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# ATOMS AND IRREDUCIBLE ELEMENTS IN ALMOST SEMILATTICES

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Abstract. The concepts of an atom in an Almost Semilattice(ASL) and atomic ASLs are introduced and proved that every finite ASL with 0 is atomic. The concepts of irreducible elements and prime elements are introduced in an ASL L and proved that if L is an ASL with 0 satisfying maximum condition, then every one of its element in L can be represented as a combination of a finite number of irreducible elements. Also, proved that every prime element is irreducible but, the converse need not be true.

## 1. Introduction

It was Garett Birkhoff's (1911 - 1996) work in the mid thirties that started the general development of the lattice theory. In a brilliant series of papers, he demonstrated the importance of the lattice theory and showed that it provides a unified frame work for unrelated developments in many mathematical disciplines. V. Glivenko, Karl Menger, John Van Neumann, Oystein Ore, George Gratzer, P. R. Halmos, E. T. Schmidt, G. Szasz, M. H. Stone , R. P. Dilworth and many others have developed enough of this field for making it attractive to the mathematicians and for its further progress. In particular, M.H. Stone has proved that any Boolean algebra made into a Boolean ring and vice versa. The traditional approach to lattice theory proceeds from partially ordered sets to general lattices, semimodular

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lattices, modular lattices and finally to distributive lattices. The class of distributive lattices has occupied in major part of the present lattice theory, since lattices were abstracted from Boolean algebras through the class of distributive lattices and the class of distributive lattices has many interesting properties which lattices, in general, do not have. For this reason, the concept of an Almost Distributive Lattice (ADL) was introduced by Swamy U.M. and Rao G.C. [**4**], as a common abstraction of existing lattice theoretic and ring theoretic generalizations of Boolean algebra. In 1939 F. Kelin introduced thee concept of semilattice which is a generalization of more prominent concept of a lattice. Latter, the concept of Almost Semilattice (ASL) was introduced by G. Nanaji Rao and Terefe Getachew Beyene [**1**] as a generalization of a more prominent concept almost distributive lattice and a Semilattice. Also, the concept of Almost Lattices was introduced by G. Nanaji Rao and Habtamu tiruneh Alemu as a generalization of a lattice and an almost distributive lattice. Moreover, they introduced the concept of irreducible and prime elements in an almost lattice [**2**].

In this paper, we introduced the concepts of atom, atomic Almost Semilattice(ASL) and proved that every finite ASL with 0 is an atomic ASL. Also, we proved that if an ASL L satisfies a minimum condition, then L is atomic. We introduced the concepts of irreducible elements in an ASL L and proved that if L is an ASL with 0 satisfying maximum condition, then every one of its element can be represented as a combination of a finite number of irreducible elements. Finally, we introduced the concept of prime elements in an ASL L which are generalizations of the concept of irreducible elements and proved that every prime element in an ASL L is irreducible but, the converse need not be true.

## 2. Preliminaries

In the text, we are using the two references [**1**] and [**2**] most frequently in which the major work of the text is based on these two papers. But, the other references [**1, 4, 5, 3**] are used as a basic foundation for the text.

In this section, we collect a few important definitions and results which are already known and which will be used more frequently in the paper.

Definition 2.1. ([**3**]) Let $(P, \leqslant)$ be a poset and $a \in P$. Then

(1) $a$ is called the least element of $P$ if $a \leqslant x$ for all $x \in P$.
(2) $a$ is called the greatest element of $P$ if $x \leqslant a$ for all $x \in P$.

It can be easily observed that, if least (greatest) element exists in a poset, then it is unique.

Definition 2.2. ([**3**]) Let $(P, \leqslant)$ be a poset and $a \in P$. Then

(1) $a$ is called a minimal element, if $x \in P$ and $x \leqslant a$, then $x = a$.
(2) $a$ is called maximal element, if $x \in P$ and $a \leqslant x$, then $a = x$.

It can be easily verified that least (greatest) element (if exists), then it is minimal (maximal) but, the converse need not be true.

Definition 2.3. ([**3**]) Let $(P, \leqslant)$ be a poset and $S$ be a non empty subset of $P$. Then

1. An element $a$ in $P$ is called a lower bound of $S$ if $a \leqslant x$ for all $x \in S$.
2. An element $a$ in $P$ is called an upper bound of $S$ if $x \leqslant a$ for all $x \in S$.
3. An element $a$ in $P$ is called the greatest lower bound (g.l.b or infimum) of $S$ if $a$ is a lower bound of $S$ and $b \in P$ such that $b$ is a lower bound of $S$, then $b \leqslant a$.
4. An element $a$ in $P$ ia called the least upper bound (l.u.b or supremum) of $S$ if $a$ is an upper bound of $S$ and $b \in P$ such that $b$ is an upper bound of $S$, then $a \leqslant b$.

Definition 2.4. ([**3**], **(Zorn's Lemma)**) If every chain of a partly ordered set $(P, \leqslant)$ has an upper bound, then $P$ has a maximal element.

Definition 2.5. ([**3**]) Let $(P, \leqslant)$ be a poset. Then

1. P is said to satisfy descending chain condition (dcc) if every descending chain in P is terminate. That is if $.... < x_n < x_{n-1} < ... < x_2 < x_1 < x_0$ is a descending chain in P, then there exists $n \in Z^+$ such that $x_n = x_{n+1} = x_{n+2} = ......$
2. P is said to satisfy ascending condition (acc) if every ascending chain in P is terminate. That is if $x_0 < x_1 < x_2 < ..... < x_n < ....$ is an ascending chain in P, then there exists $n \in Z^+$ such that $x_n = x_{n+1} = x_{n+2} = ......$
3. P is said to satisfy minimum(maximum) condition if every nonempty subset of P has a minimal(maximal) element.

Theorem 2.1 ([**3**]). *Let $(P, \leqslant)$ be a poset. Then we have the following.*

1. *P satisfies ascending chain condition(acc) if and only if P satisfies a maximum condition.*
2. *P satisfies descending chain condition(dcc) if and only if P satisfies a minimum condition.*

Theorem 2.2 ([**3**]). *Let $(P, \leqslant)$ be a poset. If P satisfying minimum(maximum) condition. Then for any $x \in P$ there exists a minimal(maximal) element $m$ in $P$ such that $m \leqslant x(x \leqslant m)$.*

Theorem 2.3 ([**3**]). *Every subchain of a partially ordered subset satisfying a minimum(maximum) condition has least(greatest) element.*

Definition 2.6. ([**3**]) Let $(P, \leqslant)$ be a poset. Then $P$ is said to be lattice ordered set if for every pair $x, y \in P$, $l.u.b\{x, y\}$ and $g.l.b\{x, y\}$ exists.

Definition 2.7. ([**1**]) An Almost Semilattice(ASL) is an algebra $(L, \circ)$ of type (2) satisfies the following conditions:

1. $(x \circ y) \circ z = x \circ (y \circ z)$ (Associative Law)
2. $(x \circ y) \circ z = (y \circ x) \circ z$ (Almost Commutative Law)
3. $x \circ x = x$, for all $x, y, z \in L$. (Idempotent)

Definition 2.8. ([**1**]) Let $L$ be an $ASL$. Then for any $a, b \in L$, we say that $a$ is less or equal to $b$ and write $a \leqslant b$, if $a \circ b = a$.

Lemma 2.1 ([**1**]). *Let $L$ be an $ASL$. Then for any $a, b \in L$, we have:*
(1) $a \circ (a \circ b) = a \circ b$
(2) $(a \circ b) \circ b = a \circ b$
(3) $b \circ (a \circ b) = a \circ b$.

Corollary 2.1 ([**1**]). *Let $L$ be an $ASL$. Then for any $a, b \in L, a \circ b \leqslant b$.*

Corollary 2.2 ([**1**]). *Let $L$ be an $ASL$. Then for any $a, b \in L, a \circ b = b \circ a$ whenever $a \leqslant b$.*

Theorem 2.4. *Let $L$ be an $ASL$. Then the relation $\leqslant$ is a partial ordering on $L$.*

Definition 2.9 ([**1**]). Let $L$ be an $ASL$. An element $a \in L$ is said to be minimal (maximal) element if for any $x \in L$, $x \leqslant a(a \leqslant x)$, then $x = a(a = x)$.

Theorem 2.5 ([**1**]). *Let $L$ be an $ASL$. Then for any $a, b \in L$ with $a \leqslant b$. Then $a \circ c \leqslant b \circ c$ and $c \circ a \leqslant c \circ b$ for all $c \in L$.*

Theorem 2.6. [**1**]*] Let $L$ be an $ASL$. Then for any $a, b, c \in L$, we have the following:*
(1) $a \leqslant b$ *and* $c \leqslant d \implies a \circ c \leqslant b \circ d$.
(2) $a, b \leqslant c \implies a \circ b, b \circ a \leqslant c$
(3) $a \leqslant b, c \implies a \leqslant b \circ c, c \circ b$.

Definition 2.10. ([**1**]) Let $L$ be an $ASL$. Then an element $m \in L$ is said to be *unimaximal* if $m \circ x = x$ for all $x \in L$.

Definition 2.11. ([**1**]) Let $L$ be an $ASL$. An element $0 \in L$ is called a zero element of $L$ if $0 \circ a = 0$ for all $a \in L$.

It can be easily seen that an $ASL$ can have at most one zero element and it will be the least element of the poset $(L, \leqslant)$. We always denote the zero element of $L$, if it exists, by '0'. If $L$ has 0, then the algebra $(L, \circ, 0)$ is called an $ASL$ with '0'.

Theorem 2.7 ([**1**]). Let $L$ be an $ASL$ and 0 be any external element of $L$. For any $x, y \in L \cup \{0\}$, define:

$$x \circ y = \begin{cases} x \circ y, \ (in\ L) & \text{if } x, y \in L. \\ 0, & \text{otherwise.} \end{cases}$$

Then $(L \cup \{0\}, \circ.0)$ is an $ASL$ with 0. We denote this $ASL$ by $L^o$.

According to definition 2.11, an $ASL$ with 0 is an algebra $(L, \circ, 0)$ of type $(2, 0)$ satisfying the following axioms:

$(AS_1)$ $(x \circ y) \circ z = x \circ (y \circ z)$ (Associative Law)
$(AS_2)$ $(x \circ y) \circ z = (y \circ x) \circ z$ (Almost Commutative Law)
$(AS_3)$ $x \circ x = x$ (Idempotent Law)
$(AS_0)$ $0 \circ x = 0$ for all $x \in L$.

For brevity, in future, we will refer to this almost semilattice with 0 as $ASL$ with 0.

Lemma 2.2 ([**1**]). *Let L be an ASL with* 0*. Then for any* $a \in L, a \circ 0 = 0$.

Lemma 2.3 ([**1**]). *Let L be an ASL with* 0*. Then for any* $a, b \in L, a \circ b = 0$ *if and only if* $b \circ a = 0$.

Corollary 2.3 ([**1**]). *Let L be an ASL with* 0*. Then for any* $a, b \in L, a \circ b = b \circ a$ *whenever* $a \circ b = 0$

## 3. Atoms and irreducible elements in ASLs

In this section, we introduce the concepts of atom, atomic Almost semilattice and prove that every finite ASL with 0 is an atomic ASL. Also, we prove that if an ASL L satisfies maximum condition, then L is atomic. Moreover, we introduce the concepts of irreducible elements in an ASL L and prove that if L is an ASL with 0 satisfying maximum condition, then every one of its element can be represented as the combination of a finite number of irreducible elements. Finally, we introduce the concepts of prime elements in an ASL L which are the generalizations of the concept of irreducible elements and prove that every prime element in an ASL L is irreducible, but the converse need not be true. First, we begin with the following definition.

Definition 3.1. Let L be an ASL and $a, b \in L$ with $a \leqslant b$. Then we say that $a$ is covered by $b$ or, $b$ covers $a$, write as $a \prec b$ if for any $c \in L$, such that $a \leqslant c \leqslant b$, then either $a = c$ or $c = b$.

Definition 3.2. Let L be an ASL with 0. Then an element $a(\neq 0) \in L$ is called an atom if 0 is covered by $a$.

Example 3.1. Let $L = \{0, a, b, c\}$. Define binary operations $\circ$ on $L$ as follows:

| ∘ | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | a | a |
| b | 0 | a | b | b |
| c | 0 | a | b | c |

Then clearly $(L, \circ, 0)$ is an ASL with 0. Also, we can observe that $a$ is an atom but $b$ *and* $c$ are not atoms.

In the following we introduce the concept of atomic Almost Semilattice

Definition 3.3. An ASL L with 0 is said to be atomic if for every nonzero $a \in L$, there exists an atom $p \in L$ such that $p \leqslant a$.

Example 3.2. Let $L = \{0, a, b, c\}$. Define $\circ$ on $L$ as follows.

| ∘ | 0 | a | b | c |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| a | 0 | a | b | c |
| b | 0 | a | b | c |
| c | 0 | c | c | c |

Then clearly $(L, \circ, 0)$ is an atomic ASL.

Next, we prove the following:

THEOREM 3.1. *Let L be a finite ASL with 0 and $a(\neq 0) \in L$. Then there exists an atom $p \in L$ such that $p \leqslant a$.*

PROOF. Suppose L is finite ASL and $a \in L$ such that $a \neq 0$. If $a$ is an atom, then there is nothing to prove. Suppose $a$ is not an atom, then there exists $a_1 \in L$ such that $0 \leqslant a_1 < a$. If $a_1$ is an atom, then the result is clear. Otherwise continue the above process. Since L is finite, there exists $a_n \in L$ such that $a_n$ is an atom and $a_n < a$ for some positive integer $n$. □

COROLLARY 3.1. *Let L be an ASL with 0 which satisfies the minimum condition. Then for every nonzero element a in L, there exists an atom p in L such that $p \leqslant a$.*

COROLLARY 3.2. *Let L be an ASL which satisfies a minimum condition. Then L is atomic.*

Now, we introduce the concept of irreducible elements in ASLs.

DEFINITION 3.4. Let L be an ASL. Then an element $a \in L$ is said to be irreducible if $a = a_1 \circ a_2 = a_2 \circ a_1$, then either $a = a_1$ or $a = a_2$. Otherwise $a$ is called reducible.

EXAMPLE 3.3. Every elements of an ASL L defined in example 3.2 are irreducible.

Note that, in an ASL L, it may happen that every element of L is reducible. An ASL of this kind is presented in the following example.

EXAMPLE 3.4. Let $Z$ denote the set of all integers and let us form the product set $Z \times Z$. Now, define the operations $\circ$ on $Z \times Z$ point wise as follows:

$$(x_1, x_2) \circ (y_1, y_2) = (min(x_1, y_1), min(x_2, y_2))$$

for any $x_1, x_2, y_1, y_2 \in Z$. Then clearly $(Z \times Z, \circ)$ is an ASL. Consider, an arbitrary element $(a_1, a_2)$ of $Z \times Z$. If $p_j$ $(j = 1, 2)$ is an (arbitrary) integer greater than $a_j$ and $q_j$ be an other element less than $a_j$ then $(a_1, a_2) = (a_1, p_2) \circ (p_1, a_2) = (p_1, a_2) \circ (a_1, p_2)$, and in the ASL $Z \times Z$, $(a_1, q_2), (q_1, a_2) \leqslant (a_1, a_2) \leqslant (a_1, p_2), (p_1, a_2)$. Therefore $a_1, a_2$ is reducible and hence every elements of $(Z \times Z, \circ)$ are reducible.

EXAMPLE 3.5. Let $L_1$ and $L_2$ be two nontrivial ASLs and $L = L_1 \times L_2$ be their product. Then L is an ASL under point wise operations. Choose $0 \neq a_1 \in L_1$ and $0 \neq a_2 \in L_2$. Then $(0, 0) = (a_1, 0) \circ (0, b_2) = (0, b_2) \circ (a_1, 0)$. Hence $(0, 0)$ is not irreducible.

THEOREM 3.2. *Let L be an ASL. If L is a chain, then every element in L is irreducible.*

PROOF. Suppose L is a chain. Let $a \in L$ such that $a = a_1 \circ a_2 = a_2 \circ a_1$ for some $a_1, a_2 \in L$. Now, since L is a chain, $a_1 \leqslant a_2$ or $a_2 \leqslant a_1$. It follows that, either $a = a_2$ or $a = a_1$. Therefore $a$ is irreducible. □

The converse of the above theorem is not true. For, if L is a discrete ASL, then clearly every element in L is irreducible. But, L is not a chain.

THEOREM 3.3. *Let L be an ASL with 0 satisfying the maximum condition. Then every one of an element in L can be represented as the combination of a finite number of irreducible elements.*

PROOF. Suppose L be an ASL with 0 satisfying the maximum condition and suppose H is the set of all elements in L which can not be represented as a combination of a finite number of irreducible elements. Then we shall prove that H is empty. Clearly, H contains no irreducible elements, since if $a$ is such an element, then $a = a \circ a$ is easily found representations of the required form that contradicts our assumption for H. Suppose $H \neq \emptyset$. Since $L$ satisfies maximum condition, H has a maximal element say m. Clearly m is not irreducible element. Then we can choose $m_1, m_2 \in L$ such that;

$$m = m_1 \circ m_2 = m_2 \circ m_1, \ (m < m_1, m_2)$$

Since m is a maximal element of H, the elements $m_1, m_2 \notin H$. Hence $m_1, m_2$ can be represented as $m_1 = q_1 \circ q_2 \circ .... \circ q_s$ and $m_2 = r_1 \circ r_2 \circ .... \circ r_t$ where all the $q_j$ and $r_j$ are irreducible. Therefore $m = \bigcirc_{j=1}^{s} q_j \circ \bigcirc_{k=1}^{t} r_k$, which is a combination of finite number of irreducible elements, a contradiction to $m \in H$. Thus $H = \emptyset$. □

In the following, we introduce the concepts of prime elements in an ASL L which are generalization of the concept of irreducible elements.

DEFINITION 3.5. An element $a$ of an ASL L is called prime if $a_1 \circ a_2 = a_2 \circ a_1 \leqslant a$ implies either $a_1 \leqslant a$ or $a_2 \leqslant a$.

EXAMPLE 3.6. In an ASL L of example 3.2, it is observed that $a$ and $b$ are prime elements.

Next, we prove that every prime element is irreducible but not converse.

THEOREM 3.4. *Every prime element in an ASL L is irreducible.*

PROOF. Suppose $a \in L$ is prime element. Let $a = a_1 \circ a_2 = a_2 \circ a_1$ for some $a_1, a_2 \in L$. Then we have $a \leqslant a_1, a_2$. On the other hand, we have $a_1 \circ a_2 = a_2 \circ a_1 \leqslant a$. It follows that, either $a_1 \leqslant a$ or $a_2 \leqslant a$. Hence either $a = a_1$ or $a = a_2$. Therefore $a$ is irreducible. □

The converse of the above theorem is not true in general. For, suppose $L_1$ and $L_2$ are two discrete ASLs with zero and each with at least three elements. Then clearly, $L = L_1 \times L_2$ is an ASL under point wise operations. Choose $0 \neq p_1 \in L_1$ and $0 \neq p_2 \in L_2$. Put $p = (p_1, p_2)$. Now, let $(q_1, q_2)$, $(r_1, r_2) \in L$ such that $p = (q_1, q_2) \circ (r_1, r_2) = (r_1, r_2) \circ (q_1, q_2)$. Then $(p_1, p_2) = (q_1 \circ r_1, q_2 \circ r_2) = (r_1 \circ q_1, r_2 \circ q_2)$ and hence $p_1 = q_1 \circ r_1 = r_1 \circ q_1$ and $p_2 = q_2 \circ r_2 = r_2 \circ q_2$. Since $p_1 \neq 0$ and $p_2 \neq 0$, it follows that, $q_1, r_1, q_2, r_2$ are non zero. Therefore $r_1 = q_1 \circ r_1 = r_1 \circ q_1 = q_1$ and $r_2 = q_2 \circ r_2 = r_2 \circ q_2 = q_2$. Therefore $(q_1, q_2) = (r_1, r_2)$. Hence $p = (p_1, p_2) = (q_1, q_2) = (r_1, r_2)$. Therefore P is irreducible. However, p is not prime. For, choose $q_i \in L_i - \{0, p_i\}$ $\forall i = 1, 2$. Then

$(0, q_2) \circ (q_1, 0) = (q_1, 0) \circ (0, q_2) = (0, 0) \leqslant (p_1, p_2) = p$. But $(0, q_2) \nleq p$ and $(q_1, 0) \nleq p$.

THEOREM 3.5. *Let $L$ be an ASL with 0 and $SI(L)$ be the set of all prime elements in $L$. For any $a \in L$, let $SI(a) = \{p \in SI(L) | a \leqslant p\}$. Then we have the following:*

(1) $a \leqslant b \Longrightarrow SI(b) \subseteq SI(a)$
(2) *If $p \in SI(a)$ and $q \in SI(L)$ such that $p \leqslant q$, then $q \in SI(a)$.*
(3) $SI(a \circ b) = SI(a) \cup SI(b)$
(4) $SI(0) = SI(L)$

PROOF. (1) Let $a, b \in L$ such that $a \leqslant b$. Suppose $p \in SI(b)$. Then $b \leqslant p$. It follows that, $p \in SI(a)$. Therefore $SI(b) \subseteq SI(a)$.
(2) Suppose $p \in SI(a)$ and $q \in SI(L)$ such that $p \leqslant q$. Then $a \leqslant p \leqslant q$. It follows that, $a \leqslant q$. Therefore $q \in SI(a)$.
(3) $p \in SI(a \wedge b) \Longleftrightarrow a \circ b \leqslant p \Longleftrightarrow a \leqslant p$ *or* $b \leqslant p$ (since p is prime) $\Longleftrightarrow P \in SI(a)$ *or* $p \in SI(b) \Longleftrightarrow p \in SI(a) \cup SI(b)$. Therefore $SI(a \circ b) = SI(a) \cup SI(b)$.
(4) Suppose $p \in SI(0)$. Then $0 \leqslant p \in SI(L)$. Hence $SI(0) \subseteq SI(L)$. Conversely, suppose $p \in SI(L)$. Clearly we have $0 \leqslant p$. Then $p \in SI(0)$. Hence $SI(L) \subseteq SI(0)$. Therefore $SI(0) = SI(L)$.

□

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