BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., Vol. **10**(2)(2020), 389-402
DOI: 10.7251/BIMVI2002389J

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

ZAGREB INDEX AND COINDEX OF K^{th} GENERALIZED TRANSFORMATION GRAPHS

Raju Jummannaver, Kishori Narayankar and Dickson Selvan

Abstract. Transformation graphs plays important role in the field of chemical graph theory. In this paper, we consider the k^{th} generalized transformation graphs G_k^{ab} and their complements and obtain expressions for Zagreb indices and coindices.

1. Introduction

A graph is a collection of points together with a number of lines connecting a subset of them. The points and lines of a graph are called vertices and edges of the graph, respectively. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, or briefly by V and E, respectively. When we regard molecules as specific chemical structures and replace atoms and bonds with vertices and edges, respectively, the graph obtained is called a molecular graph. That is, a molecular graph is a simple graph in such a way that its vertices match the atoms and its edges with the bonds. Remember that hydrogen atoms are often omitted and the remainder of the graph is sometimes called as the carbon graph of the corresponding molecule. A branch of mathematical chemistry that has a major effect on the development of molecular chemistry and QSAR / QSPR studies is the chemical graph theory which deals with the above-mentioned relationships between molecules and corresponding graphs.

Throughout this paper, we considered simple graph G with n vertices and m edges, denoted as (n, m). Let $V(G)$ and $E(G)$ be its vertex and edge sets, respectively. If u and v are adjacent vertices of G, then the edge joining them will

1991 *Mathematics Subject Classification.* 92E10; 05C07; 05C90.

Key words and phrases. Transformation graphs, Zagreb indices, k^{th} generalized transformation graphs.

Supported by Directorate of Minorities, Karnataka, India.

be denoted by uv. The *degree* of a vertex u in a graph G is the number of edges attached to it and is denoted by $d_G(u)$ or $d(u)$. The *complement* $\bar{G}$ of G is defined to be the graph which has $V(G)$ as vertex set and two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

A topological graph index is a numerical measure correlated with chemical composition for the comparison of chemical composition with different physical properties, chemical reactivity or biological activity. A large number of topological indices for chemical documentation, isomer classification, molecular complexity analysis, chirality, similarity / dissimilarity, QSAR / QSPR, drug design and database selection, lead optimization, etc. have been described and used in recent decades.

As an example, the boiling point of a molecule is directly related to the forces between the atoms. When a solution is heated, the temperature is increased and as it is increased, the kinetic energy between molecules increases. This implies that the molecular motion becomes so strong that the molecular bonds break and become a gas. The moment the liquid turns to gas is labeled as the boiling point. The boiling point can provide important clues about the physical properties of chemical structures. Molecules which strongly interact or bond with each other through a variety of intermolecular forces cannot move easily or rapidly and therefore, do not achieve the kinetic energy necessary to escape the liquid state. That is why the alkanes boiling points increase with the size of the molecules.

The first and second Zagreb indices introduced by Gutman and Trinajstić are two of the most important topological graph indices. They are denoted by $M_1(G)$ and $M_2(G)$ and were defined as [**8**]

$$M_1(G) = \sum_{uv \in E(G)} [d_G(u) + d_G(v)] \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} [d_G(u) d_G(v)].$$

Chemical applications and mathematical properties of Zagreb indices can be found in [**5, 9, 10, 15, 16, 17, 20**].

In 2008, Došlić defined the *first and second Zagreb coindices* as [**6**]

$$\overline{M_1}(G) = \sum_{uv \notin E(G)} [d_G(u) + d_G(v)] \quad \text{and} \quad \overline{M_2}(G) = \sum_{uv \notin E(G)} [d_G(u) d_G(v)]$$

respectively.

More details about Zagreb coindices can be found in [**1, 2, 12, 13, 11**] and the relations between Zagreb indices and coindices are reported in [**7, 9, 19**].

Theorem 1.1 ([**9**]). *For a graph G with n vertices and m edges,*

$$M_1(\overline{G}) = M_1(G) + n(n-1)^2 - 4m(n-1).$$

Theorem 1.2 ([**9**]). *Let G be a graph with n vertices and m edges. Then*

$$M_1(G) + \overline{M_1}(G) = 2m(n-1).$$

Theorem 1.3 ([**9**]). *For a simple graph G,*

$$\overline{M_1}(G) = \overline{M_1}(\overline{G}).$$

THEOREM 1.4 ([**9**]). *Let G be any (n, m) graph. Then*

$$\overline{M_2}(G) = 2m^2 - M_2(G) - \frac{1}{2}M_1(G).$$

THEOREM 1.5 ([**9**]). *Let G be a simple graph with n vertices and m edges. Then*

$$\overline{M_2}(G) = M_2(\overline{G}) - (n-1)M_1(\overline{G}) + \overline{m}(n-1)^2.$$

THEOREM 1.6 ([**9**]). *For a simple graph G,*

$$\overline{M_2}(\overline{G}) = m(n-1)^2 - (n-1)M_1(G) + M_2(G).$$

2. k^{th}-generalized transformation graphs G_k^{ab}

The chemical applications of transformation graphs are explained in [**3**]. In 1973, Sampathkumar and Chikkodimath introduced the new graph valued function, called as semitotal-point graph $T_2(G)$ of a graph G and is defined as follows [**18**]:

DEFINITION 2.1. The semitotal-point graph $T_2(G)$ of a graph G is a graph whose vertex set is $V(T_2(G)) = V(G) \cup E(G)$ and two vertices are adjacent in $T_2(G)$ if and only if

(i) they are adjacent vertices of G or

(ii) one is a vertex of G and other is an edge of G incident with it.

In 2012, S. R. Jog put forward $k-th$ semitotal-point graph of G and is defined as follows [**14**]:

DEFINITION 2.2. The k^{th} semitotal-point graph T_2^k of G is the graph obtained by adding k vertices to each edge of G and joining them to the end vertices of the respective edge. Obviously, this is equivalent to adding k triangles to each edge of G.

Later, Basavanagoud et al. [**3**] introduced some new graphical transformations namely generalized transformation graphs G^{xy} which generalizes the concept of semitotal -point graph.

DEFINITION 2.3. The *generalized transformation graph* G^{xy} is a graph whose vertex set is $V(G) \cup E(G)$, and $\alpha, \beta \in V(G^{xy})$. The vertices α and β are adjacent in G^{xy} if and only if (a) and (b) holds:

(a) $\alpha, \beta \in V(G)$, α, β are adjacent in G if $x = +$ and α, β are not adjacent in G if $x = -$.

(b) $\alpha \in V(G)$ and $\beta \in E(G)$, α, β are incident in G if $y = +$ and α, β are not incident in G if $y = -$.

In view of definitions 2.1-2.3, we define k^{th}-Generalized transformation graphs G_k^{ab} which generalizes the concept of generalized transformation graph G^{xy} [**3**].

Let e_i be the edge of G and let $E_1, E_2, ..., E_m$ be the distinct edge set, and each E_i is correspondent to the edge e_i and $|E_i| = k, i = 1, 2...m.$ for some positive integer k.

DEFINITION 2.4. The k^{th}-*generalized transformation graph* G_k^{xy} is a graph whose vertex set is $V(G_k^{xy}) = V(G) \cup (E' = \cup_{i=1}^{k} E_i)$ and $\alpha, \beta \in V(G_k^{xy})$. The vertices α and β are adjacent in G_k^{xy} if and only if (a) or (b) holds:

(a) $\alpha, \beta \in V(G)$ α, β are adjacent in G if $x = +$ and α, β are not adjacent in G if $x = -$.

(b) $\alpha \in V(G)$ and $\beta \in E_i$, for some $i \in m'$ (where $m' = 1, 2, ..m$), α, e_i are incident in G if $y = +$ and α, e_i are not incident in G if $y = -$.

Since there are four distinct 2-permutations of $\{+-\}$ we can obtain four-graphical transformations of G as G_k^{++}, G_k^{+-}, G_k^{-+} and G_k^{--}. Particularly, if $k = 1$ then $G^{xy} = G_1^{xy}$.

In this paper, we consider the $k^{th}-$ generalized transformation graphs G_k^{xy} and obtain expressions for their first and second Zagreb indices and coindices. Analogous expressions are also obtained for the complements of G_k^{ab}.

3. The Results

We state the following propositions to prove our main results.

PROPOSITION 3.1. *Let G be an (n, m)-graph. Then, the degrees of point and line vertices in G_k^{ab} are*

(1) $d_{G_k^{++}}(v_i) = d(v_i)(k+1)$ *and* $d_{G_k^{++}}(e_i) = 2$.
(2) $d_{G_k^{+-}}(v_i) = d(v_i) + k(m - d(v_i))$ *and* $d_{G_k^{+-}}(e_i) = n - 2$.
(3) $d_{G_k^{-+}}(v_i) = (n-1) + d(v_i)(k-1)$ *and* $d_{G_k^{-+}}(e_i) = 2$.
(4) $d_{G_k^{--}}(v_i) = n + km - 1 - (k+1)d(v_i)$ *and* $d_{G_k^{--}}(e_i) = n - 2$.

PROPOSITION 3.2. *Let G be an (n, m)-graph. Then, the order of G_k^{ab} is $(n + km)$.*

(1) *The size of G_k^{++} is $m(1 + 2k)$.*
(2) *The size of G_k^{+-} is $mk(n-2) + m$.*
(3) *The size of G_k^{-+} is $\frac{n}{2}(n-1) - m(1-2k)$.*
(4) *The size of G_k^{--} is $\frac{n}{2}(n-1) - m + mk(n-2)$.*

THEOREM 3.1. *Let G be an (n, m)-graph. Then*

$$M_1(G_k^{++}) = (k+1)^2 M_1(G) + 4mk.$$

PROOF. Since G_k^{++} has $(n + km)$ vertices.

$$\begin{aligned} M_1(G_k^{++}) &= \sum_{u \in V(G_k^{++})} d_{G_k^{++}}(u)^2 \\ &= \sum_{u \in V(G_k^{++}) \cap V(G)} d_{G_k^{++}}(u)^2 + \sum_{u \in V(G_k^{++}) \cap E'(G)} d_{G_k^{++}}(u)^2 \end{aligned}$$

By Proposition 3.1 we have,

$$= \sum_{u \in V(G_k^{++}) \cap V(G)} [d(u)(k+1)]^2 + \sum_{u \in V(G_k^{++}) \cap E'(G)} 2^2$$

$$M_1(G_k^{++}) = (k+1)^2 M_1(G) + 4mk.$$

□

Corollary 3.1. *Let G be an (n, m)-graph. Then*

$$\overline{M_1}(G_k^{++}) = 2m(n + mk - 1 + 2kn + 2mk^2 - 4k) - (k+1)^2 M_1(G).$$

Proof. From Theorem 1.2

$$\begin{aligned}\overline{M_1}(G_k^{++}) &= 2m^2k + 4m^2k^2 + 2mn - 2m + 4mkn - 8mk - (k+1)^2 M_1(G)\\ &= 2m(n + mk - 1 + 2kn + 2mk^2 - 4k) - (k+1)^2 M_1(G).\end{aligned}$$

□

Corollary 3.2. *Let G be an (n, m)-graph. Then*

$$M_1(\overline{G_k^{++}}) = (k+1)^2 M_1(G) + 4mk + (n+km-1)[(n+km)(n+km-1) - 4m(1+2k)].$$

Proof. From Theorem 1.1,

$$M_1(\overline{G_k^{++}}) = M_1(G^{++k}) + n_1(n_1 - 1)^2 - 4m_1(n_1 - 1)$$

Where n_1 and m_1 are the number of vertices and edges of G_k^{++}. By Theorem 3.1 and 3.2 we get the result. □

Corollary 3.3. *Let G be an (n, m)-graph. Then*

$$\overline{M_1}(\overline{G_k^{++}}) = 2m(n + km - 1 + 2kn + 2mk^2 - 4k) - (k+1)^2 M_1(G).$$

Proof follows from Theorem 1.3 and Corollary 3.1.

Theorem 3.2. *Let G be an (n, m)-graph. Then*

$$M_1(G_k^{+-}) = mk(nmk + 4m - 4mk) + (k-1)^2 M_1(G) + (n-2)^2 m(k).$$

Proof. Since G_k^{+-} has $(n + km)$ vertices.

$$\begin{aligned}M_1(G_k^{+-}) &= \sum_{u \in V(G_k^{+-})} d_{G_k^{+-}}(u)^2\\ &= \sum_{u \in V(G_k^{+-}) \cap V(G)} d_{G_k^{+-}}(u)^2 + \sum_{u \in V(G_k^{+-}) \cap E'(G)} d_{G_k^{+-}}(u)^2\end{aligned}$$

By Proposition 3.1, we have

$$\begin{aligned}&= \sum_{u \in V(G_k^{++}) \cap V(G)} [d(u) + k(m - d(u))]^2\\ &\quad + \sum_{u \in V(G_k^{++}) \cap E'(G)} (n-2)^2\\ &= \Big[(2mk - 2mk^2) \sum d(u) - (2k - k^2 - 1)\\ &\quad \sum d(u)^2 + nm^2k^2\Big] + \Big[\sum (n-2)^2\Big]\\ M_1(G_k^{+-}) &= mk(nmk + 4m - 4mk) + (k-1)^2 M_1(G) + (n-2)^2 m(k).\end{aligned}$$

□

COROLLARY 3.4. *Let G be an (n,m)-graph. Then*

$$M_1(\overline{G_k^{+-}}) = mk(m^2k^2 + 2mk + 4n - 3) - 4m(n-1) + n(n-1)^2 + (k-1)^2 M_1(G).$$

PROOF. From Theorem 1.1, 3.2 and Proposition 3.2 we get the required result. □

COROLLARY 3.5. *Let G be an (n,m)-graph. Then*

$$\overline{M_1}(\overline{G_k^{+-}}) = m^2k^2n - 2m^2k + mkn^2 - 2mkn + 2mn - 2m - (k-1)^2 M_1(G).$$

PROOF. From Theorem 1.3 and 1.2 we have,

$$\begin{aligned}\overline{M_1}(\overline{G_k^{+-}}) =& 2m(n-1) - M_1(G_k^{+-}) \\ =& 2(mk(n-2)+m)(n+km-1) - [mk(nmk + 4m - 4mk) \\ &+ (k-1)^2 M_1(G) + (n-2)^2 mk] \\ \overline{M_1}(\overline{G_k^{+-}}) =& m^2k^2n - 2m^2k + mkn^2 - 2mkn + 2mn - 2m - (k-1)^2 M_1(G).\end{aligned}$$

□

COROLLARY 3.6. *Let G be an (n,m)-graph. Then*

$$\overline{M_1}(G_k^{+-}) = m^2k^2n - 2m^2k + mkn^2 - 2mkn + 2mn - 2m - (k-1)^2 M_1(G).$$

PROOF. Applying Theorem 1.3 and Corollary 3.5 we get the result. □

THEOREM 3.3. *Let G be an (n,m)-graph. Then*

$$M_1(G_k^{-+}) = 4m(nk - n - k + 1) + (k-1)^2 M_1(G) + n(n-1)^2 + 4mk.$$

PROOF. Since G_k^{-+} has $(n+km)$ vertices.

$$\begin{aligned}M_1(G_k^{-+}) &= \sum_{u \in V(G_k^{-+})} d_{G_k^{-+}}(u)^2 \\ &= \sum_{u \in V(G_k^{-+}) \cap V(G)} d_{G_k^{-+}}(u)^2 + \sum_{u \in V(G_k^{-+}) \cap E(G)} d_{G^{-+k}}(u)^2.\end{aligned}$$

By Proposition 3.1, we have,

$$\begin{aligned}&= \sum_{u \in V(G_k^{-+}) \cap V(G)} \Big[(n-1) + d_{G_k^{-+}}(u)(k-1)\Big]^2 \\ &\quad + \sum_{u \in V(G_k^{++}) \cap E'(G)} 2^2 \\ &= \Big[2(nk - n - k + 1)\sum d(u) + (k^2 - 2k + 1)\sum d(u)^2 \\ &\quad + \sum n^2 - 2n + 1\Big] + \sum(4) \\ &= 4m(nk - n - k + 1) + (k-1)^2 M_1(G) + n(n-1)^2 + 4mk\end{aligned}$$

$$M_1(G^{-+k}) = 4m(nk-n-k+1)+(k-1)^2M_1(G)+n(n-1)^2+4mk.$$

□

COROLLARY 3.7. *Let G be an (n,m)-graph. Then*

$$M_1(\overline{G_k^{-+}}) = m^2k^2(mk+3n-10)+mk(4m-6n+9+n^2)+(k-1)^2M_1(G).$$

PROOF. From Theorem 1.1 implies $M_1(G_k^{-+})+(n_1-1)^2-4m_1(n_1-1)$. Where n_1 and m_1 are number of vertices and edges of G_k^{-+}. Now the results follows from Theorem 3.3 and Proposition 3.2. □

The next two Corollaries are deduced in a fully analogous manner.

COROLLARY 3.8. *Let G be an (n,m)-graph. Then*

$$\begin{aligned}\overline{M_1}(G_k^{-+}) &= 2\left(\binom{n}{2}+(2k-1)m\right)(n+km-1)-n(n-1)^2\\ &\quad -4m(k-(k-1)(n-1))-(k-1)^2M_1(G).\end{aligned}$$

COROLLARY 3.9. *Let G be an (n,m)-graph. Then*

$$\begin{aligned}\overline{M_1}(\overline{G_k^{-+}}) &= n^2mk-nmk+2nm-2m^2k+4m^2k^2-2m-4mk\\ &\quad -(k-1)^2\left[4m(nk-n-k+1)+(k-1)^2M_1(G)+n(n-1)^2+4mk\right].\end{aligned}$$

THEOREM 3.4. *Let G be an (n,m)-graph. Then*

$$M_1(G_k^{--}) = n(n+km-1)^2-4m(k+1)(n+km-1)+(k+1)^2M_1(G)+km(n-2)^2.$$

PROOF.

$$\begin{aligned}M_1(G_k^{--}) &= \sum_{u\in V(G_k^{--})} d_{G_k^{--}}(u)^2\\ &= \sum_{u\in V(G_k^{--})\cap V(G)} d_{G_k^{--}}(u)^2 + \sum_{u\in V(G_k^{--})\cap E'(G)} d_{G_k^{--}}(u)^2\end{aligned}$$

By Proposition 3.1, we have,

$$\begin{aligned}&= \sum_{u\in V(G_k^{-+})\cap V(G)} [(n+mk-1-(k+1)d(u))]^2 + \sum_{u\in V(G_k^{++})\cap E'(G)} (n-2)^2\\ &= \sum_{u\in V(G_k^{-+})\cap V(G)} [2mnk+m^2k^2-2mk+(n-1)^2+d(u)\\ &\quad (-2nk-2mk^2-2mk+2k-2n+2)+(k+1)^2d(u)^2]\\ &\quad + \sum_{u\in V(G_k^{++})\cap E(G)} (n-2)^2\\ &= n(n+km-1)^2-4m(k+1)(n+km-1)+(k+1)^2M_1(G)+km(n-2)^2.\end{aligned}$$

□

COROLLARY 3.10. *Let G be an (n,m)-graph. Then*

$$\begin{aligned} M_1(\overline{G_k^{--}}) &= n(n+km-1)^2 - 4m(k+1)(n+km-1) + (k+1)^2 M_1(G) \\ &\quad + km(n-2)^2 + (km+n-1)[(km+n)(km+n-1) \\ &\quad - 2n(n-1) + 4m - 4mk(n-2)]. \end{aligned}$$

PROOF. Theorem 1.1 results in

$$M_1(\overline{G_k^{--}}) = M_1(G_k^{--}) + n_1(n_1-1)^2 - 4m_1(n_1-1)$$

where n_1 and m_1 are numbers of vertices and edges of G_k^{--}. Bearing in mind Theorem 3.4 and Proposition 3.2 the result follows. □

The next two Corollaries are deduced in a fully analogous manner.

COROLLARY 3.11. *Let G be an (n,m)-graph. Then*

$$\overline{M_1}(G_k^{--}) = 2m^2k + nm^2k^2 + 2nm - 2m + 3nmk - 4mk - (k+1)^2 M_1(G).$$

COROLLARY 3.12. *Let G be an (n,m)-graph. Then*

$$\overline{M_1}(\overline{G_k^{--}}) = 2m^2k + nm^2k^2 + 2nm - 2m + 3nmk - 4mk - (k+1)^2 M_1(G).$$

THEOREM 3.5. *Let G be an (n,m)-graph. Then*

$$M_2(G_k^{++}) = M_2(T_2)(G) = (k+1)^2 M_2(G) + 2k(k+1)M_1(G).$$

PROOF. Partition the edge set $E(G^{++k})$ into two sets E_1 and E_2.
$E_1 = \{uv \in E(G)\}$ and $E_2 = \{ue | u \text{ is incident to edge } e\}$,
$|E_1| = m$ and $|E_2| = 2mk$.
Therefore,

$$\begin{aligned} M_2(G_k^{++}) &= \sum_{uv\in E(G^{++k})} [d_{G^{++k}}(u) d_{G^{++k}}(v)] \\ &= \sum_{uv\in E_1} (d_{G^{++k}}(u) d_{G^{++k}}(v)) + \sum_{uv\in E_2} (d_{G_k^{++}}(u) d_{G_k^{++}}(v)) \\ &= \sum_{uv\in E(G)} (d_G(u)(k+1) d_G(v)(k+1)) + \sum_{ue\in E_2} (d_G(u)(k+1)(2) \\ &= (k+1)^2 \sum_{uv\in E(G)} [d_G(u) d_G(v)] + 2k(k+1) \sum_{u\in V(G)} d_G(u)^2 \\ &= (k+1)^2 M_2(G) + 2k(k+1) M_1(G). \end{aligned}$$

□

COROLLARY 3.13. *Let G be an (n,m)-graph. Then*

$$\overline{M_2}(G_k^{++}) = 2m[m(1+2k)^2+1] - (k+1)^2 M_2(G) - (k+1)M_1(G)\left(\frac{4+(k+1)}{2}\right).$$

PROOF. We know that,

$$\begin{aligned}
\overline{M_2}(G_k^{++}) &= 2m^2 - M_2(G_k^{++}) - \frac{1}{2}M_1(G_k^{++}) \\
&= 2(m(1+2k))^2 - [(k+1)^2M_2(G) + 2k(k+1)M_1(G)] \\
&\quad -\frac{1}{2}[(k+1)^2M_1(G) + 4m] \\
&= 2m[m(1+2k)^2+1] - (k+1)^2M_2(G) \\
&\quad -(k+1)M_1(G)\left(\frac{4+(k+1)}{2}\right).
\end{aligned}$$

□

COROLLARY 3.14. *Let G be an (n,m)-graph. Then*

$$\begin{aligned}
M_2(\overline{G_k^{++}}) &= 2m[m(1+2k)^2+1] - (k+1)^2M_2(G) - (k+1)M_1(G) \\
&\quad \left(\frac{4+(k+1)}{2}\right) + (n+km-1)\big[(k+1)^2M_1(G) + 4mk \\
&\quad +(n+km-1)[(n+km)\ (n+km-1) - 4m(1+2k)]\big] \\
&\quad +\left(\binom{n+km}{2} - m(1+2k)\right)(n+km-1)^2.
\end{aligned}$$

COROLLARY 3.15. *Let G be an (n,m)-graph. Then*

$$\begin{aligned}
\overline{M_2}(\overline{G_k^{++}}) &= m(1+2k)(n+km-1)^2 - (n+km-1)^2[(k+1)^2M_1(G) + 4mk] \\
&\quad +[(k+1)^2M_2(G) + 2k(k+1)M_1(G)].
\end{aligned}$$

THEOREM 3.6. *Let G be an (n,m)-graph. Then*

$$\begin{aligned}
M_2(G_k^{+-}) =& mk(1-k)M_1(G) + (k-1)^2M_2(G) + k^2m^3 + k(n-2) \\
& \left[m^2(kn-4k+2) + (k-1)M_1((G)\right].
\end{aligned}$$

PROOF. Since G^{+-k} has $(n+km)$ vertices and $mk(n-2)+m$ edges

$$\begin{aligned}
M_2(G_k^{+-}) &= \sum_{uv\in E(G_k^{+-})} d_{G_k^{+-}}(u)d_{G_k^{+-}}(v) \\
&= \sum_{uv\in E(G)} d_{G_k^{+-}}(u)d_{G_k^{+-}}(v) + \sum_{ue\in E(G_k^{+-})-E(G)} d_{G_k^{+-}}(u)d_{G_k^{+-}}(e)
\end{aligned}$$

By Proposition 3.1, we have,

$$\begin{aligned}
M_2(G_k^{+-}) &= \sum_{uv\in E(G)} [d(u)+k(m-d(u))]\,[d(v)+k(m-d(v))] \\
&\quad + \sum_{u\in V(G)} k(m-d_G(u))(d(u)+k(m-d(u)))(n-2) \\
&= \sum_{uv\in E(G)} d(u)d(v) + km\sum_{uv\in E(G)} d(u) - k\sum_{uv\in E(G)} d(u)d(v)
\end{aligned}$$

$$+ km \sum_{uv\in E(G)} d(v) - k \sum_{uv\in E(G)} d(u)d(v)$$
$$+ k^2 \sum_{uv\in E(G)} (m^2 - (d(u)+d(v))m + d(u)d(v))$$
$$= mk(1-k)M_1(G) + (k-1)^2 M_2(G) + k^2 m^3$$
$$+ k(n-2)\left[m^2(kn-4k+2) + (k-1)M_1((G)\right].$$

□

Corollary 3.16. *Let G be an (n,m)-graph. Then*

$$\overline{M_2}(G_k^{+-}) = 2m^2(nk-2k+1)^2 - [mk(1-k)M_1(G) + (k-1)^2M_2(G) + k^2m^3$$
$$+ k(n-2)[m^2(kn-4k+2) + (k-1)M_1((G)]]$$
$$- \frac{1}{2}\left[mk(nmk+4m-4mk) + (k-1)^2M_1(G) + (n-2)^2m(k)\right].$$

Corollary 3.17. *Let G be an (n,m)-graph. Then*

$$M_2(\overline{G_k^{+-}}) = \Bigg[2m^2(nk-2k+1)^2 - [mk(1-k)M_1(G) + (k-1)^2M_2(G) + k^2m^3$$
$$+k(n-2)[m^2(kn-4k+2) + (k-1)M_1((G)]]$$
$$-\frac{1}{2}\left[mk(nmk+4m-4mk) + (k-1)^2M_1(G) + (n-2)^2m(k)\right]\Bigg]$$
$$+(n+km-1)\Bigg[mk(m^2k^2+2mk+4n-3) - 4m(n-1)$$
$$+n(n-1)^2 + (k-1)^2M_1(G)\Bigg] + \left(\binom{n+km}{2} - (mnk-2mk+m)\right)$$
$$(n+km-1)^2.$$

Corollary 3.18. *Let G be an (n,m)-graph. Then*

$$\overline{M_2}(\overline{G_k^{+-}}) = (mnk-2mk+m)(n+km-1)^2 - (n+km-1)\Bigg[mk(nmk$$
$$+4m-4mk) + (k-1)^2M_1(G) + (n-2)^2m(k)\Bigg] +$$
$$\Bigg[mk(1-k)M_1(G) + (k-1)^2M_2(G) + k^2m^3$$
$$+k(n-2)\left[m^2(kn-4k+2) + (k-1)M_1(G)\right]\Bigg].$$

Theorem 3.7. *Let G be an (n,m)-graph. Then*

$$M_2(G_k^{-+}) = \left[\frac{n(n-1)}{2} - m\right](n-1)^2 + (n-1)(k-1)M_1(\overline{G}) + (k-1)^2M_2(\overline{G})$$
$$+ 2k(2(n-1)m + (k-1)M_1(G)).$$

PROOF. Since G_k^{-+} has $(n+km)$ vertices and $\frac{n}{2}(n-1)-m(1-2k)$ edges

$$M_2(G_k^{-+}) = \sum_{uv\in E(G_k^{-+})} d_{G_k^{+-}}(u)d_{G_k^{-+}}(v)$$

$$= \sum_{uv\in E(G_k^{-+})\cap E(\overline{G})} d_{G_k^{-+}}(u)d_{G_k^{-+}}(v) + \sum_{uv\in E(G_k^{-+})-E(\overline{G})} d_{G_k^{-+}}(u)d_{G_k^{+-}}(v)$$

By Proposition 3.1, we have,

$$M_2(G_k^{+-}) = \sum_{uv\in E(\overline{G})} [n-1+d(v_i)(k-1)]\,[n-1+d(u_i)(k-1)]$$

$$+ \sum_{u\in V(G)} [(n-1)+d(u)(k-1)](2)kd(u)$$

$$= \left[\frac{n(n-1)}{2}-m\right](n-1)^2+(n-1)(k-1)M_1(\overline{G})+(k-1)^2M_2(\overline{G})$$

$$+ 2k(2(n-1)m+(k-1)M_1(G)).$$

□

COROLLARY 3.19. *Let G be an (n,m)-graph. Then*

$$\overline{M_2}(G_k^{-+}) =2\left(\frac{n(n-1)}{2}-m(1-2k)\right)^2-\left[\frac{n(n-1)}{2}-m\right](n-1)^2-(n-1)$$

$$(k-1)M_1(\overline{G})-(k-1)^2M_2(\overline{G})-2k(2(n-1)m-(k-1)M_1(G))$$

$$-\frac{1}{2}\left[4m(nk-n-k+1)+(k-1)^2M_1(G)+n(n-1)^2+4mk\right].$$

COROLLARY 3.20. *Let G be an (n,m)-graph. Then*

$$M_2(\overline{G_k^{-+}}) = \left[2\left(\frac{n(n-1)}{2}-m(1-2k)\right)^2-\left[\frac{n(n-1)}{2}-m\right](n-1)^2-\right.$$

$$(n-1)(k-1)M_1(\overline{G})-(k-1)^2M_2(\overline{G})-2k(2(n-1)m-(k-1)$$

$$M_1(G)-\frac{1}{2}\left[4m(nk-n-k+1)+(k-1)^2M_1(G)+n(n-1)^2+\right.$$

$$\left.\left.4mk\right]\right]+(n+km-1)\left[m^2k^2(mk+3n-10)+mk(4m-6n+9+n^2)\right.$$

$$\left.+(k-1)^2M_1(G)\right]+\left[\binom{n+km}{2}-\binom{n}{2}+m(1-2k)\right](n+km-1)^2.$$

COROLLARY 3.21. *Let G be an (n,m)-graph. Then*

$$\overline{M_2}(\overline{G_k^{-+}}) = \left[\binom{n}{2}-m(1-2k)\right](n+km-1)^2-(n+km-1)\left[4m(nk-n\right.$$

$$\left.-k+1)+(k-1)^2M_1(G)+n(n-1)^2+4mk\right]+\left[\left(\frac{n(n-1)}{2}\right.\right.$$

$$-m\Big)(n-1)^2+(n-1)(k-1)M_1(\overline{G})+(k-1)^2M_2(\overline{G})$$
$$+2k(2(n-1)m+(k-1)M_1(G))\Big].$$

THEOREM 3.8. *Let G be an (n,m)-graph. Then*

$$\begin{aligned} M_2(G_k^{--}) &= (n+km-1)\Bigg[(n+km-1)\left(\binom{n}{2}-m\right)+km(n-2)^2- \\ &\quad (k+1)M_1(\overline{G})\Bigg]+k(k+1)(n-2)(-2m^2+M_1(G))+(k+1)^2M_2(\overline{G}). \end{aligned}$$

.

PROOF. Since G_k^{--} has $(n+km)$ vertices and $\frac{n}{2}(n-1)-m-mk(n-2)$ edges.

$$\begin{aligned} M_2(G_k^{--}) &= \sum_{uv\in E(G_k^{--})} d_{G_k^{--}}(u)d_{G^{--k}}(v) \\ &= \sum_{uv\in E(\overline{G})} d_{G_k^{--}}(u)d_{G_k^{--}}(v)+\sum_{uv\in E(G_k^{--})-E(\overline{G})} d_{G_k^{--}}(u)d_{G_k^{--}}(e) \end{aligned}$$

By Proposition 3.1, we have,

$$\begin{aligned} M_2(G_k^{--}) &= \sum_{uv\in E(\overline{G})}[n+km-1-(k+1)d(v_i)]\,[n+km-1-(k+1)d(u_i)] \\ &\quad +\sum_{uv\in E(G_k^{--})-E(\overline{G})}(n-2)[n+km-1-(k+1)d(v)] \\ &= (n+km-1)^2\left(\binom{n}{2}-m\right)-(n+km-1)(k+1)M_1(\overline{G}) \\ &\quad +(k+1)^2M_2(\overline{G}) \\ &\quad +\sum_{u\in V(G)}(n+km-1-(k+1)d(u))(n-2)k(m-d(u)) \\ &= (n+km-1)\Bigg[(n+km-1)\left(\binom{n}{2}-m\right)+km(n-2)^2 \\ &\quad -(k+1)M_1(\overline{G})\Bigg]+k(k+1)(n-2)(-2m^2+M_1(G))+(k+1)^2M_2(\overline{G}). \end{aligned}$$

□

COROLLARY 3.22. *Let G be an (n,m)-graph. Then*

$$\begin{aligned} \overline{M_2}(G_k^{--}) &= 2\left(\frac{n}{2}(n-1)-m+mk(n-2)\right)^2 \\ &\quad -(n+km-1)\Bigg[(n+km-1)\left(\binom{n}{2}-m\right)+km(n-2)^2- \end{aligned}$$

$$\left.(k+1)M_1(\overline{G})\right] - k(k+1)(n-2)(-2m^2+M_1(G)) - (k+1)^2M_2(\overline{G})$$
$$-\frac{1}{2}\left[mk(nkm-4mk-4m+3n^2-10n+8)-4m(n-1)\right.$$
$$\left.+n(n-1)^2+(k+1)^2M_1(G)\right].$$

COROLLARY 3.23. *Let G be an (n,m)-graph. Then*

$$\begin{aligned}
M_2(\overline{G_k^{--}}) &= \left[2\left(\frac{n}{2}(n-1)-m+mk(n-2)\right)^2\right. \\
&-(n+km-1)\left[(n+km-1)\left(\binom{n}{2}-m\right)+km(n-2)^2\right. \\
&\left.-(k+1)M_1(\overline{G})\right]-k(k+1)(n-2)(-2m^2+M_1(G)) \\
&-(k+1)^2M_2(\overline{G})-\frac{1}{2}\left[mk(nkm-4mk-4m+3n^2-10n+8)\right. \\
&\left.\left.-4m(n-1)+n(n-1)^2+(k+1)^2M_1(G)\right]\right]+(n+km-1) \\
&\left[M_1(G_k^{--})+(km+n-1)\left[(km+n)(km+n-1)-2n(n-1)\right.\right. \\
&\left.\left.+4m-4mk(n-2)\right]\right]+\left[\binom{n+km}{2}-\binom{n}{2}+m-mk(n-2)\right] \\
&(n+km-2)^2.
\end{aligned}$$

COROLLARY 3.24. *Let G be an (n,m)-graph. Then*

$$\begin{aligned}
\overline{M_2}(\overline{G_k^{--}}) &= \left[\binom{n}{2}-m+mk(n-2)\right](n+km-2)^2-(n+km-2) \\
&\left[mk(nkm-4mk-4m+3n^2-10n+8)-4m(n-1)\right. \\
&\left.+n(n-1)^2+(k+1)^2M_1(G)\right]+\left[(n+km-1)\right. \\
&\left((n+km-1)\left(\binom{n}{2}-m\right)+km(n-2)^2-(k+1)M_1(\overline{G})\right) \\
&\left.+k(k+1)(n-2)(-2m^2+M_1(G))+(k+1)^2M_2(\overline{G})\right].
\end{aligned}$$

References

[1] A. R. Ashrafi, T. Došlić and A. Hamzeh. The Zagreb coindices of graph operations. *Discrete Appl. Math.*, **158**(15)(2010), 1571–1578.

[2] A. R. Ashrafi, T. Došlić and A. Hamzeh. Extremal graphs with respect to the Zagreb coindices. *MATCH Commun. Math. Comput. Chem.*, **65**(1)(2011), 85–92.

[3] B. Basavanagoud, I. Gutman and V. R. Desai. Zagreb indices of generalized transformation graphs and their complements. *Kragujevac J. Sci.*, **37**(2015), 99–112.

[4] B. Borovićanin, K. C. Das, B. Furtula and I. Gutman. Bounds for Zagreb indices. *MATCH Commun. Math. Comput. Chem.*, **78**(1)(2017), 17–100.

[5] K. C. Das and I. Gutman. Some properties of the second Zagreb index. *MATCH Commun. Math. Comput. Chem.*, **52**(2004), 103–112.

[6] T. Došlić. Vertex-weighted Wiener polynomials for composite graphs. *Ars Math. Contemp.*, **1**(2008), 66–80.

[7] T. Došlić, B. Furtula, A. Graovac, I. Gutman, S. Moradi and Z. Yarahmadi. On vertex-degree-based molecular structure descriptors. *MATCH Commun. Math. Comput Chem.*, **66**92)(2011), 613–626.

[8] I. Gutman and N. Trinajstić. Graph theory and molecular orbitals. Total φ electron energy of alternant hydrocarbons. *Chem. Phys. Lett.*, **17**(4)(1972), 535–538.

[9] I. Gutman, B. Furtula, Z. Kovijanić-Vukićević, and G. Popivoda. Zagreb indices and coindices. *MATCH Commun. Math. Comput. Chem.*, **74**(1)(2015), 5–16.

[10] I. Gutman and K. C. Das. The first Zagreb index 30 years after. *MATCH Commun. Math. Comput. Chem.*, **50**(2004), 83–92.

[11] S. Hossein–Zadeh, A. Hamzeh and A. R. Ashrafi. Extermal properties of Zagreb coindices and degree distance of graphs. *Miskolc Math. Notes*, **11**92)(2010), 129–138.

[12] H. Hua, A. Ashrafi and L. Zhang. More on Zagreb coindices of graphs. *Filomat*, **26**(6)(2012), 1210–1220.

[13] H. Hua and S. Zhang. Relations between Zagreb coindices and some distance-based topological indices. *MATCH Commun. Math. Comput. Chem.*, **68**(1)(2012), 199–208.

[14] S. R. Jog, S. P. Hande, I. Gutman and S. B. Bozkurt. Derived graphs of some graphs. *Kragujevac J. Math.*, **36**(2)(2012), 309–314.

[15] R. B. Jummannaver, I. Gutman and R. A. Mundewadi. On Zagreb indices and coindices of cluster graphs. *Bull. Int. Math. Virtual Inst.*, **8**(3)(2018), 477–485.

[16] P. N. Kishori and S. B. Lokesh. Peripheral Wiener index of a graph. *Commun. Comb. Optim.*, **2**(1)(2017), 43–56.

[17] S. Nikolić, G. Kovačević, A. Miličević and N. Trinajstić. The Zagreb indices 30 years after. *Croat. Chem. Acta.*, **76**(2)(2003), 113–124.

[18] E. Sampathkumar and S. B. Chikkodimath. Semitotal graphs of a graph - I. *J. Karnatak Univ. Sci.*, **18**(1973), 274–280.

[19] G. Su, L. Xiong and L. Xu. The Nordhaus-Gaddum-type inequalities for the Zagreb index and coindex of graphs. *Appl. Math. Lett.*, **25**(11)(2012), 1701–1707.

[20] R. Todeschini and V. Consonni. *Handbook of Molecular Descriptors*. Wiley–VCH, Weinheim, 2009.

Received by editors 22.01.2020; Revised version 29.01.2020; Available online 03.02.2020.

Department of Mathematics, Karnatak Science College, Karnataka, India
E-mail address: `rajesh.rbj065@gmail.com`

Department of Mathematics, Mangalore University, Mangalore, India
E-mail address: `kishori_pn@yahoo.co.in`

Department of Mathematics, Mangalore University, Mangalore, India
E-mail address: `dickson.selvan@gmail.com`