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PARITY OF CLASSICAL AND DYNAMIC INEQUALITIES MAGNIFIED ON TIME SCALES

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Abstract. In this paper, we present an extension of dynamic Lyapunov's inequality and its reverse version on time scales. Furthermore, we find generalizations of the well–known classical Radon's inequality by using Specht's ratio and Rogers–Hölder's inequality by using Kantorovich's ratio on time scales. Our investigations unify and extend some continuous inequalities and their corresponding discrete analogues.

1. Introduction

We introduce here some well–known classical inequalities.

If $x_k > 0$, $y_k > 0$, $k = 1, 2, \ldots, n$ and $0 < \beta_1 < \beta_2 < \beta_3 < \infty$, then

$$\left(\sum_{k=1}^{n} x_k y_k^{\beta_2}\right)^{\beta_3-\beta_1} \leqslant \left(\sum_{k=1}^{n} x_k y_k^{\beta_1}\right)^{\beta_3-\beta_2} \left(\sum_{k=1}^{n} x_k y_k^{\beta_3}\right)^{\beta_2-\beta_1}. \tag{1.1}$$

The inequality (1.1) is called, in literature, Lyapunov's inequality as given in [**14**].

If $n \in \mathbb{N}$, $x_k \geqslant 0$, $y_k > 0$, $k \in \{1, 2, \ldots, n\}$, $\beta \geqslant 0$ and $\gamma \geqslant 1$, then

$$\frac{\left(\sum\limits_{k=1}^{n} x_k y_k^{\gamma-1}\right)^{\beta+\gamma}}{\left(\sum\limits_{k=1}^{n} y_k^{\gamma}\right)^{\beta+\gamma-1}} \leqslant \sum_{k=1}^{n} \frac{x_k^{\beta+\gamma}}{y_k^{\beta}}, \tag{1.2}$$

with equality if and only if $\frac{x_1}{y_1} = \frac{x_2}{y_2} = \ldots = \frac{x_n}{y_n}$.

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The inequality (1.2) is called generalized Radon's inequality as given in [**11**].

If $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and x_k, y_k for all $k = 1, 2, \ldots, n$ are positive real numbers, then

$$\sum_{k=1}^{n} x_k y_k \leqslant \left(\sum_{k=1}^{n} x_k^p \right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} y_k^q \right)^{\frac{1}{q}}. \tag{1.3}$$

The inequality (1.3) is called Rogers–Hölder's inequality as given in [**13**].

We will unify and extend these results on time scales. The calculus of time scales was initiated by Stefan Hilger as given in [**12**]. A time scale is an arbitrary nonempty closed subset of the real numbers. The theory of time scales is applied to combine results in one comprehensive form. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$ where $q > 1$. The time scales calculus is studied as delta calculus, nabla calculus and diamond–α calculus. This hybrid theory is also widely applied on dynamic inequalities.

In this paper, it is assumed that all considerable integrals exist and are finite and $\mathbb{T}$ is a time scale, $a, b \in \mathbb{T}$ with $a < b$ and an interval $[a, b]_{\mathbb{T}}$ means the intersection of a real interval with the given time scale.

2. Preliminaries

We need here basic concepts of delta calculus. The results of delta calculus are adapted from monographs [**6, 7**].

For $t \in \mathbb{T}$, the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}.$$

The mapping $\mu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\mu(t) := \sigma(t) - t$ is called the forward graininess function. The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

The mapping $\nu : \mathbb{T} \to \mathbb{R}_0^+ = [0, +\infty)$ such that $\nu(t) := t - \rho(t)$ is called the backward graininess function. If $\sigma(t) > t$, we say that t is right–scattered, while if $\rho(t) < t$, we say that t is left–scattered. Also, if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right–dense, and if $t > \inf \mathbb{T}$ and $\rho(t) = t$, then t is called left–dense. If $\mathbb{T}$ has a left–scattered maximum M, then $\mathbb{T}^k = \mathbb{T} - \{M\}$, otherwise $\mathbb{T}^k = \mathbb{T}$.

For a function $f : \mathbb{T} \to \mathbb{R}$, the delta derivative f^{Δ} is defined as follows:

Let $t \in \mathbb{T}^k$. If there exists $f^{\Delta}(t) \in \mathbb{R}$ such that for all $\epsilon > 0$, there is a neighborhood U of t, such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \leqslant \epsilon |\sigma(t) - s|,$$

for all $s \in U$, then f is said to be delta differentiable at t, and $f^{\Delta}(t)$ is called the delta derivative of f at t.

A function $f : \mathbb{T} \to \mathbb{R}$ is said to be right–dense continuous (rd–continuous), if it is continuous at each right–dense point and there exists a finite left–sided limit at every left–dense point. The set of all rd–continuous functions is denoted by $C_{rd}(\mathbb{T},\mathbb{R})$.

The next definition is given in [**6, 7**].

DEFINITION 2.1. A function $F:\mathbb{T}\to\mathbb{R}$ is called a delta antiderivative of $f:\mathbb{T}\to\mathbb{R}$, provided that $F^{\Delta}(t)=f(t)$ holds for all $t\in\mathbb{T}^k$. Then the delta integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b)-F(a).$$

The following results of nabla calculus are taken from [**2, 6, 7**].

If $\mathbb{T}$ has a right–scattered minimum m, then $\mathbb{T}_k=\mathbb{T}-\{m\}$, otherwise $\mathbb{T}_k=\mathbb{T}$. A function $f:\mathbb{T}_k\to\mathbb{R}$ is called nabla differentiable at $t\in\mathbb{T}_k$, with nabla derivative $f^{\nabla}(t)$, if there exists $f^{\nabla}(t)\in\mathbb{R}$ such that given any $\epsilon>0$, there is a neighborhood V of t, such that

$$|f(\rho(t))-f(s)-f^{\nabla}(t)(\rho(t)-s)|\leqslant \epsilon|\rho(t)-s|,$$

for all $s\in V$.

A function $f:\mathbb{T}\to\mathbb{R}$ is said to be left–dense continuous (ld–continuous), provided it is continuous at all left–dense points in $\mathbb{T}$ and its right–sided limits exist (finite) at all right–dense points in $\mathbb{T}$. The set of all ld–continuous functions is denoted by $C_{ld}(\mathbb{T},\mathbb{R})$.

The next definition is given in [**2, 6, 7**].

DEFINITION 2.2. A function $G:\mathbb{T}\to\mathbb{R}$ is called a nabla antiderivative of $g:\mathbb{T}\to\mathbb{R}$, provided that $G^{\nabla}(t)=g(t)$ holds for all $t\in\mathbb{T}_k$. Then the nabla integral of g is defined by

$$\int_a^b g(t)\nabla t = G(b)-G(a).$$

Now we present short introduction of the diamond–α derivative as given in [**1, 18**].

DEFINITION 2.3. Let $\mathbb{T}$ be a time scale and $f(t)$ be differentiable on $\mathbb{T}$ in the Δ and ∇ senses. For $t\in\mathbb{T}_k^k$, where $\mathbb{T}_k^k=\mathbb{T}^k\cap\mathbb{T}_k$, the diamond–$\alpha$ dynamic derivative $f^{\diamond_\alpha}(t)$ is defined by

$$f^{\diamond_\alpha}(t)=\alpha f^{\Delta}(t)+(1-\alpha)f^{\nabla}(t),\ 0\leqslant\alpha\leqslant 1.$$

Thus f is diamond–α differentiable if and only if f is Δ and ∇ differentiable.

The diamond–α derivative reduces to the standard Δ–derivative for $\alpha=1$, or the standard ∇–derivative for $\alpha=0$. It represents a weighted dynamic derivative for $\alpha\in(0,1)$.

THEOREM 2.1 ([**18**]). *Let $f,g:\mathbb{T}\to\mathbb{R}$ be diamond–α differentiable at $t\in\mathbb{T}$ and $f^{\sigma}(t)=f(\sigma(t))$, $g^{\sigma}(t)=g(\sigma(t))$, $f^{\rho}(t)=f(\rho(t))$ and $g^{\rho}(t)=g(\rho(t))$. Then*

(i) *$f \pm g : \mathbb{T} \to \mathbb{R}$ is diamond–α differentiable at $t \in \mathbb{T}$, with*

$$(f \pm g)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t) \pm g^{\diamond_\alpha}(t).$$

(ii) *$fg : \mathbb{T} \to \mathbb{R}$ is diamond–α differentiable at $t \in \mathbb{T}$, with*

$$(fg)^{\diamond_\alpha}(t) = f^{\diamond_\alpha}(t)g(t) + \alpha f^\sigma(t)g^\Delta(t) + (1-\alpha)f^\rho(t)g^\nabla(t).$$

(iii) *For $g(t)g^\sigma(t)g^\rho(t) \neq 0$, $\frac{f}{g} : \mathbb{T} \to \mathbb{R}$ is diamond–α differentiable at $t \in \mathbb{T}$, with*

$$\left(\frac{f}{g}\right)^{\diamond_\alpha}(t) = \frac{f^{\diamond_\alpha}(t)g^\sigma(t)g^\rho(t) - \alpha f^\sigma(t)g^\rho(t)g^\Delta(t) - (1-\alpha)f^\rho(t)g^\sigma(t)g^\nabla(t)}{g(t)g^\sigma(t)g^\rho(t)}.$$

Definition 2.4. ([**18**]) Let $a, t \in \mathbb{T}$ and $h : \mathbb{T} \to \mathbb{R}$. Then the diamond–$\alpha$ integral from a to t of h is defined by

$$\int_a^t h(s) \diamond_\alpha s = \alpha \int_a^t h(s)\Delta s + (1-\alpha)\int_a^t h(s)\nabla s, \ 0 \leqslant \alpha \leqslant 1,$$

provided that there exist delta and nabla integrals of h on $\mathbb{T}$.

Theorem 2.2 ([**18**]). *Let $a, b, t \in \mathbb{T}$, $c \in \mathbb{R}$. Assume that $f(s)$ and $g(s)$ are $\diamond_\alpha$–integrable functions on $[a, b]_\mathbb{T}$. Then*

(i) $\int_a^t [f(s) \pm g(s)] \diamond_\alpha s = \int_a^t f(s) \diamond_\alpha s \pm \int_a^t g(s) \diamond_\alpha s$;

(ii) $\int_a^t cf(s) \diamond_\alpha s = c\int_a^t f(s) \diamond_\alpha s$;

(iii) $\int_a^t f(s) \diamond_\alpha s = -\int_t^a f(s) \diamond_\alpha s$;

(iv) $\int_a^t f(s) \diamond_\alpha s = \int_a^b f(s) \diamond_\alpha s + \int_b^t f(s) \diamond_\alpha s$;

(v) $\int_a^a f(s) \diamond_\alpha s = 0$.

We need the following results.

Theorem 2.3 ([**1**]). *Let $a, b \in \mathbb{T}$ with $a < b$ and $w, f, h \in C([a, b]_\mathbb{T}, \mathbb{R})$ be $\diamond_\alpha$–integrable functions with $\int_a^b |w(x)||h(x)|^q \diamond_\alpha x > 0$. If $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, then*

$$\int_a^b |w(x)||f(x)h(x)| \diamond_\alpha x \leqslant \left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||h(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}}. \tag{2.1}$$

If $\frac{1}{p} + \frac{1}{q} = 1$ with $p < 0$ or $q < 0$, then inequality (2.1) *is reversed.*

The Specht's ratio [**9, 19**] is defined by

$$S(h) = \frac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}} \ for \ h > 0, \ h \neq 1.$$

The following inequality is due to Furuichi [**10**] and provides a refinement for Young's inequality

$$S\left(\left(\frac{a}{b}\right)^{\delta}\right) a^{\frac{1}{p}} b^{\frac{1}{q}} \leqslant \frac{a}{p}+\frac{b}{q} \tag{2.2}$$

for $a, b>0$, $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$ and $\delta=\min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

We also consider the Kantorovich's ratio defined by

$$K(h):=\frac{(h+1)^{2}}{4 h}, \ h>0.$$

The function K is decreasing on $(0,1)$ and increasing on $[1,+\infty)$, $K(h) \geqslant 1$ for any $h>0$ and $K(h)=K\left(\frac{1}{h}\right)$ for any $h>0$.

The following Young's inequality [**21**] in terms of Kantorovich's ratio holds

$$K^{\delta}\left(\frac{a}{b}\right) a^{\frac{1}{p}} b^{\frac{1}{q}} \leqslant \frac{a}{p}+\frac{b}{q} \tag{2.3}$$

where $a, b>0$, $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$ and $\delta=\min\left\{\frac{1}{p}, \frac{1}{q}\right\}$.

3. Lyapunov's Inequality

In order to present our main results, first we give a simple proof for an extension of dynamic Lyapunov's inequality and its reverse version on time scales.

THEOREM 3.1. *Let $w, f, g, h \in C\left([a, b]_{\mathbb{T}}, \mathbb{R}-\{0\}\right)$ be $\diamond_{\alpha}$–integrable functions. Further assume that $|f|^{\beta_{1}(\beta_{3}-\beta_{2})}|g|^{\beta_{2}(\beta_{1}-\beta_{3})}|h|^{\beta_{3}(\beta_{2}-\beta_{1})}=M$ for $\beta_{1}, \beta_{2}, \beta_{3} \in \mathbb{R}$, where M is a positive real number.*

(i) *If $\beta_{1}<\beta_{2}<\beta_{3}$, then*

$$\left(\int_{a}^{b}|w(x)||f(x)|^{\beta_{1}} \diamond_{\alpha} x\right)^{\beta_{3}-\beta_{2}}\left(\int_{a}^{b}|w(x)||g(x)|^{\beta_{2}} \diamond_{\alpha} x\right)^{\beta_{1}-\beta_{3}} \times\left(\int_{a}^{b}|w(x)||h(x)|^{\beta_{3}} \diamond_{\alpha} x\right)^{\beta_{2}-\beta_{1}} \geqslant M. \tag{3.1}$$

(ii) *If $\beta_{2}<\beta_{1}<\beta_{3}$, then*

$$\left(\int_{a}^{b}|w(x)||f(x)|^{\beta_{1}} \diamond_{\alpha} x\right)^{\beta_{3}-\beta_{2}}\left(\int_{a}^{b}|w(x)||g(x)|^{\beta_{2}} \diamond_{\alpha} x\right)^{\beta_{1}-\beta_{3}} \times\left(\int_{a}^{b}|w(x)||h(x)|^{\beta_{3}} \diamond_{\alpha} x\right)^{\beta_{2}-\beta_{1}} \leqslant M. \tag{3.2}$$

PROOF. Case (i). We note that $\beta_{1}-\beta_{2}+\beta_{3}-\beta_{1}+\beta_{2}-\beta_{3}=0$. Set $p=\frac{\beta_{3}-\beta_{1}}{\beta_{3}-\beta_{2}}>1$ and $q=\frac{\beta_{3}-\beta_{1}}{\beta_{2}-\beta_{1}}>1$. Then $\frac{1}{p}+\frac{1}{q}=1$.

Applying Rogers–Hölder's inequality given in (2.1), we get

$$\int_a^b |w(x)||f(x)h(x)| \diamond_\alpha x \leqslant \left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||h(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}}. \tag{3.3}$$

Replacing $|f(x)|$ by $|f(x)|^{\frac{\beta_1}{p}}$ and $|h(x)|$ by $|h(x)|^{\frac{\beta_3}{q}}$ in (3.3), we obtain

$$\int_a^b |w(x)||f(x)|^{\beta_1\left(\frac{\beta_3-\beta_2}{\beta_3-\beta_1}\right)}|h(x)|^{\beta_3\left(\frac{\beta_2-\beta_1}{\beta_3-\beta_1}\right)} \diamond_\alpha x \leqslant \left(\int_a^b |w(x)||f(x)|^{\beta_1} \diamond_\alpha x\right)^{\frac{\beta_3-\beta_2}{\beta_3-\beta_1}} \left(\int_a^b |w(x)||h(x)|^{\beta_3} \diamond_\alpha x\right)^{\frac{\beta_2-\beta_1}{\beta_3-\beta_1}}. \tag{3.4}$$

Taking power $\beta_3 - \beta_1 > 0$ on both sides of inequality (3.4), we get

$$\left(\int_a^b |w(x)||f(x)|^{\beta_1\left(\frac{\beta_3-\beta_2}{\beta_3-\beta_1}\right)}|h(x)|^{\beta_3\left(\frac{\beta_2-\beta_1}{\beta_3-\beta_1}\right)} \diamond_\alpha x\right)^{\beta_3-\beta_1} \leqslant \left(\int_a^b |w(x)||f(x)|^{\beta_1} \diamond_\alpha x\right)^{\beta_3-\beta_2} \left(\int_a^b |w(x)||h(x)|^{\beta_3} \diamond_\alpha x\right)^{\beta_2-\beta_1}. \tag{3.5}$$

Using the condition that $|f|^{\beta_1(\beta_3-\beta_2)}|g|^{\beta_2(\beta_1-\beta_3)}|h|^{\beta_3(\beta_2-\beta_1)} = M$ for $\beta_1 < \beta_2 < \beta_3$, where M is a positive real number, the inequality (3.5) becomes

$$\left(\int_a^b |w(x)|M^{\frac{1}{\beta_3-\beta_1}}|g(x)|^{\beta_2} \diamond_\alpha x\right)^{\beta_3-\beta_1} \leqslant \left(\int_a^b |w(x)||f(x)|^{\beta_1} \diamond_\alpha x\right)^{\beta_3-\beta_2} \left(\int_a^b |w(x)||h(x)|^{\beta_3} \diamond_\alpha x\right)^{\beta_2-\beta_1}. \tag{3.6}$$

This directly yields (3.1). Similarly, we can prove the Case (ii) by applying reverse Rogers–Hölder's inequality. Thus, the proof of Theorem 3.1 is now complete. □

Remark 3.1. Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w(k) = x_k \in (0, +\infty)$ and $f(k) = g(k) = h(k) = y_k \in (0, +\infty)$ for $k \in \{1, 2, \ldots, n\}$. Then we get $M = 1$ and inequality (3.1) reduces to

$$\left(\sum_{k=1}^n x_k y_k^{\beta_1}\right)^{\beta_3-\beta_2} \left(\sum_{k=1}^n x_k y_k^{\beta_2}\right)^{\beta_1-\beta_3} \left(\sum_{k=1}^n x_k y_k^{\beta_3}\right)^{\beta_2-\beta_1} \geqslant 1. \tag{3.7}$$

This directly yields (1.1).

Next, we conclude that some dynamic inequalities are equivalent.

Theorem 3.2. *The following inequalities are equivalent on time scales:*

(1) *Generalized Radon's inequality,*
(2) *Radon's inequality,*
(3) *The weighted power mean inequality,*
(4) *Schlömilch's inequality,*
(5) *Rogers–Hölder's inequality,*
(6) *Bernoulli's inequality,*
(7) *Lyapunov's inequality.*

Proof. It is clear from [**15**, Theorem 3.13] that dynamic inequalities (1) to (6) are equivalent. Now we prove that (5) is equivalent to (7).

We have proved in Theorem 3.1 that Rogers–Hölder's inequality $\Rightarrow$ Lyapunov's inequality. To complete the proof of Theorem 3.2, we show that Lyapunov's inequality $\Rightarrow$ Rogers–Hölder's inequality.

Let $\beta_1 < \beta_2 < \beta_3$ for $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$. Take $p = \frac{\beta_3-\beta_1}{\beta_3-\beta_2} > 1$ and $q = \frac{\beta_3-\beta_1}{\beta_2-\beta_1} > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$.

Replacing $|f(x)|$, $|g(x)|$ and $|h(x)|$ by $F(x)$, $G(x)$ and $H(x)$, respectively, and letting $F(x) = |f(x)|^{\frac{\beta_3-\beta_1}{\beta_1(\beta_3-\beta_2)}}$, $G(x) = |f(x)h(x)|^{\frac{1}{\beta_2}}$ and $H(x) = |h(x)|^{\frac{\beta_3-\beta_1}{\beta_3(\beta_2-\beta_1)}}$ in Theorem 3.1, we obtain $M = 1$. Thus, the inequality (3.1) becomes

$$(3.8)\quad \left(\int_a^b |w(x)||f(x)|^{\frac{\beta_3-\beta_1}{\beta_3-\beta_2}} \diamond_\alpha x\right)^{\beta_3-\beta_2} \left(\int_a^b |w(x)||f(x)h(x)| \diamond_\alpha x\right)^{\beta_1-\beta_3} \times \left(\int_a^b |w(x)||h(x)|^{\frac{\beta_3-\beta_1}{\beta_2-\beta_1}} \diamond_\alpha x\right)^{\beta_2-\beta_1} \geqslant 1.$$

Taking power $\beta_3 - \beta_1 > 0$ on both sides of inequality (3.8), we have

$$(3.9)\quad \int_a^b |w(x)||f(x)h(x)| \diamond_\alpha x \leqslant \left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}} \left(\int_a^b |w(x)||h(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}}.$$

The inequality (3.9) is known as Rogers–Hölder's inequality on time scales. □

4. Radon's Inequality

Now, we present an extension of dynamic Radon's inequality on time scales with Specht's ratio.

THEOREM 4.1. *Let $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be $\diamond_\alpha$–integrable on $[a,b]_{\mathbb{T}}$. If $\beta > 0$ and $\gamma \geqslant 1$, then*

$$\frac{\left(\int_a^b S\left(\left(\frac{\Omega|f(x)|^{\beta+\gamma}}{\Lambda|g(x)|^{\beta+\gamma}}\right)^{\delta}\right)|w(x)||f(x)||g(x)|^{\gamma-1}\diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^b |w(x)||g(x)|^{\gamma}\diamond_\alpha x\right)^{\beta+\gamma-1}} \leqslant \int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}}\diamond_\alpha x, \tag{4.1}$$

where $\Lambda = \int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta}}\diamond_\alpha x$, $\Omega = \int_a^b |w(x)||g(x)|^{\gamma}\diamond_\alpha x$, $\delta = \min\left\{\frac{1}{p},\frac{1}{q}\right\}$ with $p = \beta+\gamma > 1$, $q = \frac{\beta+\gamma}{\beta+\gamma-1} > 1$, and $S(.)$ is the Specht's ratio.

PROOF. Set

$$\Phi(x) = \frac{|w(x)||f(x)|^p}{\int_a^b |w(x)||f(x)|^p\diamond_\alpha x} \text{ and } \Psi(x) = \frac{|w(x)||g(x)|^q}{\int_a^b |w(x)||g(x)|^q\diamond_\alpha x} \text{ on } [a,b]_{\mathbb{T}}.$$

Then Young's inequality from (2.2) becomes

$$\frac{S\left(\left(\frac{\Omega|f(x)|^p}{\Lambda|g(x)|^q}\right)^{\delta}\right)|w(x)||f(x)g(x)|}{\left(\int_a^b |w(x)||f(x)|^p\diamond_\alpha x\right)^{\frac{1}{p}}\left(\int_a^b |w(x)||g(x)|^q\diamond_\alpha x\right)^{\frac{1}{q}}} \leqslant \frac{|w(x)||f(x)|^p}{p\left(\int_a^b |w(x)||f(x)|^p\diamond_\alpha x\right)} + \frac{|w(x)||g(x)|^q}{q\left(\int_a^b |w(x)||g(x)|^q\diamond_\alpha x\right)}, \tag{4.2}$$

where $\Lambda = \int_a^b |w(x)||f(x)|^p\diamond_\alpha x$ and $\Omega = \int_a^b |w(x)||g(x)|^q\diamond_\alpha x$.

By integrating both sides of inequality (4.2) over x from a to b, we obtain

$$\frac{\int_a^b S\left(\left(\frac{\Omega|f(x)|^p}{\Lambda|g(x)|^q}\right)^{\delta}\right)|w(x)||f(x)g(x)|\diamond_\alpha x}{\left(\int_a^b |w(x)||f(x)|^p\diamond_\alpha x\right)^{\frac{1}{p}}\left(\int_a^b |w(x)||g(x)|^q\diamond_\alpha x\right)^{\frac{1}{q}}} \leqslant \frac{1}{p}+\frac{1}{q} = 1. \tag{4.3}$$

Therefore

$$\int_a^b S\left(\left(\frac{\Omega|f(x)|^p}{\Lambda|g(x)|^q}\right)^{\delta}\right)|w(x)||f(x)g(x)|\diamond_\alpha x \leqslant \left(\int_a^b |w(x)||f(x)|^p\diamond_\alpha x\right)^{\frac{1}{p}}\left(\int_a^b |w(x)||g(x)|^q\diamond_\alpha x\right)^{\frac{1}{q}}. \tag{4.4}$$

Replacing $|f(x)|$ and $|g(x)|$ by $F(x)$ and $G(x)$, respectively, and letting $F(x) = \left|\frac{f(x)}{g(x)}\right|^{\frac{1}{\beta+\gamma}}$ and $G(x) = |f(x)|^{\frac{\beta+\gamma-1}{\beta+\gamma}}|g(x)|^{\frac{1}{\beta+\gamma}}$ in (4.4), we have

$$\begin{aligned}(4.5)\quad &\int_a^b S\left(\left(\frac{\Omega}{\Lambda|g(x)|^{\frac{\beta+\gamma}{\beta+\gamma-1}}}\right)^{\delta}\right)|w(x)||f(x)|\diamond_\alpha x\\ &\leqslant\left(\int_a^b\frac{|w(x)||f(x)|}{|g(x)|}\diamond_\alpha x\right)^{\frac{1}{\beta+\gamma}}\left(\int_a^b|w(x)||f(x)||g(x)|^{\frac{1}{\beta+\gamma-1}}\diamond_\alpha x\right)^{\frac{\beta+\gamma-1}{\beta+\gamma}},\end{aligned}$$

where $\Lambda = \int_a^b \frac{|w(x)||f(x)|}{|g(x)|}\diamond_\alpha x$ and $\Omega = \int_a^b |w(x)||f(x)||g(x)|^{\frac{1}{\beta+\gamma-1}}\diamond_\alpha x$.

Taking power $\beta+\gamma$ on both sides of inequality (4.5), we get

$$(4.6)\quad \frac{\left(\int_a^b S\left(\left(\frac{\Omega}{\Lambda|g(x)|^{\frac{\beta+\gamma}{\beta+\gamma-1}}}\right)^{\delta}\right)|w(x)||f(x)|\diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^b|w(x)||f(x)||g(x)|^{\frac{1}{\beta+\gamma-1}}\diamond_\alpha x\right)^{\beta+\gamma-1}}\leqslant\int_a^b\frac{|w(x)||f(x)|}{|g(x)|}\diamond_\alpha x.$$

Replacing $|g(x)|$ by $\left|\frac{g(x)}{f(x)}\right|^{\beta+\gamma-1}$ in inequality (4.6), we get

$$(4.7)\quad \frac{\left(\int_a^b S\left(\left(\frac{\Omega|f(x)|^{\beta+\gamma}}{\Lambda|g(x)|^{\beta+\gamma}}\right)^{\delta}\right)|w(x)||f(x)|\diamond_\alpha x\right)^{\beta+\gamma}}{\left(\int_a^b|w(x)||g(x)|\diamond_\alpha x\right)^{\beta+\gamma-1}}\leqslant\int_a^b\frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta+\gamma-1}}\diamond_\alpha x,$$

where $\Lambda = \int_a^b \frac{|w(x)||f(x)|^{\beta+\gamma}}{|g(x)|^{\beta+\gamma-1}}\diamond_\alpha x$ and $\Omega = \int_a^b |w(x)||g(x)|\diamond_\alpha x$.

Replacing $|w(x)|$ by $|w(x)||g(x)|^{\gamma-1}$ in inequality (4.7), we get the desired claim. The proof of Theorem 4.1 is completed. □

REMARK 4.1. If $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k \in (0,+\infty)$ and $g(k) = y_k \in (0,+\infty)$ for $k \in \{1,2,\dots,n\}$, then discrete version of inequality (4.1) reduces to

$$(4.8)\quad \frac{\left(\sum\limits_{k=1}^{n} S\left(\left(\frac{\Omega x_k^{\beta+\gamma}}{\Lambda y_k^{\beta+\gamma}}\right)^{\delta}\right)x_k y_k^{\gamma-1}\right)^{\beta+\gamma}}{\left(\sum\limits_{k=1}^{n} y_k^{\gamma}\right)^{\beta+\gamma-1}}\leqslant\sum_{k=1}^{n}\frac{x_k^{\beta+\gamma}}{y_k^{\beta}},$$

where $\Lambda = \sum\limits_{k=1}^{n}\frac{x_k^{\beta+\gamma}}{y_k^{\beta}}$ and $\Omega = \sum\limits_{k=1}^{n} y_k^{\gamma}$.

The inequality (4.8) is obtained by using Specht's ratio in (1.2). The integral version of reverse Radon's inequality with Specht's ratio was obtained in [**20**].

Next, we give the following extension of dynamic Bergström's inequality on time scales.

COROLLARY 4.1. *Let $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be $\diamond_\alpha$–integrable on $[a,b]_{\mathbb{T}}$. Then*

$$\frac{\left(\int_a^b S\left(\left(\frac{\Omega|f(x)|^2}{\Lambda|g(x)|^2}\right)^{\frac{1}{2}}\right)|w(x)||f(x)|\diamond_\alpha x\right)^2}{\int_a^b |w(x)||g(x)|\diamond_\alpha x} \leqslant \int_a^b \frac{|w(x)||f(x)|^2}{|g(x)|}\diamond_\alpha x, \tag{4.9}$$

where $\Lambda = \int_a^b \frac{|w(x)||f(x)|^2}{|g(x)|}\diamond_\alpha x$ and $\Omega = \int_a^b |w(x)||g(x)|\diamond_\alpha x$.

PROOF. Putting $\beta = \gamma = 1$ in Theorem 4.1, the inequality (4.9) follows. $\square$

REMARK 4.2. If $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k \in \mathbb{R}-\{0\}$ and $g(k) = y_k \in (0,+\infty)$ for $k \in \{1,2,\dots,n\}$, then inequality (4.9) reduces to

$$\frac{\left(\sum\limits_{k=1}^{n} S\left(\left(\frac{\Omega x_k^2}{\Lambda y_k^2}\right)^{\frac{1}{2}}\right)x_k\right)^2}{\sum\limits_{k=1}^{n} y_k} \leqslant \sum_{k=1}^{n}\frac{x_k^2}{y_k}, \tag{4.10}$$

where $\Lambda = \sum\limits_{k=1}^{n}\frac{x_k^2}{y_k}$ and $\Omega = \sum\limits_{k=1}^{n} y_k$.

The inequality (4.10) is obtained with Specht's ratio, which is a discrete version of the classical Bergström's inequality as given in [**3, 4, 5, 14**]. Bergström's inequality is called, in literature, Titu Andreescu's inequality, or Engel's inequality.

5. Rogers–Hölder's Inequality

In order to conclude our main results, we prove an extension of dynamic Rogers–Hölder's inequality by using Kantorovich's ratio on time scales.

THEOREM 5.1. *Let $w, f, g \in C([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ such that $|f(x)|^p$ and $|g(x)|^q$ are $\diamond_\alpha$–integrable on $[a,b]_{\mathbb{T}}$. If $\frac{1}{p}+\frac{1}{q}=1$ with $p>1$, then*

$$\begin{aligned} &\int_a^b K^{\delta}\left(\frac{\Omega|f(x)|^p}{\Lambda|g(x)|^q}\right)|w(x)||f(x)g(x)|\diamond_\alpha x \\ &\qquad \leqslant \left(\int_a^b |w(x)||f(x)|^p\diamond_\alpha x\right)^{\frac{1}{p}}\left(\int_a^b |w(x)||g(x)|^q\diamond_\alpha x\right)^{\frac{1}{q}}, \end{aligned} \tag{5.1}$$

where $\Lambda = \int_a^b |w(x)||f(x)|^p\diamond_\alpha x$, $\Omega = \int_a^b |w(x)||g(x)|^q\diamond_\alpha x$, $\delta = \min\left\{\frac{1}{p},\frac{1}{q}\right\}$, and $K(.)$ is the Kantorovich's ratio.

PROOF. Setting

$$\Phi(x) = \frac{|w(x)||f(x)|^p}{\int_a^b |w(x)||f(x)|^p\diamond_\alpha x} \text{ and } \Psi(x) = \frac{|w(x)||g(x)|^q}{\int_a^b |w(x)||g(x)|^q\diamond_\alpha x}$$

on $[a,b]_{\mathbb{T}}$. Then Young's inequality from (2.3) becomes

$$\frac{K^{\delta}\left(\frac{\Omega|f(x)|^p}{\Lambda|g(x)|^q}\right)|w(x)||f(x)g(x)|}{\left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}}\left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}}} \leqslant \frac{|w(x)||f(x)|^p}{p\left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)} + \frac{|w(x)||g(x)|^q}{q\left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)}. \tag{5.2}$$

By integrating both sides of inequality (5.2) over x from a to b, we obtain

$$\frac{\int_a^b K^{\delta}\left(\frac{\Omega|f(x)|^p}{\Lambda|g(x)|^q}\right)|w(x)||f(x)g(x)| \diamond_\alpha x}{\left(\int_a^b |w(x)||f(x)|^p \diamond_\alpha x\right)^{\frac{1}{p}}\left(\int_a^b |w(x)||g(x)|^q \diamond_\alpha x\right)^{\frac{1}{q}}} \leqslant \frac{1}{p}+\frac{1}{q}=1. \tag{5.3}$$

Thus, inequality (5.1) follows from inequality (5.3). □

Remark 5.1. If $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $b = n+1$, $w \equiv 1$, $f(k) = x_k \in (0,+\infty)$ and $g(k) = y_k \in (0,+\infty)$ for $k \in \{1,2,\dots,n\}$, then discrete version of inequality (5.1) reduces to

$$\sum_{k=1}^{n} K^{\delta}\left(\frac{\Omega x_k^p}{\Lambda y_k^q}\right) x_k y_k \leqslant \left(\sum_{k=1}^{n} x_k^p\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} y_k^q\right)^{\frac{1}{q}}, \tag{5.4}$$

where $\Lambda = \sum\limits_{k=1}^{n} x_k^p$ and $\Omega = \sum\limits_{k=1}^{n} y_k^q$.

The inequality (5.4) is obtained by using Kantorovich's ratio in (1.3).

6. Conclusion and Future Work

There have been recent developments of the theory and applications of dynamic inequalities on time scales. Basic work on dynamic inequalities is done by Ravi Agarwal, George Anastassiou, Martin Bohner, Allan Peterson, Donal O'Regan, and Samir Saker and many other authors. In this research article, we have presented some dynamic inequalities for the diamond–α integral, which is the linear combination of the delta and nabla integrals. If we set $\alpha = 1$, then we get the delta versions and if we set $\alpha = 0$, then we get the nabla versions of diamond–α integral operator inequalities presented in this article. Also, if we set $\mathbb{T} = \mathbb{Z}$, then we get the discrete versions and if we set $\mathbb{T} = \mathbb{R}$, then we get the continuous versions of diamond–α integral operator inequalities presented in this article. Some generalizations and applications of Rogers–Hölder's inequality, Radon's inequality, Bergström's inequality, Nesbitt's inequality and other dynamic inequalities on time scales are also given in [**1, 8, 15, 16, 17**].

In the future research, we will continue to explore other dynamic inequalities on time scales. We can consider the dynamic inequalities by using Specht's ratio, Kantorovich's ratio, a functional generalization, n–tuple diamond–α integral, the fractional derivatives, the fractional Riemann–Liouville integrals, the quantum calculus, and the α, β–symmetric quantum calculus.

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