k}a^k b^{m-k}+2\nu^m(ab)^{\frac{m}{2}}\\
=\;&\sum_{i=0}^{m}\mu_i y_i,
\end{aligned}\tag{2.6}$$

where the numbers y_i and the parameters μ_i are given by:

$$y_0:=b^m,\quad\text{with}\quad\mu_0:=\Big((1-\nu)^m-\nu^m\Big),$$

$$y_k:=a^k b^{m-k},\text{ with }\mu_k:=\binom{m}{k}\nu^k(1-\nu)^{m-k}\quad\text{for all integer } k\in\{1,2,\dots,m-1\},$$

and

$$y_m=(ab)^{\frac{m}{2}}\quad\text{with}\quad\mu_m=2\nu^m.$$

We observe that

(i) $y_i>0$ for all $i\in\{0,1,\dots m\}$ and

(ii) $\mu_i \geqslant 0$ for all $i \in \{0, 1, \dots m\}$ with $\sum_{k=0}^{m} \mu_k = 1$.

Therefore, (2.6) is a convex combination of positive numbers. So, we may apply the arithmetic-geometric mean inequality and obtain

$$(\nu a + (1-\nu)b)^m - r_0^m (a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \geqslant \prod_{i=0}^{m} y_i^{\mu_i} = a^{\alpha(m)} b^{\beta(m)}$$

where (by using the lemma 2.1), we obtain the following identities:

$$\begin{aligned} \alpha(m) &:= \sum_{k=1}^{m-1} \binom{m}{k} k\nu^k (1-\nu)^{m-k} + m\nu^m \\ &= \sum_{k=1}^{m} \binom{m}{k} k\nu^k (1-\nu)^{m-k} = m\nu, \end{aligned}$$

and

$$\begin{aligned} \beta(m) &:= \sum_{k=1}^{m-1} \binom{m}{k} (m-k)\nu^k (1-\nu)^{m-k} + m(1-\nu)^m \\ &= \sum_{k=0}^{m} \binom{m}{k} (m-k)\nu^k (1-\nu)^{m-k} = m(1-\nu). \end{aligned}$$

Hence, $a^{\alpha(m)} b^{\beta(m)} = (a^\nu b^{1-\nu})^m$. This completes the proof when $r_0 = \nu$.

When $r_0 = (1-\nu)$, the proof of $(M-K)$ is done by a similar manner.

This completes the proof of the inequality $(M-K)$. □

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