\(\alpha\)-IDEALS IN
0-DISTRIBUTIVE ALMOST SEMILATTICES

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Abstract. The concept of \(\alpha\)-ideal in 0-distributive almost semilattice (0-distributive ASL) is introduced and proved certain properties of \(\alpha\)-ideals. Obtained several charaterizations for an S-ideal to become \(\alpha\)-ideals in 0-distributive ASL and derived a set of identities for an S-ideal to become \(\alpha\)-ideal in 0-distributive ASL. Finally, we proved that the image of an \(\alpha\)-ideal is again an \(\alpha\)-ideal under annihilator preserving homomorphism.

1. Introduction

Varlet [11] has introduced the concept of 0-distributive lattices. Generalizing the concept of 0-distributivity in semilattices a theory of 0-distributive semilattices is developed. Using the definition of an ideal (in a semilattice) given by Venkata-narasimhan [12], special types of ideals called \(\alpha\)-ideals, in 0-distributive semilattices are defined and several characterizations of \(\alpha\)-ideals in 0-distributive semilattices and 0-distributive lattices are furnished, which generalize the results of Cornish [1] and supplement to those of Jayaram [2]. The concept of an \(\alpha\)-ideal in almost distributive lattice \(R\) was introduced by Rao, G.C. and Sambasiva Rao, M. [10] and they proved that there is a one-to-one correspondence between the set \(I_\alpha(R)\) of all \(\alpha\)-ideals in \(R\) and the set of all \(\alpha\)-ideals of the lattice \(PZ(R)\) of all principal ideals of \(R\). Later, the concept of 0-distributive almost semilattices was introduced by Nanaji Rao and Swapna [3] and proved some basic properties of 0-distributive almost semilattices. Also, derived several characterization for an ASL with 0 to become 0-distributive ASL.

1991 Mathematics Subject Classification. 06D99, 06D15.
Key words and phrases. 0-distributive ASL, \(\alpha\)-ideal, prime S-ideal, minimal prime S-ideal, maximal filter, annihilator S-ideal, dense S-ideal, semi-ideal, annihilator preserving homomorphism.
In this paper, we introduced the concept of $\alpha$-ideal in $0$-distributive ASL and proved certain properties of $\alpha$-ideals. Obtained several characterizations for an S-ideal to become $\alpha$-ideals in $0$-distributive ASL. Next, we proved that for any annihilator S-ideal $I$ and a filter $F$ in a $0$-distributive ASL $L$ such that $I \cap F = \emptyset$, there exists a prime $\alpha$-ideal $P$ in $L$ containing $I$ and disjoint with $F$. Also, we derived a set of identities for an S-ideal to become $\alpha$-ideal in $0$-distributive ASL. Finally, we proved that the image of an $\alpha$-ideal is again an $\alpha$-ideal under annihilator preserving homomorphism.

2. Preliminaries

In this section we collect a few important definitions and results which are already known and which will be used more frequently in the text.

Definition 2.1. ([6]) An ASL with 0 is an algebra $(L, \circ, 0)$ of type $(2, 0)$ satisfies the following conditions:
1. $(x \circ y) \circ z = x \circ (y \circ z)$
2. $(x \circ y) \circ z = (y \circ x) \circ z$
3. $x \circ x = x$
4. $0 \circ x = 0$, for all $x, y, z \in L$.

Definition 2.2. ([5]) Let $L$ be an ASL. A nonempty subset $I$ of $L$ is said to be an S-ideal if it satisfies the following conditions:
1) If $x \in I$ and $a \in L$, then $x \circ a \in I$.
2) If $x, y \in I$, then there exists $d \in I$ such that $d \circ x = x, d \circ y = y$.

Definition 2.3. ([5]) Let $L$ be an ASL and $a \in L$. Then $\{a \circ x : x \in L\}$ is an S-ideal of $L$ and is called principal S-ideal generated by $a$.

Definition 2.4. ([8]) A nonempty subset $F$ of an ASL $L$ is said to be a filter if $F$ satisfies the following conditions:
1) If $x, y \in F$, then $x \circ y \in F$.
2) If $x \in F$ and $a \in L$ such that $a \circ x = x$, then $a \in F$.

Definition 2.5. ([5]) A proper S-ideal $P$ of an ASL $L$ is said to be a prime S-ideal if for any $x, y \in L$, $x \circ y \in P$ imply $x \in P$ or $y \in P$.

Definition 2.6. ([8]) A proper filter $F$ of $L$ is said to be a prime filter if for any filters $F_1$ and $F_2$ of $L$, $F_1 \cap F_2 \subseteq F$ imply $F_1 \subseteq F$ or $F_2 \subseteq F$.

Definition 2.7. ([8]) A proper filter $F$ of $L$ is said to be maximal if for any filter $G$ of $L$ such that $F \subseteq G \subseteq L$, then either $F = G$ or $G = L$.

Definition 2.8. ([8]) An element $m \in L$ is said to be unimaximal if $m \circ x = x$ for all $x \in L$.

Definition 2.9. ([3]) Let $L$ be an ASL with 0. Then $L$ is said to be 0-distributive ASL if for any $x, y, z \in L$, $x \circ y = 0$ and $x \circ z = 0$ then there exists $d \in L$ such that $d \circ y = y, d \circ z = z$ and $d \circ x = 0$. 
Definition 2.10. ([7]) Let L be an ASL with 0. Then for any nonempty subset A of L, \( A^* = \{ x \in L : x \circ a = 0 \mbox{ for all } a \in A \} \) is called the annihilator of A, and is denoted by \( A^* \). Note that if \( A = \{ a \} \), then we denote \( A^* = \{ a \}^* \) by \( [a]^* \).

Theorem 2.1 ([7]). Let L be an ASL with 0. Then for any nonempty subsets I, J of L, we have the following:

1. \( I^* = \bigcap_{a \in I} [a]^* \)
2. \( (I \cap J)^* = (J \cap I)^* \)
3. \( I \subseteq J \implies J^* \subseteq I^* \)
4. \( I^* \cap J^* \subseteq (I \cap J)^* \)
5. \( I \subseteq I^{**} \)
6. \( I^{***} = I^* \)
7. \( I^* \subseteq J^* \iff J^{**} \subseteq I^{**} \)
8. \( I \cap J = \{ 0 \} \iff I \subseteq J^* \iff J \subseteq I^* \)
9. \( (I \cup J)^* = I^* \cap J^* \)

Theorem 2.2 ([7]). Let L be an ASL with 0. Then for any \( x, y \in L \), we have the following.

1. \( x \leq y \implies [y]^* \subseteq [x]^* \)
2. \( [x]^* \subseteq [y]^* \implies [y]^{**} \subseteq [x]^{**} \)
3. \( x \in [x]^{**} \)
4. \( [x]^* = [x]^* \)
5. \( [x] \cap [x]^* = \{ 0 \} \)
6. \( x \circ [y]^* = [y \circ x]^* \)
7. \( [x]^* \cap [y]^* \subseteq [x \circ y]^* \)
8. \( x \circ [y]^{**} = [x]^{**} \cap [y]^{**} \)
9. \( [x]^{***} = [x]^* \)
10. \( [x]^* \subseteq [y]^* \iff [y]^{**} \subseteq [x]^{**} \)

Theorem 2.3 ([3]). Let L be an ASL with 0. A proper filter M of L is maximal if and only if for any \( a \in L - M \), there exists \( b \in M \) such that \( a \circ b = 0 \).

Theorem 2.4 ([3]). Let L be an ASL with 0, in which intersection of any family of S-ideals is again an S-ideal. Then the following are equivalent:

1. \( L \) is 0-distributive ASL.
2. \( A^* \) is an S-ideal, for all \( A(\neq \emptyset) \subseteq L \).
3. SI(L) is pseudo-complemented semilattice.
4. SI(L) is 0-distributive semilattice.
5. PSI(L) is 0-distributive semilattice.

Theorem 2.5 ([4]). Every proper filter in ASL L is contained in a maximal filter.

Theorem 2.6 ([3]). Let L be 0-distributive ASL. Then every maximal filter of L is a prime filter.

Definition 2.11. ([7]) An element \( a \) in an ASL L with 0 is said to be dense element if \( [a]^* = \{ 0 \} \).
Lemma 2.1 ([3]). The set $D$, of all dense elements in an ASL $L$ with unimaximal element is a filter.

Definition 2.12. ([4]) An S-ideal $I$ in a 0-distributive ASL $L$ is called dense if $I^* = \{0\}$.

Lemma 2.2 ([4]). Let $L$ be an ASL. Then a subset $P$ of $L$ is a prime S-ideal if and only if $L - P$ is a prime filter.

Theorem 2.7 ([4]). Let $L$ be a 0-distributive ASL. Then a subset $M$ of $L$ is a minimal prime S-ideal if and only if $L - M$ is a maximal filter.

Theorem 2.8 ([4]). Let $L$ be a 0-distributive ASL. Then a prime S-ideal $M$ of $L$ is minimal if and only if $[x] \cap M \neq \emptyset$ for any $x \in M$.

Theorem 2.9 ([3]). Let $L$ be a 0-distributive ASL with unimaximal element in which intersection of any family of S-ideals is again an S-ideal. Then for any filter $F$ of $L$ and for any annihilator ideal $I$ of $L$ such that $F \cap I = \emptyset$, there exists a prime filter containing $F$ and disjoint with $I$.

Corollary 2.1 ([4]). Let $L$ be a 0-distributive ASL. Then a prime S-ideal $M$ of $L$ is minimal if and only if it contains precisely one of $\{x\}, [x]^*$ for every $x \in L$.

Definition 2.13. ([7]) Let $L$ and $L'$ be two ASLs with 0 and 0' respectively. Then a mapping $f : L \rightarrow L'$ is called a homomorphism if it satisfies the following:

1. $f(a \circ b) = f(a) \circ f(b)$ for all $a, b \in L$
2. $f(0) = 0'$.

Definition 2.14. ([7]) Let $L, L'$ be an ASLs with 0 and 0' respectively and let $f : L \rightarrow L'$ be a homomorphism. Then $f$ is called annihilator preserving if $f(A^*) = (f(A))^*$, for any $(0) \subset A \subset L$.

3. $\alpha$-ideals in 0-distributive Almost Semilattices

In this section, we introduce the concept of an $\alpha$-ideal in 0-distributive ASL and prove that every minimal prime S-ideal in 0-distributive ASL is an $\alpha$-ideal. Also, prove that for any filter $F$ in a 0-distributive ASL $L$, the set

$$O(F) = \{x \in L : x \circ y = 0 \text{ for some } y \in F\}$$

is an $\alpha$-ideal. If $F$ is a maximal filter in a 0-distributive ASL then we prove that $O(F)$ is a minimal prime S-ideal and hence is an $\alpha$-ideal. Next, we prove that every annihilator S-ideal in a 0-distributive ASL is an $\alpha$-ideal and prove that for any annihilator S-ideal $I$ and a filter $F$ in a 0-distributive ASL $L$ such that $I \cap F = \emptyset$, there exists a prime $\alpha$-ideal in $L$ containing $I$ and disjoint with $F$. Also, we introduce the concept of semi-ideal in an ASL with 0 and prove that if $I$ is a dense S-ideal in $L$ such that the semi-ideal $I' = \{x \in L : x \in [a]^*, \text{ for some } a \in I\}$ is an S-ideal.
then I contains a dense element. Also, we derive a set of identities for an S-ideal to become an \( \alpha \)-ideal in 0-distributive ASL. We give a necessary and sufficient condition for an S-ideal to become an \( \alpha \)-ideal in 0-distributive ASL. Finally, we prove that the image of an \( \alpha \)-ideal under annihilator preserving homomorphism. First we begin this section with the following definition.

**Definition 3.1.** Let \( L \) be a 0-distributive ASL. Then an S-ideal \( I \) in \( L \) is called an \( \alpha \)-ideal if \( [x]^* \subseteq I \) for each \( x \in I \).

**Example 3.1.** Let \( L = \{0, a, b, c\} \) and define a binary operation \( \circ \) on \( L \) as follows:

\[
\begin{array}{ccc|ccc}
\circ & 0 & a & b & c \\
0 & 0 & 0 & 0 & 0 \\
a & 0 & a & 0 & a \\
b & 0 & 0 & b & b \\
c & 0 & a & b & c \\
\end{array}
\]

Then clearly \((L, \circ)\) is an 0-distributive ASL. Now, put \( I = \{0, a\} \). Then clearly \( I \) is an S-ideal and also \([x]^* \subseteq I \) for each \( x \in I \). Therefore \( I \) is an \( \alpha \)-ideal.

It can be easily observe that intersection of any two \( \alpha \)-ideals in a 0-distributive ASL is again an \( \alpha \)-ideal and hence the set \( I_\alpha(L) \) of all \( \alpha \)-ideals in a 0-distributive ASL is a subsemilattice of the semilattice \( SI(L) \). In the following we prove that every minimal prime S-ideal is an \( \alpha \)-ideal.

**Theorem 3.1.** Let \( L \) be a 0-distributive ASL in which intersection of any family of S-ideals is again an S-ideal. Then every minimal prime S-ideal in \( L \) is an \( \alpha \)-ideal and for any prime S-ideal \( P \) in \( L \), the set

\[ O(P) = \{ x \in L : x \circ y = 0 \text{ for some } y \notin P \} \]

is an \( \alpha \)-ideal.

**Proof.** Suppose \( M \) is a minimal prime S-ideal in \( L \) and \( x \in M \). Now, we shall prove that \([x]^* \subseteq M \). Since \( M \) is minimal, \([x]^* - M \neq \emptyset \). Therefore there exists \( y \in [x]^* \) and \( y \notin M \). Let \( z \in [x]^* \). Then \( y \circ z \in [x]^* \cap [x]^* = \{0\} \). Therefore \( y \circ z = 0 \). Since \( M \) is prime and \( y \notin M \), \( z \in M \). Thus \([x]^* \subseteq M \). Therefore \( M \) is an \( \alpha \)-ideal. Next, we shall prove that \( O(P) = \{ x \in L : x \circ y = 0 \text{ for some } y \notin P \} \) is an \( \alpha \)-ideal. Since \( P \) is prime, and hence is proper. Therefore, we can choose \( y \in L \) such that \( y \notin P \) and also \( 0 \circ y = 0 \). Hence 0 \in \( O(P) \). Therefore \( O(P) \) is a nonempty subset of \( L \). Let \( x \in O(P) \) and \( t \in L \). Then \( x \circ t = 0 \text{ for some } y \notin P \) and \( t \in L \). Consider \((x \circ t) \circ y = (t \circ x) \circ y = t \circ (x \circ y) = t \circ 0 = 0 \). Therefore \( x \circ t \in O(P) \). Let \( x, y \in O(P) \). Then \( x \circ a = 0 \text{ for some } a \notin P \) and \( y \circ b = 0 \text{ for some } b \notin P \). Therefore, we get \( x \circ (a \circ b) = 0 \) and \( y \circ (a \circ b) = 0 \). Since \( L \) is 0-distributive, there exists \( d \in L \) such that \( d \circ x = x \), \( d \circ y = y \) and \( d \circ (a \circ b) = 0 \). It follows that \( d \circ (a \circ b) = 0 \) and \( a \circ b \notin P \), since \( P \) is prime. Hence \( d \in O(P) \). Let \( x \in O(P) \). Then \( x \circ y = 0 \text{ for some } y \notin P \). Therefore \( y \in [x]^* \). Let \( z \in [x]^* \). Then \( y \circ z = 0 \). Since \( y \notin P \), \( z \in O(P) \). Therefore \([x]^* \subseteq O(P) \). Thus \( O(P) \) is an \( \alpha \)-ideal. \( \square \)
Let $L$ be a 0-distributive ASL. Then each prime $S$-ideal in $L$ contains an $\alpha$-ideal.

**Proof.** Suppose $P$ is a prime $S$-ideal of $L$ and $x \in O(P)$. Then there exists $y \notin P$ such that $x \circ y = 0$. Now, $x \circ y = 0 \in P$, we get $x \in P$ since $P$ is a prime $S$-ideal. Thus $O(P) \subseteq P$. □

Next, we introduce the following notation. For any filter $F$ in a 0-distributive ASL $L$, $O(F) = \{x \in L : x \circ y = 0 \text{ for some } y \in F\}$. In the following we prove that for any filter $F$ in $L$, $O(F)$ is a $\alpha$-ideal.

**Theorem 3.2.** Let $L$ be a 0-distributive ASL. Then for any filter $F$ in $L$, $O(F)$ is an $\alpha$-ideal in $L$.

**Proof.** Suppose $F$ is a filter in $L$. Now, we shall prove that $O(F)$ is an $\alpha$-ideal in $L$. Since $0 \in O(F)$, $O(F) \neq \emptyset$. Let $x \in O(F)$ and $t \in L$. Then $x \circ y = 0$ for some $y \in F$. Consider $(x \circ t) \circ y = (t \circ x) \circ y = t \circ (x \circ y) = t \circ 0 = 0$. Therefore $x \circ t \in O(F)$. Let $x, y \in O(F)$. Then $x \circ a = 0$, $y \circ b = 0$ for some $a, b \in F$. Therefore, we get $x \circ (a \circ b) = 0$ and $y \circ (a \circ b) = 0$. Since $L$ is 0-distributive, there exists $d \in L$ such that $d \circ x = x$, $d \circ y = y$ and $d \circ (a \circ b) = 0$. It follows that $d \circ (a \circ b) = 0$ and $a \circ b \in F$. Hence $d \in O(F)$. Let $x \in O(F)$. Then $x \circ y = 0$ for some $y \in F$. Therefore $y \in [x]^*$. Let $z \in [x]^**$. Then $y \circ z = 0$. Therefore $z \in O(F)$ since $y \in F$. Thus $O(F)$ is an $\alpha$-ideal in $L$. □

Next, we prove that if $F$ is a maximal filter in 0-distributive ASL, then $O(F) = L - F$ is a minimal prime $S$-ideal and hence $O(F)$ is an $\alpha$-ideal.

**Theorem 3.3.** Let $L$ be a 0-distributive ASL and $F$ be a maximal filter in $L$. Then $O(F)$ minimal prime $S$-ideal in $L$.

**Proof.** Suppose $F$ is a maximal filter in $L$. Then we have $L - F$ is a minimal prime $S$-ideal in $L$. Now, we shall prove that $O(F)$ is a minimal prime $S$-ideal in $L$. That is enough to prove that $O(F) = L - F$. Let $x \in O(F)$. Then $x \circ y = 0$ for some $y \in F$. Suppose $x \in F$. Then $x \circ y \in F$. It follows that $0 \in F$. Hence $F = L$, a contradiction to $F$ is a maximal filter. Therefore $x \notin F$. Hence $x \in L - F$. Thus $O(F) \subseteq L - F$. Conversely, suppose $x \in L - F$. Then $x \notin F$. It follows that $F \nsubseteq F \cup [x]$. Since $F$ is maximal, $F \cup [x] = L$. Now, we have $0 \in L = F \cup [x]$. Therefore $0 \circ (a \circ b) = a \circ b$ for some $a \in F$, $b \in [x]$. This implies $a \circ b = 0$, $a \in F$ and $b \in [x]$. Now, $b \in [x]$ and hence $b \circ x = x$. Therefore $a \circ x = a \circ (b \circ x) = (a \circ b) \circ x = 0 \circ x = 0$. Hence $x \in O(F)$. Therefore $L - F \subseteq O(F)$. Thus $O(F) = L - F$. □

**Theorem 3.4.** Let $L$ be a 0-distributive ASL. Then every annihilator $S$-ideal $I$ in $L$ is an $\alpha$-ideal in $L$.

**Proof.** Suppose $I$ is an annihilator $S$-ideal in $L$ and $x \in I$. Then $I = I^*$. Now, let $x \in I$. Then $x \in I^*$. It follows that $x \circ y = 0$ for all $y \in I^*$. Therefore $y \in [x]^*$ for all $y \in I^*$. Hence $I^* \subseteq [x]^*$. This implies $I^{**} \supseteq [x]^{**}$. Therefore $I \supseteq [x]^{**}$. Thus $I$ is an $\alpha$-ideal in $L$. □
Let $L$ be a finite 0-distributive ASL, in which intersection of any family of S-ideals is again an S-ideal. Then for any annihilator S-ideal $I$ and a filter $F$ in $L$ such that $I \cap F = \emptyset$, there exists a prime $\alpha$-ideal $P$ in $L$ containing $I$ and disjoint with $F$.

**Theorem 3.5.** Let $L$ be a finite 0-distributive ASL, in which intersection of any family of S-ideals is again an S-ideal. Then for any annihilator S-ideal $I$ and a filter $F$ in $L$ such that $I \cap F = \emptyset$, there exists a prime $\alpha$-ideal $P$ in $L$ containing $I$ and disjoint with $F$.

**Proof.** Suppose $I$ is an annihilator S-ideal and $F$ is a filter of $L$ such that $I \cap F = \emptyset$. Since $I$ is an annihilator S-ideal, $I = A^*$ for some nonempty subset $A$ of $L$. Since $A$ is finite, we can write $A = \{a_1, a_2, \ldots, a_n\}$ ($n$ is finite). Clearly, $A^* = \bigcap_{i=1}^n [a_i]^*$. Now, suppose $[a_i]^* \cap F \neq \emptyset$ for all $i = 1, \ldots, n$. Then we can choose $x_i \in [a_i]^* \cap F$ for each $i$, $1 \leq i \leq n$. It follows that $(\bigcap_{i=1}^n x_i) \circ a_i = \{0\}$ for each $i$, $1 \leq i \leq n$. This implies $\bigcap_{i=1}^n x_i \in [a_i]^*$ for all $i = 1, \ldots, n$. This implies $\bigcap_{i=1}^n x_i \in F = A^* \cap F$. Hence $A^* \cap F \neq \emptyset$. Therefore $I \cap F \neq \emptyset$, a contradiction to our assumption. Thus we can choose $i_0, 1 \leq i_0 \leq n$, $[a_{i_0}]^* \cap F = \emptyset$. Now, put $\mathcal{F} = \{G : G$ is a filter of $L, F \subseteq G$ and $F \cap [a_i]^* = \emptyset\}$. Then clearly $\mathcal{F} \neq \emptyset$, since $F \in \mathcal{F}$. Clearly, $\mathcal{F}$ is a poset w.r.t. set inclusion and also $\mathcal{F}$ satisfies the hypothesis of Zorn’s lemma. Therefore by Zorn’s lemma, $\mathcal{F}$ has a maximal element (say) $H$. Then clearly $H$ is a proper filter, $F \subseteq H$ and $H \cap [a_i]^* = \emptyset$. Suppose $a_i \not\in H$. Then the filter $H \cup \{a_i\}$ contains $H$ properly. Now, if $(H \cup \{a_i\}) \cap [a_i]^* = \emptyset$ then $H \cup \{a_i\} \in \mathcal{F}$, a contradiction to the maximality of $H$ in $\mathcal{F}$. Therefore $(H \cup \{a_i\}) \cap [a_i]^* \neq \emptyset$. Choose $t \in L$ such that $t \in H \cup \{a_i\}$ and $t \not\in [a_i]^*$. This implies $t \circ (a \circ b) = a \circ b$, for some $a \in H$, $b \in [a_i]$ and $a \circ b = 0$. Now, we have $t \circ (a \circ b) = a \circ b$ and hence $(t \circ (a \circ b)) \circ a_1 = (a \circ b) \circ a_1$. It follows that $(a \circ b) \circ a_i = 0$. Therefore $a \circ a_i = 0$. Hence $a \in [a_i]^*$ and also $a \in H$. Therefore $H \cap [a_i]^* \neq \emptyset$, a contradiction to $H \in \mathcal{F}$. Thus $a_i \in H$. Now, we shall prove that $H$ is a maximal filter. Let $z \in L$ such that $z \not\in M$. Since $H$ is maximal element in $\mathcal{F}$, $(H \cup \{z\}) \cap [a_i]^* \neq \emptyset$. Therefore we can choose $c \in L$ such that $c \in H \cup \{z\}$ and $c \in [a_i]^*$. This implies $c \circ (x \circ y) = x \circ y$, for some $x \in H$, $y \in \{z\}$ and $c \circ a_i = 0$. Now, we have $c \circ (x \circ y) = x \circ y$. This implies $c \circ (x \circ y) = (x \circ y) \circ a_i$. It follows that $(x \circ y) \circ a_i = 0$. This implies $(x \circ a_i) \circ y = 0$. Hence we get $((x \circ a_i) \circ y) \circ z = 0$. It follows that $(x \circ a_i) \circ y \circ z = 0$. Therefore $(x \circ a_i) \circ z = 0$, since $y \in \{z\}$ and hence $y \circ z = z$. Thus $x \circ a_i \in H$ such that $(x \circ a_i) \circ z = 0$. Therefore $H$ is a maximal filter. Hence $L - H$ is a minimal prime S-ideal. Therefore by theorem 3.1, $L - H$ is a prime $\alpha$-ideal. Since $[a_i] \cap H = \emptyset$, $[a_i]^* \subseteq L - H$. This implies $\bigcap_{i=1}^n [a_i]^* \subseteq L - H$. Hence $A^* \subseteq L - H$. Therefore $I \subseteq L - H$. Also, since $F \subseteq H$, $F \cap (L - H) = \emptyset$. □

Next, we introduce the concept of semi-ideal in an ASL $L$ and prove that if $I$ is an S-ideal in a 0-distributive ASL $L$ then the set $I' = \{x \in L : x \in [a]^{**}$, for some $a \in I\}$ is a semi-ideal.
Definition 3.2. Let $L$ be a ASL with 0. Then a nonempty subset $I$ of $L$ is said to be semi-ideal if $x \leq y$ and $y \in I$ then $x \in I$.

Lemma 3.1. Let $L$ be an ASL with 0 and $I$ be an $S$-ideal of $L$. Then the set $I' = \{x \in L : x \in [a]^{**}, \text{ for some } a \in I\}$ is a semi-ideal of $L$.

Proof. Let $y \in I'$ and $y \in L$ such that $x \leq y$. Then $y \in [a]^{**}$ for some $a \in I$. Therefore $y \circ z = 0$ for all $z \in [a]^*$. Since $x \leq y$, $x \circ z \leq y \circ z$. Therefore $x \circ z = 0$ and $z \in [a]^*$. Hence $x \in I'$. Thus $I'$ is a semi-ideal of $L$.

Recall that the set $D$ of all dense element in an ASL with 0 with unimaximal element is a filter. In the following we prove that if $I$ is a dense $S$-ideal in 0-distributive ASL and if $I'$ is an $S$-ideal then $I$ contains a dense element.

Theorem 3.6. Let $L$ be a finite 0-distributive ASL with unimaximal element, in which intersection of any family of $S$-ideals is again an $S$-ideal such that every $\alpha$-ideal in $L$ is an annihilator $S$-ideal. Let $I$ be a dense $S$-ideal in $L$. Then $I$ contains a dense element if the semi-ideal $I' = \{x \in L : x \in [a]^{**}, \text{ for some } a \in I\}$ is an $S$-ideal in $L$.

Proof. Suppose $I$ is a dense $S$-ideal in $L$ and suppose $I'$ is an $S$-ideal in $L$. Now, we shall prove that $I$ contains a dense element.

Claim I: Every minimal prime $S$-ideal in $L$ is non-dense.

Suppose $M$ is a minimal prime $S$-ideal in $L$. Then by theorem 3.1, $M$ is an $\alpha$-ideal and hence is an annihilator $S$-ideal. Thus $M = M^{**}$. Suppose $M$ is dense. Then $M^* = \{0\}$. Now, $M = M^{**} = \{M^*\}^* = \{0\}^* = L$, a contradiction. Therefore $M$ is non-dense.

Claim II: $I \cap D = \emptyset$ implies $I' \cap D = \emptyset$.

Suppose $I \cap D = \emptyset$ and suppose $I' \cap D \neq \emptyset$. Then there exists $d \in L$ such that $d \in I'$ and $d \in D$. Therefore there exists $a \in I$ such that $d \in [a]^{**}$ and $d \in D$. Since $[a]^* = [a]^* \subseteq [d]^* = \{0\}$, $[a]^* = \{0\}$. It follows that $a \in D$. Hence $I \cap D \neq \emptyset$, a contradiction to $I \cap D = \emptyset$. Thus $I' \cap D = \emptyset$.

Claim III: $I'$ is an $\alpha$-ideal in $L$.

Suppose $x \in I'$. Now, we shall prove that $[x]^{**} \subseteq I'$. Since $x \in I'$, there exists $a \in I$ such that $x \in [a]^{**}$. It follows that $[x]^{**} \subseteq [a]^{**} \subseteq I'$, $a \in I$. Therefore $[x]^{**} \subseteq I'$. Thus $I'$ is an $\alpha$-ideal.

Claim IV: $I' \cap D = \emptyset$ implies $I \subseteq M$ for some minimal prime $S$-ideal $M$ in $L$.

Suppose $I' \cap D \neq \emptyset$. Then by claim III, $I'$ is an $\alpha$-ideal and hence by hypothesis, $I'$ is an annihilator $S$-ideal. Therefore there exists a maximal filter $F$ containing $D$ and disjoint with $I'$. It follows that $M = L - F$ is a minimal prime $S$-ideal containing $I'$. Since $I \subseteq I'$, $I \subseteq M$.

Claim V: $I$ contains a dense element.

Suppose $I \cap D = \emptyset$. Then by claim II, $I' \cap D = \emptyset$. Therefore claim IV, $I \subseteq M$. This implies $M^* \subseteq I^* = \{0\}$. Hence $M^* = \{0\}$, a contradiction to claim I. Thus $I \cap D \neq \emptyset$. Therefore $I$ contains a dense element.
Recall that set of all minimal prime S-ideals in 0-distributive ASL denoted by $\mathfrak{M}$. Now, we introduce the following notation.

$$h(x) = \{M \in \mathfrak{M} : x \in M\} (x \in L)$$

In the following, we derive a set of identities for an S-ideal to become $\alpha$-ideal. First, we prove the following.

**Lemma 3.2.** Let $L$ be a 0-distributive ASL and let $a(\neq 0) \in L$. Then there exists a minimal prime S-ideal not containing $a$.

**Proof.** Suppose $a \in L$ and $a \neq 0$. Now, put $F = \{a\}$. Then clearly, $F$ is a proper filter of $L$ and hence $F$ is contained in a maximal prime filter (say) $K$. Therefore by theorem 2.7, $L - K$ is a minimal prime S-ideal of $L$ and $a \notin L - K$. \hfill $\square$

**Lemma 3.3.** Let $L$ be a 0-distributive ASL. Then for any $x \in L$, $[x]^* = \cap \{M \in \mathfrak{M} : x \notin M\}$.

**Proof.** Suppose $t \in [x]^*$. Then $t \circ x = 0$. Suppose $M \in \mathfrak{M}$ such that $x \notin M$. Since $t \circ x = 0 \in M$, $t \in M$. Thus $t \in \cap \{M \in \mathfrak{M} : x \notin M\}$. Hence $[x]^* \subseteq \cap \{M \in \mathfrak{M} : x \notin M\}$. Conversely, suppose $t \notin [x]^*$. Then $t \circ x \neq 0$. Therefore there exists a minimal prime S-ideal (say) $M$ in $L$ such that $t \circ x \notin M$. This implies $t \notin M$ and $x \notin M$. It follows that $t \notin \cap \{M \in \mathfrak{M} : x \notin M\}$. Therefore $\cap \{M \in \mathfrak{M} : x \notin M\} \subseteq [x]^*$. Thus $[x]^* = \cap \{M \in \mathfrak{M} : x \notin M\}$. \hfill $\square$

Lemma 3.4. Let $L$ be a 0-distributive ASL. Then for any $x, y \in L$, $[x]^* \subseteq [y]^*$ if and only if $h(x) \subseteq h(y)$.

**Proof.** Suppose $[x]^* \subseteq [y]^*$ and $P \in h(x)$. Then $x \in P$. Hence by corollary 2.1, $[x]^* \notin P$. Therefore $[y]^* \notin P$. Hence $y \in P$. Therefore $P \in h(y)$. Thus $h(x) \subseteq h(y)$. Conversely, suppose $h(x) \subseteq h(y)$. Let $t \notin [y]^*$. Then $t \circ y \neq 0$. Hence by lemma 3.2, there exists a minimal prime S-ideal $P$ of $L$ such that $t \circ y \notin P$. Therefore $t \notin P$ and $y \notin P$. It follows that $t \notin P$ and $P \notin h(y)$. Hence $t \notin P$ and $P \notin h(x)$. This implies that $t \notin P$ and $x \notin P$. Therefore $t \circ x \notin P$ since $P$ is a prime S-ideal. It follows that $t \circ x \neq 0$. Hence $t \notin [x]^*$. Thus $[x]^* \subseteq [y]^*$. \hfill $\square$

Now, we prove the following.

**Theorem 3.7.** Let $L$ be a 0-distributive ASL, in which intersection of any family of S-ideals is again an S-ideal. Then for any S-ideal $I$ of $L$, following are equivalent:

1. $I$ is an $\alpha$-ideal.
2. $I = \bigcup_{x \in I} [x]^*.$
3. For any $x, y \in L$, $[x]^* = [y]^*$ and $x \in I \implies y \in I.$
4. For any $x, y \in L$, $h(x) = h(y)$, $x \in I \implies y \in I.$

**Proof.** (1) $\Rightarrow$ (2): Suppose $I$ is an $\alpha$-ideal. Then $[x]^* \subseteq I$ for each $x \in I$.

Now, we shall prove that $I = \bigcup_{x \in I} [x]^*$. Since $I$ is an $\alpha$-ideal, $[x]^* \subseteq I$ for each $x \in I$, $\bigcup_{x \in I} [x]^* \subseteq I$. 


Conversely, suppose \( t \in I \). Then \([t]\) \( \subseteq I \). But, we have \( t \in [t]\) \( \subseteq \bigcup_{x \in I} [x] \). Thus \( I = \bigcup_{x \in I} [x] \).

Converse is clear.

\((2) \Rightarrow (3) : \) Suppose \( y \notin I \). Then \( y \notin \bigcup_{x \in I} [x] \). This implies \( y \notin [x] \) for some \( x \in I \). It follows that \( y \circ z \neq 0 \) for some \( z \in [x] \). Since \([x] = [y] \), \( z \in [y] \). Hence \( y \circ z = 0 \), a contradiction. Therefore \( y \in I \).

\((3) \Rightarrow (2) : \) Assume \((3) \). Clearly, \( I \subseteq \bigcup_{x \in I} [x] \).

Conversely, suppose \( y \in \bigcup_{x \in I} [x] \). Then \( y \in [x] \) for some \( x \in I \). This implies \( y \circ z = 0 \) for all \( z \in [x] \). Therefore \([x] \subseteq [y] \). Suppose \( t \in [x \circ y] \). Then \( t \circ (x \circ y) = 0 \). It follows that \( t \circ y \in [x] \). This implies \( t \circ y \in [y] \). Hence \((t \circ y) \circ y = 0 \). Therefore \( t \circ y = 0 \). Hence \( t \in [y] \). Therefore \([x \circ y] \subseteq [y] \). On the other hand, since \( x \circ y \subseteq y \), \([y] \subseteq [x \circ y] \). Therefore \([x \circ y] = [y] \) and \( x \circ y \in I \).

Thus by assumption, \( y \in I \). Therefore \( \bigcup_{x \in I} [x] \) \( \subseteq I \). Hence \( I = \bigcup_{x \in I} [x] \).

Proof of \((3) \Leftrightarrow (4) \) follows by Lemma 3.4.

Recall that \( I^c = \{\{a \} : a \in I \} \) is an ideal in \( PSI(L) \). Next, we derive a necessary and sufficient condition for an \( S \)-ideal to become an \( \alpha \)-ideal in 0-distributive ASL.

First, we prove the following.

**Lemma 3.5.** Let \( L \) be a 0-distributive ASL. Then \( \{\{a\} \} \) is an ideal in \( PSI(L) \).

**Proof.** Since \( (a) \cap \{0\} = (a \circ 0) = \{0\}, \{0\} \in \{\{a\} \} \). Therefore \( \{\{a\} \} \) is a nonempty subset of \( PSI(L) \). Now, let \( x \in \{\{a\} \} \) and \( \{t\} \in PSI(L) \) such that \( \{t\} \subseteq \{x\} \). Then \( \{a\} \cap \{x\} = \{0\} \). Now, since \( \{t\} \subseteq \{x\}, \{t\} \cap \{a\} \subseteq \{x\} \cap \{a\} \). It follows that \( \{t\} \cap \{a\} = \{0\} \). Hence \( \{t\} \in \{\{a\} \} \). Let \( \{x\}, \{y\} \in \{\{a\} \} \). Then \( \{a\} \cap \{x\} = \{0\} \) and \( \{a\} \cap \{y\} = \{0\} \). It follows that \( a \circ x = 0, a \circ y = 0 \). Since \( L \) is 0-distributive, there exists \( d \in L \) such that \( d \circ x = x, d \circ y = y \) and \( d \circ a = 0 \). It follows that \( \{d\} \cap \{x\} = \{d \circ x\} = \{x\} \) and \( \{d\} \cap \{y\} = \{d \circ y\} = \{y\} \). Therefore \( \{x\} \subseteq \{d\} \), \( \{y\} \subseteq \{d\} \) and \( d \in \{\{a\} \} \) since \( d \circ a = 0 \). Thus \( \{\{a\} \} \) is an ideal in \( PSI(L) \).

**Lemma 3.6.** Let \( L \) be an ASL with 0. Then for any \( a, b \in L \), we have the following.

1. \( x \in [a] \Leftrightarrow x \in \{\{a\} \} \)
2. \( [a] = [b] \Leftrightarrow \{\{a\} \} = \{\{b\} \} \)

**Proof.** (1) We have \( x \in [a] \Leftrightarrow x \circ a = 0 \Leftrightarrow \{x\} \cap \{a\} = \{0\} \Leftrightarrow \{x\} \in \{\{a\} \} \).

Therefore \( [a] = \{\{a\} \} \).

(2) Suppose \( [a] = [b] \). Then \( x \in \{\{a\} \} \Leftrightarrow x \in \{\{b\} \} \)
(3) \( x \circ a = 0 \)
(4) \( x \in [a] \)
(5) \( x \in [b] \)
(6) \( x \circ b = 0 \)
\( L \) be a 0-distributive ASL and let \( I \) be an S-ideal of \( L \). Then
\[
\begin{align*}
\& \iff (x \circ b) = (0) \\
\& \iff (x) \cap (b) = (0) \\
\& \iff (x) \subseteq \{(a)\}^*.
\end{align*}
\]
Therefore \( \{(a)\}^* = \{(b)\}^* \). Conversely, suppose \( \{(a)\}^* = \{(b)\}^* \). Then
\[
\begin{align*}
\& x \in [a]^* \iff (x) \subseteq \{(a)\}^* \\
\& \iff (x) \subseteq \{(b)\}^* \\
\& \iff (x) \cap (b) = (0) \\
\& \iff (x \circ b) = (0) \\
\& \iff x \circ b = 0 \\
\& \iff x \in [b]^*
\end{align*}
\]
Therefore \([a]^* = [b]^*\).

**Theorem 3.8.** Let \( L \) be a 0-distributive ASL and let \( I \) be an S-ideal of \( L \). Then \( I \) is an \( \alpha \)-ideal in \( L \) if and only if \( I \) is an \( \alpha \)-ideal in \( PSI(L) \).

**Proof.** Suppose \( I \) is an \( \alpha \)-ideal in \( L \). Now, we shall prove that \( I^\circ := \{(a) : a \in I\} \) is an \( \alpha \)-ideal in \( PSI(L) \). Clearly, \( I^\circ \) is an ideal in \( PSI(L) \) ([5], lemma 4.1). Let \( (a), (b) \in PSI(L) \) such that \( \{(a)\}^* = \{(b)\}^* \) and \( (a) \in I^\circ \). It follows that \( a \in I \). Now since \( \{(a)\}^* = \{(b)\}^* \), by lemma 3.6, \( [a]^* = [b]^* \). Again, since \( I \) is an \( \alpha \)-ideal of \( L \), \( b \in I \). Hence \( (b) \in I^\circ \). Therefore \( I^\circ \) is an \( \alpha \)-ideal in \( PSI(L) \) since by theorem 3.7. Conversely, suppose \( I^\circ \) is an \( \alpha \)-ideal in \( PSI(L) \). Let \( a, b \in L \) such that \( [a]^* = [b]^* \) and \( a \in I \). Then \( \{(a)\}^* = \{(b)\}^* \) and \( (a) \in I^\circ \). Since \( I^\circ \) is an \( \alpha \)-ideal in \( PSI(L) \), \( (b) \in I \). Hence \( b \in I \). Therefore \( I \) is an \( \alpha \)-ideal in \( L \).

Recall that an ASL homomorphism \( f : L \to L' \) is said to be annihilator preserving homomorphism if for any subset \( A \) of \( L \), \( \{0\} \subseteq A \subseteq L \), \( f(A^*) = (f(A))^* \). In the following we prove that the image of an \( \alpha \)-ideal is again an \( \alpha \)-ideal under annihilator preserving homomorphism. For this first we need the following lemma.

**Lemma 3.7.** Let \( L \) and \( L' \) be two ASLs with \( 0 \) and \( 0' \) respectively and let \( f : L \to L' \) be a homomorphism. Then \( f([a]) \subseteq (f(a)) \) (\( a \in L \)). Moreover, if \( f \) is onto, then \( f([a]) = (f(a)) \).

**Proof.** Let \( f(x) \in f([a]) \). Then \( x \in [a] \) and hence \( x = a \circ x \). It follows that \( f(x) = f(a \circ x) = f(a) \circ f(x) \). Therefore \( f(x) \in (f(a)) \). Thus \( f([a]) \subseteq (f(a)) \). Now, suppose \( f \) is onto. Let \( t \in (f(a)) \). Since \( f \) is onto, there exists \( x \in L \) such that \( f(x) = t \). It follows that \( f(x) = f(a) \circ f(x) = f(a \circ x) \in f([a]) \). Therefore \( t = f(a) \in f([a]) \). Hence \( f([a]) \subseteq f([a]) \). Thus \( f([a]) = (f(a)) \).

**Theorem 3.9.** Let \( L, L' \) be 0-distributive ASLs and let \( f : L \to L' \) be an annihilator preserving epimorphism. If \( J \) is an \( \alpha \)-ideal in \( L \), then \( f(J) \) is an \( \alpha \)-ideal in \( L' \).

**Proof.** Suppose \( J \) is an \( \alpha \)-ideal of \( L \). Now, we shall prove that \( f(J) \) is an \( \alpha \)-ideal in \( L' \). First, we shall prove that \( f(J) \) is an S-ideal in \( L' \). We have \( f(J) = \{f(x) : x \in J\} \). Since \( 0' = f(0) \in f(J), 0' \in f(J) \). Therefore \( f(J) \) is a nonempty subset of \( L' \). Let \( f(a) \in f(J) \) and \( t \in L' \). Since \( f \) is onto, there exists \( s \in L \) such
that $f(s) = t$. Again, since $aos \in J$, $f(aos) \in f(J)$. This implies $f(a) \circ f(s) \in f(J)$. It follows that $f(a) \circ t \in f(J)$. Let $f(a), f(b) \in f(J)$. Then $a, b \in J$. Since $J$ is an $S$-ideal in $L$, there exists $d \in J$ such that $d \circ a = a$ and $d \circ b = b$. This implies $f(d) \circ f(a) = f(a)$, $f(d) \circ f(b) = f(b)$ and $f(d) \in f(J)$. Therefore $f(J)$ is an $S$-ideal in $L'$. Let $x \in f(J)$. Then $x = f(a)$, for some $a \in J$. Since $J$ is an $\alpha$-ideal and $a \in J$, $[a]^{**} \subseteq J$. This implies $f([a]^{**}) \subseteq f(J)$. Therefore $[f(a)]^{**} \subseteq f(J)$ and hence $(f(a))^{**} \subseteq f(J)$. Hence we get $[x]^{**} = [f(a)]^{**} \subseteq f(J)$. Therefore $f(J)$ is an $\alpha$-ideal in $L$.

$\square$

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Received by editors 06.11.2019; Revised version 11.02.2020; Available online 17.02.2020.

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