BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., Vol. **10**(3)(2020), 413-424
DOI: 10.7251/BIMVI2003413R

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

α-IDEALS IN 0-DISTRIBUTIVE ALMOST SEMILATTICES

G. Nanaji Rao and Ch. Swapna

Abstract. The concept of α-ideal in 0-distributive almost semilattice (0-distributive ASL) is introduced and proved certain properties of α-ideals. Obtained several characterizations for an S-ideal to become α-ideals in 0-distributive ASL and derived a set of identities for an S-ideal to become α-ideal in 0-distributive ASL. Finally, we proved that the image of an α-ideal is again an α-ideal under annihilator preserving homomorphism.

1. Introduction

Varlet [**11**] has introduced the concept of 0-distributive lattices. Generalizing the concept of 0-distributivity in semilattices a theory of 0-distributive semilattices is developed. Using the definition of an ideal (in a semilattice) given by Venkatanarasimhan [**12**], special types of ideals called α-ideals, in 0-distributive semilattices are defined and several characterizations of α-ideals in 0-distributive semilattices and 0-distributive lattices are furnished, which generalize the results of Cornish [**1**] and supplement to those of Jayaram [**2**]. The concept of an α-ideal in almost distributive lattice R was introduced by Rao, G.C. and Sambasiva Rao, M. [**10**] and they proved that there is a one-to-one correspondence between the set $I_\alpha(R)$ of all α-ideals in R and the set of all α-ideals of the lattice $\mathcal{PI}(R)$ of all principal ideals of R. Later, the concept of 0-distributive almost semilattices was introduced by Nanaji Rao and Swapna [**3**] and proved some basic properties of 0-distributive almost semilattices. Also, derived several characterization for an ASL with 0 to become 0-distributive ASL.

1991 *Mathematics Subject Classification.* 06D99, 06D15.
Key words and phrases. 0-distributive ASL, α-ideal, prime S-ideal, minimal prime S-ideal, maximal filter, annihilator S-ideal, dense S-ideal, semi-ideal, annihilator preserving homomorphism.

In this paper, we introduced the concept of α-ideal in 0-distributive ASL and proved certain properties of α-ideals. Obtained several characterizations for an S-ideal to become α-ideals in 0-distributive ASL. Next, we proved that for any annihilator S-ideal I and a filter F in a 0-distributive ASL L such that $I \cap F = \emptyset$, there exists a prime α-ideal P in L containing I and disjoint with F. Also, we derived a set of identities for an S-ideal to become α-ideal in 0-distributive ASL. Finally, we proved that the image of an α-ideal is again an α-ideal under annihilator preserving homomorphism.

2. Preliminaries

In this section we collect a few important definitions and results which are already known and which will be used more frequently in the text.

Definition 2.1. ([**6**]) An ASL with 0 is an algebra $(L, \circ, 0)$ of type (2, 0) satisfies the following conditions:

1. $(x \circ y) \circ z = x \circ (y \circ z)$
2. $(x \circ y) \circ z = (y \circ x) \circ z$
3. $x \circ x = x$
4. $0 \circ x = 0,\ for\ all\ x, y, z \in L.$

Definition 2.2. ([**5**]) Let L be an ASL. A nonempty subset I of L is said to be an S-ideal if it satisfies the following conditions:

1) If $x \in I\ and\ a \in L$, then $x \circ a \in I$.
2) If $x, y \in I$, then there exists $d \in I$ such that $d \circ x = x, d \circ y = y$.

Definition 2.3. ([**5**]) Let L be an ASL and $a \in L$. Then $(a] = \{a \circ x : x \in L\}$ is an S-ideal of L and is called principal S-ideal generated by a.

Definition 2.4. ([**8**]) A nonempty subset F of an ASL L is said to be a $filter$ if F satisfies the following conditions:

(1) $x,\ y \in F$ implies $x \circ y \in F$
(2) If $x \in F\ and\ a \in L\ such\ that\ a \circ x = x, then\ a \in F$

Definition 2.5. ([**5**]) A proper S-ideal P of an ASL L is said to be a prime S-ideal if for any $x, y \in L, x \circ y \in P$ imply $x \in P\ or\ y \in P$.

Definition 2.6. ([**8**]) A proper filter F of L is said to be a prime filter if for any filters $F_1\ and\ F_2$ of L, $F_1 \cap F_2 \subseteq F$ imply $F_1 \subseteq F\ or\ F_2 \subseteq F$.

Definition 2.7. ([**8**]) A proper filter F of L is said to be *maximal* if for any filter G of L such that $F \subseteq G \subseteq L$, then either $F = G$ or $G = L$.

Definition 2.8. ([**8**]) An element $m \in L$ is said to be *unimaximal* if $m \circ x = x$ for all $x \in L$.

Definition 2.9. ([**3**]) Let L be an ASL with 0. Then L is said to be 0-distributive ASL if for any $x, y, z \in L,\ x \circ y = 0\ and\ x \circ z = 0$ then there exists $d \in L$ such that $d \circ y = y,\ d \circ z = z\ and\ d \circ x = 0$.

Definition 2.10. ([**7**]) Let L be an ASL with 0. Then for any nonempty subset A of L, $A^* = \{x \in L : x \circ a = 0 \; for \; all \; a \in A\}$ is called the annihilator of A, and is denoted by A^*. Note that if $A = \{a\}$, then we denote $A^* = \{a\}^*$ by $[a]^*$.

Theorem 2.1 ([**7**]). *Let L be an ASL with 0. Then for any nonempty subsets I, J of L, we have the following:*
(1) $I^* = \bigcap\limits_{a \in I} [a]^*$
(2) $(I \cap J)^* = (J \cap I)^*$
(3) $I \subseteq J \implies J^* \subseteq I^*$
(4) $I^* \cap J^* \subseteq (I \cap J)^*$
(5) $I \subseteq I^{**}$
(6) $I^{***} = I^*$
(7) $I^* \subseteq J^* \Leftrightarrow J^{**} \subseteq I^{**}$
(8) $I \cap J = (0] \Leftrightarrow I \subseteq J^* \Leftrightarrow J \subseteq I^*$
(9) $(I \cup J)^* = I^* \cap J^*$

Theorem 2.2 ([**7**]). *Let L be an ASL with 0. Then for any $x, y \in L$, we have the following.*
(1) $x \leqslant y \Rightarrow [y]^* \subseteq [x]^*$
(2) $[x]^* \subseteq [y]^* \Rightarrow [y]^{**} \subseteq [x]^{**}$
(3) $x \in [x]^{**}$
(4) $(x]^* = [x]^*$
(5) $(x] \cap [x]^* = \{0\}$
(6) $[x \circ y]^* = [y \circ x]^*$
(7) $[x]^* \cap [y]^* \subseteq [x \circ y]^*$
(8) $[x \circ y]^{**} = [x]^{**} \cap [y]^{**}$
(9) $[x]^{***} = [x]^*$
(10) $[x]^* \subseteq [y]^* \Leftrightarrow [y]^{**} \subseteq [x]^{**}$

Theorem 2.3 ([**3**]). *Let L be an ASL with 0. A proper filter M of L is maximal if and only if for any $a \in L - M$, there exists $b \in M$ such that $a \circ b = 0$.*

Theorem 2.4 ([**3**]). *Let L be an ASL with 0, in which intersection of any family of S-ideals is again an S-ideal. Then the following are equivalent:*
(1) *L is 0-distributive ASL.*
(2) *A^* is an S-ideal, for all $A(\neq \emptyset) \subseteq L$.*
(3) *$SI(L)$ is pseudo-complemented semilattice.*
(4) *$SI(L)$ is 0-distributive semilattice.*
(5) *$PSI(L)$ is 0-distributive semilattice.*

Theorem 2.5 ([**4**]). *Every proper filter in ASL L is contained in a maximal filter.*

Theorem 2.6 ([**3**]). *Let L be 0-distributive ASL. Then every maximal filter of L is a prime filter.*

Definition 2.11. ([**7**]) An element a in an ASL L with 0 is said to be dense element if $[a]^* = \{0\}$.

LEMMA 2.1 ([**3**]). *The set D, of all dense elements in an ASL L with unimaximal element is a filter.*

DEFINITION 2.12. ([**4**]) An S-ideal I in a 0-distributive ASL L is called *dense* if $I^* = \{0\}$.

LEMMA 2.2 ([**4**]). *Let L be an ASL. Then a subset P of L is a prime S-ideal if and only if $L - P$ is a prime filter.*

THEOREM 2.7 ([**4**]). *Let L be a 0-distributive ASL. Then a subset M of L is a minimal prime S-ideal if and only if $L - M$ is a maximal filter.*

THEOREM 2.8 ([**4**]). *Let L be a 0-distributive ASL. Then a prime S-ideal M of L is minimal if and only if $[x]^* - M \neq \emptyset$ for any $x \in M$.*

THEOREM 2.9 ([**3**]). *Let L be a 0-distributive ASL with unimaximal element in which intersection of any family of S-ideals is again an S-ideal. Then for any filter F of L and for any annihilator ideal I of L such that $F \cap I = \emptyset$, there exists a prime filter containing F and disjoint with I.*

COROLLARY 2.1 ([**4**]). *Let L be a 0-distributive ASL. Then a prime S-ideal M of L is minimal if and only if it contains precisely one of $\{x\}, [x]^*$ for every $x \in L$.*

THEOREM 2.10 ([**4**]). *Let L be a 0-distributive ASL, in which intersection of any family of S-ideals is again an S-ideal. Then for any S-ideal I of L, I^* is the intersection of all minimal prime S-ideals not containing I.*

DEFINITION 2.13. ([**7**]) Let L and $L^{'}$ be two *ASLs* with 0 and $0^{'}$ respectively. Then a mapping $f : L \to L^{'}$ is called a homomorphism if it satisfies the following:
(1) $f(a \circ b) = f(a) \circ f(b)$ for all $a, b \in L$
(2) $f(0) = 0^{'}$.

DEFINITION 2.14. ([**7**]) Let $L, L^{'}$ be an *ASLs* with 0 and $0^{'}$ respectively and let $f : L \to L^{'}$ be a homomorphism. Then f is called *annihilator preserving* if $f(A^\star) = (f(A))^\star$, for any $\{0\} \subset A \subset L$.

3. α-ideals in 0-distributive Almost Semilattices

In this section, we introduce the concept of an α-ideal in 0-distributive ASL and prove that every minimal prime S-ideal in 0-distributive ASL is an α-ideal. Also, prove that for any filter F in a 0-distributive ASL L, the set

$$O(F) = \{x \in L : x \circ y = 0 \ for\ some\ y \in F\}$$

is an α-ideal. If F is a maximal filter in a 0-distributive ASL then we prove that $O(F)$ is a minimal prime S-ideal and hence is an α-ideal. Next, we prove that every annihilator S-ideal in a 0-distributive ASL is an α-ideal and prove that for any annihilator S-ideal I and a filter F in a 0-distributive ASL L such that $I \cap F = \emptyset$, there exists a prime α-ideal in L containing I and disjoint with F. Also, we introduce the concept of semi-ideal in an ASL with 0 and prove that if I is a dense S-ideal in L such that the semi-ideal $I^{'} = \{x \in L : x \in [a]^{**},\ for\ some\ a \in I\}$ is an S-ideal then I contains a dense element. Also, we derive a set of identities for an S-ideal to become an α-ideal in 0-distributive ASL. We give a necessary and sufficient condition for an S-ideal to become an α-ideal in 0-distributive ASL. Finally, we prove that the image of an α-ideal is again an α-ideal under annihilator preserving homomorphism. First we begin this section with the following definition.

Definition 3.1. Let L be a 0-distributive ASL. Then an S-ideal I in L is called an α-ideal if $[x]^{**} \subseteq I$ *for each* $x \in I$.

Example 3.1. Let L = { 0, a, b, c} and define a binary operation $\circ$ on L as follows:

$\circ$	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Then clearly $(L, \circ)$ is an 0-distributive ASL. Now, put $I = \{0, a\}$. Then clearly I is an S-ideal and also $[x]^{**} \subseteq I$ *for each* $x \in I$. Therefore I is an α-ideal.

It can be easily observe that intersection of any two α-ideals in a 0-distributive ASL is again an α-ideal and hence the set $I_\alpha(L)$ of all α-ideals in a 0-distributive ASL is a subsemilattice of the semilattice $SI(L)$. In the following we prove that every minimal prime S-ideal is an α-ideal.

Theorem 3.1. *Let L be a 0-distributive ASL in which intersection of any family of S-ideals is again an S-ideal. Then every minimal prime S-ideal in L is an α-ideal and for any prime S-ideal P in L, the set*

$$O(P) = \{x \in L : x \circ y = 0 \ for\ some\ y \notin P\}$$

is an α-ideal.

Proof. Suppose M is a minimal prime S-ideal in L and $x \in M$. Now, we shall prove that $[x]^{**} \subseteq M$. Since M is minimal, $[x]^{*} - M \neq \emptyset$. Therefore there exists $y \in [x]^{*}$ *and* $y \notin M$. Let $z \in [x]^{**}$. Then $y \circ z \in [x]^{*} \cap [x]^{**} = \{0\}$. Therefore $y \circ z = 0$. Since M is prime and $y \notin M,\ z \in M$. Thus $[x]^{**} \subseteq M$. Therefore M is an α-ideal. Next, we shall prove that $O(P) = \{x \in L : x \circ y = 0\ for\ some\ y \notin P\}$ is an α-ideal. Since P is prime, and hence is proper. Therefore, we can choose $y \in L$ such that $y \notin P$ and also $0 \circ y = 0$. Hence $0 \in O(P)$. Therefore $O(P)$ is a nonempty subset of L. Let $x \in O(P)$ *and* $t \in L$. Then $x \circ y = 0$ *for some* $y \notin P$ *and* $t \in L$. Consider $(x \circ t) \circ y = (t \circ x) \circ y = t \circ (x \circ y) = t \circ 0 = 0$. Therefore $x \circ t \in O(P)$. Let $x, y \in O(P)$. Then $x \circ a = 0$ *for some* $a \notin P$ *and* $y \circ b = 0$ *for some* $b \notin P$. Therefore, we get $x \circ (a \circ b) = 0$ *and* $y \circ (a \circ b) = 0$. Since L is 0-distributive, there exists $d \in L$ such that $d \circ x = x,\ d \circ y = y$ *and* $d \circ (a \circ b) = 0$. It follows that $d \circ (a \circ b) = 0$ *and* $a \circ b \notin P$, since P is prime. Hence $d \in O(P)$. Let $x \in O(P)$. Then $x \circ y = 0$ *for some* $y \notin P$. Therefore $y \in [x]^{*}$. Let $z \in [x]^{**}$. Then $y \circ z = 0$. Since $y \notin P,\ z \in O(P)$. Therefore $[x]^{**} \subseteq O(P)$. Thus $O(P)$ is an α-ideal. $\square$

COROLLARY 3.1. *Let L be a 0-distributive ASL. Then each prime S-ideal in L contains an α-ideal.*

PROOF. Suppose P is a prime S-ideal of L and $x \in O(P)$. Then there exists $y \notin P$ such that $x \circ y = 0$. Now, $x \circ y = 0 \in P$, we get $x \in P$ since P is a prime S-ideal. Thus $O(P) \subseteq P$. $\square$

Next, we introduce the following notation. For, any filter F in a 0-distributive ASL L, $O(F) = \{x \in L : x \circ y = 0 \ for\ some\ y \in F\}$. In the following we prove that for any filter F in L, $O(F)$ is a α-ideal.

THEOREM 3.2. *Let L be a 0-distributive ASL. Then for any filter F in L, $O(F)$ is an α-ideal in L.*

PROOF. Suppose F is a filter in L. Now, we shall prove that $O(F)$ is an α-ideal in L. Since $0 \in O(F)$, $O(F) \neq \emptyset$. Let $x \in O(F)\ and\ t \in L$. Then $x \circ y = 0\ for\ some\ y \in F$. Consider $(x \circ t) \circ y = (t \circ x) \circ y = t \circ (x \circ y) = t \circ 0 = 0$. Therefore $x \circ t \in O(F)$. Let $x, y \in O(F)$. Then $x \circ a = 0,\ y \circ b = 0\ for\ some\ a, b \in F$. Therefore, we get $x \circ (a \circ b) = 0\ and\ y \circ (a \circ b) = 0$. Since L is 0-distributive, there exists $d \in L$ such that $d \circ x = x,\ d \circ y = y\ and\ d \circ (a \circ b) = 0$. It follows that $d \circ (a \circ b) = 0\ and\ a \circ b \in F$. Hence $d \in O(F)$. Let $x \in O(F)$. Then $x \circ y = 0\ for\ some\ y \in F$. Therefore $y \in [x]^*$. Let $z \in [x]^{**}$. Then $y \circ z = 0$. Therefore $z \in O(F)$ since $y \in F$. Thus $O(F)$ is an α-ideal in L. $\square$

Next, we prove that if F is a maximal filter in 0-distributive ASL, then $O(F) = L - F$ is a minimal prime S-ideal and hence $O(F)$ is an α-ideal.

THEOREM 3.3. *Let L be a 0-distributive ASL and F be a maximal filter in L. Then $O(F)$ minimal prime S-ideal in L.*

PROOF. Suppose F is a maximal filter in L. Then we have $L - F$ is a minimal prime S-ideal in L. Now, we shall prove that $O(F)$ is a minimal prime S-ideal in L. That is enough to prove that $O(F) = L - F$. Let $x \in O(F)$. Then $x \circ y = 0\ for\ some\ y \in F$. Suppose $x \in F$. Then $x \circ y \in F$. It follows that $0 \in F$. Hence $F = L$, a contradiction to F is a maximal filter. Therefore $x \notin F$. Hence $x \in L - F$. Thus $O(F) \subseteq L - F$. Conversely, suppose $x \in L - F$. Then $x \notin F$. It follows that $F \subsetneqq F \vee [x)$. Since F is maximal, $F \vee [x) = L$. Now, we have $0 \in L = F \vee [x)$. Therefore $0 \circ (a \circ b) = a \circ b\ for\ some\ a \in F,\ b \in [x)$. This implies $a \circ b = 0,\ a \in F\ and\ b \in [x)$. Now, $b \in [x)$ and hence $b \circ x = x$. Therefore $a \circ x = a \circ (b \circ x) = (a \circ b) \circ x = 0 \circ x = 0$. Hence $x \in O(F)$. Therefore $L - F \subseteq O(F)$. Thus $O(F) = L - F$. $\square$

THEOREM 3.4. *Let L be a 0-distributive ASL. Then every annihilator S-ideal I in L is an α-ideal in L.*

PROOF. Suppose I is an annihilator S-ideal in L and $x \in I$. Then $I = I^{**}$. Now, let $x \in I$. Then $x \in I^{**}$. It follows that $x \circ y = 0\ for\ all\ y \in I^*$. Therefore $y \in [x]^*\ for\ all\ y \in I^*$. Hence $I^* \subseteq [x]^*$. This implies $I^{**} \supseteq [x]^{**}$. Therefore $I \supseteq [x]^{**}$. Thus I is an α-ideal in L. $\square$

But, the converse of the theorem 3.4, is not true. For, example a proper dense α-ideal is not an annihilator S-ideal. In the following we prove that for an annihilator S-ideal I disjoint with a filter F in 0-distributive ASL L, there exists a prime α-ideal in L containing I and disjoint with F.

THEOREM 3.5. *Let L be a finite 0-distributive ASL, in which intersection of any family of S-ideals is again an S-ideal. Then for any annihilator S-ideal I and a filter F in L such that $I \cap F = \emptyset$, there exists a prime α-ideal P in L containing I and disjoint with F.*

PROOF. Suppose I is an annihilator S-ideal and F is a filter of L such that $I \cap F = \emptyset$. Since I is an annihilator S-ideal, $I = A^*$ for some nonempty subset A of L. Since L is finite, we can write $A = \{a_1, a_2, ..., a_n\}(n\ is\ finite)$. Clearly, $A^* = \bigcap\limits_{i=1}^{n} [a_i]^*$. Now, suppose $[a_i]^* \cap F \neq \emptyset\ for\ all\ i = 1, ..., n$. Then we can choose $x_i \in [a_i]^* \cap F\ for\ each\ i, 1 \leqslant i \leqslant n$. It follows that $(\bigcirc_{i=1}^{n} x_i) \circ a_i = \{0\}\ for\ each\ i,\ 1 \leqslant i \leqslant n$. This implies $\bigcirc_{i=1}^{n} x_i \in [a_i]^*\ for\ all\ i = 1, ..., n$. This implies $\bigcirc_{i=1}^{n} x_i \in \bigcap\limits_{i=1}^{n} [a_i]^*$, also $\bigcirc_{i=1}^{n} x_i \in F$. Therefore $\bigcirc_{i=1}^{n} x_i \in \bigcap\limits_{i=1}^{n} [a_i]^* \cap F = A^* \cap F$. Hence $A^* \cap F \neq \emptyset$. Therefore $I \cap F \neq \emptyset$, a contradiction to our assumption. Thus we can choose $i_0, 1 \leqslant i_0 \leqslant n,\ [a_{i_0}]^* \cap F = \emptyset$. Now, put $\mathscr{F} = \{G : G\ is\ a\ filter\ of\ L, F \subseteq G\ and\ G \cap [a_{i_0}]^* = \emptyset\}$. Then clearly $\mathscr{F} \neq \emptyset,\ since F \in \mathscr{F}$. Clearly, $\mathscr{F}$ is a poset w.r.to set inclusion and also $\mathscr{F}$ satisfies the hypothesis of Zorn's lemma. Therefore by Zorn's lemma, $\mathscr{F}$ has a maximal element (say) H. Then clearly H is a proper filter, $F \subseteq H\ and\ H \cap [a_{i_0}]^* = \emptyset$. Suppose $a_i \notin H$. Then the filter $H \vee [a_i)$ contains H properly. Now, if $(H \vee [a_i)) \cap [a_i]^* = \emptyset$ then $H \vee [a_i) \in \mathscr{F}$, a contradiction to the maximality of H in $\mathscr{F}$. Therefore $(H \vee [a_i)) \cap [a_i]^* \neq \emptyset$. Choose $t \in L$ such that $t \in H \vee [a_i)\ and\ t \in [a_i]^*$. This implies $t \circ (a \circ b) = a \circ b,\ for\ some\ a \in H,\ b \in [a_i)\ and\ t \circ a_i = 0$. Now, we have $t \circ (a \circ b) = a \circ b$ and hence $(t \circ (a \circ b)) \circ a_i = (a \circ b) \circ a_i$. It follows that $(a \circ b) \circ a_i = 0$. Therefore $a \circ a_i = 0$. Hence $a \in [a_i]^*\ and\ also\ a \in H$. Therefore $H \cap [a_i]^* \neq \emptyset$, a contradiction to $H \in \mathscr{F}$. Thus $a_i \in H$. Now, we shall prove that H is a maximal filter. Let $z \in L$ such that $z \notin M$. Since H is maximal element in $\mathscr{F}$, $(H \vee [z)) \cap [a_i]^* \neq \emptyset$. Therefore we can choose $c \in L$ such that $c \in H \vee [z)\ and\ c \in [a_i]^*$. This implies $c \circ (x \circ y) = x \circ y,\ for\ some\ x \in H,\ y \in [z)\ and\ c \circ a_i = 0$. Now, we have $c \circ (x \circ y) = x \circ y$. This implies $(c \circ (x \circ y)) \circ a_i = (x \circ y) \circ a_i$. It follows that $(x \circ y) \circ a_i = 0$. This implies $(x \circ a_i) \circ y = 0$. Hence we get $((x \circ a_i) \circ y) \circ z = 0$. It follows that $(x \circ a_i) \circ (y \circ z) = 0$. Therefore $(x \circ a_i) \circ z = 0, since\ y \in [z)\ and\ hence\ y \circ z = z$. Thus $x \circ a_i \in H\ such\ that\ (x \circ a_i) \circ z = 0$. Therefore H is a maximal filter. Hence $L - H$ is a minimal prime S-ideal. Therefore by theorem 3.1, $L - H$ is a prime α-ideal. Since $[a_i] \cap H = \emptyset,\ [a_i]^* \subseteq L - H$. This implies $\bigcap\limits_{i=1}^{n} [a_i]^* \subseteq L - H$. Hence $A^* \subseteq L - H$. Therefore $I \subseteq L - H$. Also, since $F \subseteq H,\ F \cap (L - H) = \emptyset$. □

Next, we introduce the concept of semi-ideal in an ASL L and prove that if I is an S-ideal in a 0-distributive ASL L then the set $I^{'} = \{x \in L : x \in [a]^{**},\ for\ some\ a \in I\}$ is a semi-ideal.

DEFINITION 3.2. Let L be a ASL with 0. Then a nonempty subset I of L is said to be semi-ideal if $x \leqslant y$ *and* $y \in I$ then $x \in I$.

LEMMA 3.1. *Let L be an ASL with 0 and I be an S-ideal of L. Then the set* $I^{'} = \{x \in L : x \in [a]^{**},\ for\ some\ a \in I\}$ *is a semi-ideal of L.*

PROOF. Let $y \in I^{'}$ *and* $y \in L$ *such that* $x \leqslant y$. Then $y \in [a]^{**}$ *for some* $a \in I$. Therefore $y \circ z = 0$ *for all* $z \in [a]^{*}$. Since $x \leqslant y,\ x \circ z \leqslant y \circ z$. Therefore $x \circ z = 0$ *and* $z \in [a]^{*}$. Hence $x \in I^{'}$. Thus $I^{'}$ is a semi-ideal of L. □

Recall that the set D of all dense element in an ASL with 0 with unimaximal element is a filter. In the following we prove that if I is a dense S-ideal in 0-distributive ASL and if $I^{'}$ is an S-ideal then I contains a dense element.

THEOREM 3.6. *Let L be a finite 0-distributive ASL with unimaximal element, in which intersection of any family of S-ideals is again an S-ideal such that every α-ideal in L is an annihilator S-ideal. Let I be a dense S-ideal in L. Then I contains a dense element if the semi-ideal* $I^{'} = \{x \in L : x \in [a]^{**},\ for\ some\ a \in I\}$ *is an S-ideal in L.*

PROOF. Suppose I is a dense S-ideal in L and suppose $I^{'}$ is an S-ideal in L. Now, we shall prove that I contains a dense element.

Claim I : Every minimal prime S-ideal in L is non-dense.

Suppose M is a minimal prime S-ideal in L. Then by theorem 3.1, M is an α-ideal and hence is an annihilator S-ideal. Thus $M = M^{**}$. Suppose M is dense. Then $M^{*} = \{0\}$. Now, $M = M^{**} = \{M^{*}\}^{*} = \{0\}^{*} = L$, a contradiction. Therefore M is non-dense.

Claim II : $I \cap D = \emptyset$ implies $I^{'} \cap D = \emptyset$.

Suppose $I \cap D = \emptyset$ and suppose $I^{'} \cap D \neq \emptyset$. Then there exists $d \in L$ such that $d \in I^{'}$ *and* $d \in D$. Therefore there exists $a \in I$ *such that* $d \in [a]^{**}$ *and* $d \in D$. Since $[a]^{*} = [a]^{***} \subseteq [d]^{*} = \{0\},\ [a]^{*} = \{0\}$. It follows that $a \in D$. Hence $I \cap D \neq \emptyset$, a contradiction to $I \cap D = \emptyset$. Thus $I^{'} \cap D = \emptyset$.

Claim III : $I^{'}$ is an α-ideal in L.

Suppose $x \in I^{'}$. Now, we shall prove that $[x]^{**} \subseteq I^{'}$. Since $x \in I^{'}$, there exists $a \in I$ such that $x \in [a]^{**}$. It follows that $[x]^{**} \subseteq [a]^{**}$. Hence $[x]^{**} \subseteq [a]^{**} \subseteq I^{'},\ a \in I$. Therefore $[x]^{**} \subseteq I^{'}$. Thus $I^{'}$ is an α-ideal.

Claim IV : $I^{'} \cap D = \emptyset$ implies $I \subseteq M$ for some minimal prime S-ideal M in L.

Suppose $I^{'} \cap D \neq \emptyset$. Then by claim III, $I^{'}$ is an α-ideal and hence by hypothesis, $I^{'}$ is an annihilator S-ideal.Therefore there exists a maximal filter F containing D and disjoint with $I^{'}$. It follows that $M = L - F$ is a minimal prime S-ideal containing $I^{'}$. Since $I \subseteq I^{'},\ I \subseteq M$.

Claim V : I contains a dense element

Suppose $I \cap D = \emptyset$. Then by claim II, $I^{'} \cap D = \emptyset$. Therefore claim IV, $I \subseteq M$. This implies $M^{*} \subseteq I^{*} = (0]$. Hence $M^{*} = (0]$, a contradiction to claim I. Thus $I \cap D \neq \emptyset$. Therefore I contains a dense element. □

Recall that set of all minimal prime S-ideals in 0-distributive ASL denoted by $\mathfrak{M}$. Now, we introduce the following notation.

$$h(x) = \{M \in \mathfrak{M} : x \in M\}(x \in L)$$

In the following, we derive a set of identities for an S-ideal to become α-ideal. First, we prove the following.

LEMMA 3.2. *Let L be a 0-distributive ASL and let $a(\neq 0) \in L$. Then there exists a minimal prime S-ideal not containing a.*

PROOF. Suppose $a \in L$ *and* $a \neq 0$. Now, put $F = [a)$. Then clearly, F is a proper filter of L and hence F is contained in a maximal prime filter (say) K. Therefore by theorem 2.7, $L-K$ is a minimal prime S-ideal of L and $a \notin L-K$. $\square$

LEMMA 3.3. *Let L be a 0-distributive ASL. Then for any $x \in L$, $[x]^* = \cap\{M \in \mathfrak{M} : x \notin M\}$.*

PROOF. Suppose $t \in [x]^*$. Then $t \circ x = 0$. Suppose $M \in \mathfrak{M}$ such that $x \notin M$. Since $t \circ x = 0 \in M$, $t \in M$. Thus $t \in \cap\{M \in \mathfrak{M} : x \notin M\}$. Hence $[x]^* \subseteq \cap\{M \in \mathfrak{M} : x \notin M\}$. Conversely, suppose $t \notin [x]^*$. Then $t \circ x \neq 0$. Therefore there exists a minimal prime S-ideal (say) M in L such that $t \circ x \notin M$. This implies $t \notin M$ and $x \notin M$. It follows that $t \notin \cap\{M \in \mathfrak{M} : x \notin M\}$. Therefore $\cap\{M \in \mathfrak{M} : x \notin M\} \subseteq [x]^*$. Thus $[x]^* = \cap\{M \in \mathfrak{M} : x \notin M\}$. $\square$

LEMMA 3.4. *Let L be a 0-distributive ASL. Then for any $x, y \in L$, $[x]^* \subseteq [y]^*$ if and only if $h(x) \subseteq h(y)$.*

PROOF. Suppose $[x]^* \subseteq [y]^*$ *and* $P \in h(x)$. Then $x \in P$. Hence by corollary 2.1, $[x]^* \not\subseteq P$. Therefore $[y]^* \not\subseteq P$. Hence $y \in P$. Therefore $P \in h(y)$. Thus $h(x) \subseteq h(y)$. Conversely, suppose $h(x) \subseteq h(y)$. Let $t \notin [y]^*$. Then $t \circ y \neq 0$. Hence by lemma 3.2, there exists a minimal prime S-ideal P of L such that $t \circ y \notin P$. Therefore $t \notin P$ *and* $y \notin P$. It follows that $t \notin P$ *and* $P \notin h(y)$. Hence $t \notin P$ *and* $P \notin h(x)$. This implies that $t \notin P$ *and* $x \notin P$. Therefore $t \circ x \notin P$ since P is a prime S-ideal. It follows that $t \circ x \neq 0$. Hence $t \notin [x]^*$. Thus $[x]^* \subseteq [y]^*$. $\square$

Now, we prove the following.

THEOREM 3.7. *Let L be a 0-distributive ASL, in which intersection of any family of S-ideals is again an S-ideal. Then for any S-ideal I of L, following are equivalent:*

(1) *I is an α-ideal.*
(2) *$I = \bigcup\limits_{x \in I} [x]^{**}$.*
(3) *For any $x, y \in L$, $[x]^* = [y]^*$ and $x \in I \implies y \in I$.*
(4) *For any $x, y \in L$, $h(x) = h(y)$, $x \in I \implies y \in I$.*

PROOF. (1) $\Rightarrow$ (2) : Suppose I is an α-ideal. Then $[x]^{**} \subseteq I$ *for each* $x \in I$. Now, we shall prove that $I = \bigcup\limits_{x \in I} [x]^{**}$. Since I is an α-ideal, $[x]^{**} \subseteq I$ *for each* $x \in I$, $\bigcup\limits_{x \in I} [x]^{**} \subseteq I$.

Conversely, suppose $t \in I$. Then $[t]^{**} \subseteq I$. But, we have $t \in [t]^{**} \subseteq \bigcup_{x \in I} [x]^{**}$. This implies $t \in \bigcup_{x \in I} [x]^{**}$. Hence $I \subseteq \bigcup_{x \in I} [x]^{**}$. Thus $I = \bigcup_{x \in I} [x]^{**}$.

Converse is clear.

(2) $\Rightarrow$ (3) : Suppose $y \notin I$. Then $y \notin \bigcup_{x \in I} [x]^{**}$. This implies $y \notin [x]^{**}$ for some $x \in I$. It follows that $y \circ z \neq 0$ *for some* $z \in [x]^*$. Since $[x]^* = [y]^*$, $z \in [y]^*$. Hence $y \circ z = 0$, a contradiction. Therefore $y \in I$.

(3) $\Rightarrow$ (2) : Assume (3). Clearly, $I \subseteq \bigcup_{x \in I} [x]^{**}$.

Conversely, suppose $y \in \bigcup_{x \in I} [x]^{**}$. Then $y \in [x]^{**}$ *for some* $x \in I$. This implies $y \circ z = 0$ *for all* $z \in [x]^*$. Therefore $[x]^* \subseteq [y]^*$. Suppose $t \in [x \circ y]^*$. Then $t \circ (x \circ y) = 0$. It follows that $t \circ y \in [x]^*$. This implies $t \circ y \in [y]^*$. Hence $(t \circ y) \circ y = 0$. Therefore $t \circ y = 0$. Hence $t \in [y]^*$. Therefore $[x \circ y]^* \subseteq [y]^*$. On the other hand, since $x \circ y \leqslant y$, $[y]^* \subseteq [x \circ y]^*$. Therefore $[x \circ y]^* = [y]^*$ and $x \circ y \in I$. Thus by assumption, $y \in I$. Therefore $\bigcup_{x \in I} [x]^{**} \subseteq I$. Hence $I = \bigcup_{x \in I} [x]^{**}$.

Proof of (3) $\Leftrightarrow$ (4) follows by Lemma 3.4. $\square$

Recall that $I^e = \{(a] : a \in I\}$ is an ideal in $PSI(L)$. Next, we derive a necessary and sufficient condition for an S-ideal to become an α-ideal in 0-distributive ASL. First, we prove the following.

Lemma 3.5. *Let L be a 0-distributive ASL. Then* $\{(a]\}^*$ *is an ideal in* $PSI(L)$.

Proof. Since $(a] \cap (0] = (a \circ 0] = (0]$, $(0] \in \{(a]\}^*$. Therefore $\{(a]\}^*$ is a nonempty subset of $PSI(L)$. Now, let $(x] \in \{(a]\}^*$ *and* $(t] \in PSI(L)$ *such that* $(t] \subseteq (x]$. Then $(a] \cap (x] = (0]$. Now, since $(t] \subseteq (x]$, $(t] \cap (a] \subseteq (x] \cap (a]$. It follows that $(t] \cap (a] = (0]$. Hence $(t] \in \{(a]\}^*$. Let $(x], (y] \in \{(a]\}^*$. Then $(a] \cap (x] = (0]$ *and* $(a] \cap (y] = (0]$. It follows that $a \circ x = 0$, $a \circ y = 0$. Since L is 0-distributive, there exists $d \in L$ such that $d \circ x = x$, $d \circ y = y$ *and* $d \circ a = 0$. It follows that $(d] \cap (x] = (d \circ x] = (x]$ *and* $(d] \cap (y] = (d \circ y] = (y]$. Therefore $(x] \subseteq (d]$, $(y] \subseteq (d]$ *and* $d \in \{(a]\}^*$ *since* $d \circ a = 0$. Thus $\{(a]\}^*$ is an ideal in $PSI(L)$. $\square$

Lemma 3.6. *Let L be an ASL with 0. Then for any* $a, b \in L$, *we have the following.*

(1) $x \in [a]^* \Leftrightarrow (x] \in \{(a]\}^*$

(2) $[a]^* = [b]^* \Leftrightarrow \{(a]\}^* = \{(b]\}^*$

Proof. (1) We have $x \in [a]^* \Leftrightarrow x \circ a = 0 \Leftrightarrow (x] \cap (a] = (0] \Leftrightarrow (x] \in \{(a]\}^*$. Therefore $[a]^* = \{(a]\}^*$.

(2) Suppose $[a]^* = [b]^*$. Then

$(x] \in \{(a]\}^* \Leftrightarrow (x] \cap (a] = (0]$

$\Leftrightarrow x \circ a = 0$

$\Leftrightarrow x \in [a]^*$

$\Leftrightarrow x \in [b]^*$

$\Leftrightarrow x \circ b = 0$

$\Leftrightarrow (x \circ b] = (0]$
$\Leftrightarrow (x] \cap (b] = (0]$
$\Leftrightarrow (x] \in \{(a]\}^*$.
Therefore $\{(a]\}^* = \{(b]\}^*$.
Conversely, suppose $\{(a]\}^* = \{(b]\}^*$. Then
$x \in [a]^* \Leftrightarrow (x] \in \{(a]\}^*$
$\Leftrightarrow (x] \in \{(b]\}^*$
$\Leftrightarrow (x] \cap (b] = (0]$
$\Leftrightarrow (x \circ b] = (0]$
$\Leftrightarrow x \circ b = 0$
$\Leftrightarrow x \in [b]^*$
Therefore $[a]^* = [b]^*$. □

Theorem 3.8. *Let L be a 0-distributive ASL and let I be an S-ideal of L. Then I is an α-ideal in L if and only if I is an α-ideal in $PSI(L)$.*

Proof. Suppose I is an α-ideal in L. Now, we shall prove that $I^e := \{(a] : a \in I\}$ is an α-ideal in $PSI(L)$. Clearly, I^e is an ideal in $PSI(L)$ ([5], lemma 4.1). Let $(a], (b] \in PSI(L)$ such that $\{(a]\}^* = \{(b]\}^*$ *and* $(a] \in I^e$. It follows that $a \in I$. Now since $\{(a]\}^* = \{(b]\}^*$, by lemma 3.6, $[a]^* = [b]^*$. Again, since I is an α-ideal of L, $b \in I$. Hence $(b] \in I^e$. Therefore I^e is an α-ideal in $PSI(L)$ since by theorem 3.7. Conversely, suppose I^e is an α-ideal in $PSI(L)$. Let $a, b \in L$ such that $[a]^* = [b]^*$ *and* $a \in I$. Then $\{[a]\}^* = \{[b]\}^*$ *and* $(a] \in I^e$. Since I^e is an α-ideal in $PSI(L)$, $(b] \in I^e$. Hence $b \in I$. Therefore I is an α-ideal in L. □

Recall that an ASL homomorphism $f : L \to L^{'}$ is said to be annihilator preserving homomorphism if for any subset A of L, $\{0\} \subseteq A \subseteq L$, $f(A^*) = (f(A))^*$. In the following we prove that the image of an α-ideal is again an α-ideal under annihilator preserving homomorphism. For this first we need the following lemma.

Lemma 3.7. *Let L and $L^{'}$ be two ASLs with 0 and $0^{'}$ respectively and let $f : L \to L^{'}$ be a homomorphism. Then $f((a]) \subseteq (f(a)]$ $(a \in L)$. Moreover, if f is onto, then $f((a]) = (f(a)]$.*

Proof. Let $f(x) \in f((a])$. Then $x \in (a]$ and hence $x = a \circ x$. It follows that $f(x) = f(a \circ x) = f(a) \circ f(x)$. Therefore $f(x) \in (f(a)]$. Thus $f((a]) \subseteq (f(a)]$. Now, suppose f is onto. Let $t \in (f(a)]$. Since f is onto, there exists $x \in L$ such that $f(x) = t$. It follows that $f(x) = f(a) \circ f(x) = f(a \circ x) \in f((a])$. Therefore $t = f(x) \in f((a])$. Hence $(f(a)] \subseteq f((a])$. Thus $f((a]) = (f(a)]$. □

Theorem 3.9. *Let $L, L^{'}$ be 0-distributive ASLs and let $f : L \to L^{'}$ be an annihilator preserving epimorphism. If J is an α-ideal in L, then $f(J)$ is an α-ideal in $L^{'}$.*

Proof. Suppose J is an α-ideal of L. Now, we shall prove that $f(J)$ is an α-ideal in $L^{'}$. First, we shall prove that $f(J)$ is an S-ideal in $L^{'}$. We have $f(J) = \{f(x) : x \in J\}$. Since $0^{'} = f(0) \in f(J)$, $0^{'} \in f(J)$. Therefore $f(J)$ is a nonempty subset of $L^{'}$. Let $f(a) \in f(J)$ *and* $t \in L^{'}$. Since f is onto, there exists $s \in L$ such that $f(s) = t$. Again, since $a \circ s \in J$, $f(a \circ s) \in f(J)$. This implies $f(a) \circ f(s) \in f(J)$. It follows that $f(a) \circ t \in f(J)$. Let $f(a), f(b) \in f(J)$. Then $a, b \in J$. Since J is an S-ideal in L, there exists $d \in J$ such that $d \circ a = a$ *and* $d \circ b = b$. This implies $f(d) \circ f(a) = f(a)$, $f(d) \circ f(b) = f(b)$ *and* $f(d) \in f(J)$. Therefore $f(J)$ is an S-ideal in $L^{'}$. Let $x \in f(J)$. Then $x = f(a)$, *for some* $a \in J$. Since J is an α-ideal and $a \in J$, $[a]^{**} \subseteq J$. This implies $f((a]^{**}) \subseteq f(J)$. Therefore $(f(a])^{**} \subseteq f(J)$ and hence $(f(a)]^{**} \subseteq f(J)$. Hence we get $[x]^{**} = [f(a)]^{**} \subseteq f(J)$. Therefore $f(J)$ is an α-ideal in $L^{'}$. □

References

[1] Cornish, W. H. Annulets and α-ideals in Distributive Lattices. *J. Aust. Math. Soc.*, **15**(1)(1975), 70–77.

[2] Jayaram, C. Quasicomplemented Semilattices. *Acta. Math, Acad. Sci. Hungar.*, **39**(1-3)(1982), 39–47.

[3] Nanaji Rao, G. and Swapna, Ch. 0-Distributive Almost Semilattices. *Journal of Emerging Technologies and Innovative Research*, **6**(2019), 525–536.

[4] Nanaji Rao, G. and Swapna, Ch. Minimal Prime S-ideals in 0-distributive Almost Semilattices. *Bull. Int. Math. Virtual Inst.*, **10**(2)(2019), 249–261.

[5] Nanaji Rao, G., Swapna, Ch. and Terefe Getchew Beyene. S-ideals in Almost Semilattices. *J. Int. Math. Virtual Inst.*, **9**(2)(2019), 225 –239.

[6] Nanaji Rao G. and Terefe G. B. Almost Semilattice. *International Journal of Mathematical Archive*, **7**(3)(2016), 52–67.

[7] Nanaji Rao G. and Terefe G. B. Annihilator Ideals in Almost Semilattices. *Bull. Int. Math. Virtual Inst.*, **7**(2)(2017), 339–352.

[8] Nanaji Rao G. and Terefe G. B. Relative Annihilators and Filters in Almost Semilattices. *Southeast Asian Bull. Math.*, **43**(2019), 553–576.

[9] Pawar, Y. S. and Thakare, N. K. Minimal Prime Ideals in 0-distributive Semilattices. *Period. Math. Hungar.*, **13**(3)(1982), 237-246.

[10] Rao, G. C. and Sambasiva Rao, M. α-ideals in Almost Distributive Lattices. *Int. J. Contemp. Math. Sciences*, **4**(10)(2009), 457–466.

[11] Varlet, J. C. A Generalization of the Notion of Pseudo-complementedness. *Bull. Soc. Roy. Sci. Liege*, **37** (1968), 149–158.

[12] Venkatanarasimhan, P. V. Pseudo-complements in posets. *Proc. Amer. Math. Soc.*, **28**(1)(1971), 9–17.

Received by editors 06.11.2019; Revised version 11.02.2020; Available online 17.02.2020.

G. Nanaji Rao: Department of Mathematics, Andhra University, Visakhapatnam-530003, India.

E-mail address: nani6us@yahoo.com, drgnanajirao.math@auvsp.edu.in,

Ch. Swapna: Department of Mathematics, Andhra University, Visakhapatnam-530003, India,

E-mail address: swapna_chettupilli@yahoo.com