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APPROXIMATE FIXED POINT PROPERTIES IN DIGITAL TOPOLOGY

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ABSTRACT. We study the approximate fixed point property (AFPP) for continuous single-valued functions and for continuous multivalued functions in digital topology. We extend what is known about these notions and discuss errors that have appeared in the literature.

1. Introduction

In digital topology, we study properties of digital images that are inspired by classical topology. A digital version of continuous functions has been developed, and researchers have had success in studying digital versions of connectedness, homotopy, fundamental groups, homology, et al., such that digital images resemble the Euclidean objects they model with respect to these properties.

However, the fixed point properties of digital images are often quite different from those of their Euclidean inspirations. E.g., while there are many examples of topological spaces with fixed point property (FPP), it is known [**5**] that a digital image X has the FPP if and only if X has a single point. Therefore, the study of *almost* [**12, 14**] or *approximate* [**5**] *fixed points* and the *almost/approximate fixed point property* (AFPP) is often of interest.

The paper [**7**] introduced the AFPP for continuous multivalued functions on digital images and obtained some results for this property, but provided no examples of digital images with this property. We provide examples in this paper along with additional general results concerning the AFPP for continuous single-valued and multivalued functions on digital images. We also discuss errors that appeared in the papers [**11, 7**].

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2. Preliminaries

Much of this section is quoted or paraphrased from the references.

We use $\mathbb{Z}$ to indicate the set of integers.

2.1. Adjacencies. The c_u-adjacencies are commonly used. Let $x, y \in \mathbb{Z}^n$, $x \neq y$, where we consider these points as n-tuples of integers:

$$x = (x_1, \dots, x_n), \ y = (y_1, \dots, y_n).$$

Let $u \in \mathbb{Z}$, $1 \leqslant u \leqslant n$. We say x and y are c_u*-adjacent* if

- there are at most u indices i for which $|x_i - y_i| = 1$, and
- for all indices j such that $|x_j - y_j| \neq 1$ we have $x_j = y_j$.

Often, a c_u-adjacency is denoted by the number of points adjacent to a given point in $\mathbb{Z}^n$ using this adjacency. E.g.,

- In $\mathbb{Z}^1$, c_1-adjacency is 2-adjacency.
- In $\mathbb{Z}^2$, c_1-adjacency is 4-adjacency and c_2-adjacency is 8-adjacency.
- In $\mathbb{Z}^3$, c_1-adjacency is 6-adjacency, c_2-adjacency is 18-adjacency, and c_3-adjacency is 26-adjacency.
- In $\mathbb{Z}^n$, c_1-adjacency is $2n$-adjacency and c_n-adjacency is $(3^n - 1)$-adjacency.

For κ-adjacent x, y, we write $x \leftrightarrow_\kappa y$ or $x \leftrightarrow y$ when κ is understood. We write $x \leftrightarroweq_\kappa y$ or $x \leftrightarroweq y$ to mean that either $x \leftrightarrow_\kappa y$ or $x = y$. We say subsets A, B of a digital image X are (κ-)adjacent, $A \leftrightarroweq_\kappa B$ or $A \leftrightarroweq B$ when κ is understood, if there exist $a \in A$ and $b \in B$ such that $a \leftrightarroweq_\kappa b$.

We say $\{x_n\}_{n=0}^k \subset (X, \kappa)$ is a κ*-path* (or a *path* if κ is understood) from x_0 to x_k if $x_i \leftrightarroweq_\kappa x_{i+1}$ for $i \in \{0, \dots, k-1\}$, and k is the *length* of the path.

Another adjacency we will use is the following.

Definition 2.1. ([**13**]) In a digital image (X, λ), $x \leftrightarrow_{\lambda^k} y$ if there is a λ-path of length at most k in X from x to y.

A subset Y of a digital image (X, κ) is κ*-connected* [**12**], or *connected* when κ is understood, if for every pair of points $a, b \in Y$ there exists a κ-path in Y from a to b.

2.2. Digitally continuous functions. The following generalizes a definition of [**12**].

Definition 2.2. ([**1**]) Let (X, κ) and (Y, λ) be digital images. A single-valued function $f : X \to Y$ is (κ, λ)*-continuous* if for every κ-connected $A \subset X$ we have that $f(A)$ is a λ-connected subset of Y. If $(X, \kappa) = (Y, \lambda)$, we say such a function is κ*-continuous*, denoted $f \in C(X, \kappa)$.

When the adjacency relations are understood, we will simply say that f is *continuous*. Continuity can be expressed in terms of adjacency of points:

Theorem 2.1 ([**12, 1**]). *A single-valued function $f : X \to Y$ is continuous if and only if $x \leftrightarrow x'$ in X implies $f(x) \leftrightarroweq f(x')$.*

Composition preserves continuity, in the sense of the following.

THEOREM 2.2 ([**1**]). *Let (X, κ), (Y, λ), and (Z, μ) be digital images. Let $f : X \to Y$ be (κ, λ)-continuous and let $g : Y \to Z$ be (λ, μ)-continuous. Then $g \circ f : X \to Z$ is (κ, μ)-continuous.*

Given $X = \Pi_{i=1}^{v} X_i$, we denote throughout this paper the projection onto the i^{th} factor by p_i; i.e., $p_i : X \to X_i$ is defined by $p_i(x_1, \ldots, x_v) = x_i$, where $x_j \in X_j$.

2.3. Digitally continuous multivalued functions. A *multivalued function* f from X to Y assigns a subset of Y to each point of x. We will write $f : X \multimap Y$. For $A \subset X$ and a multivalued function $f : X \multimap Y$, let $f(A) = \bigcup_{x \in A} f(x)$.

The papers [**8, 9**] define continuity for multivalued functions between digital images based on subdivisions. (These papers make an error with respect to compositions, that is corrected in [**10**].) We have the following.

DEFINITION 2.3. ([**8, 9**]) For any positive integer r, the *r-th subdivision* of $\mathbb{Z}^n$ is

$$\mathbb{Z}_r^n = \{(z_1/r, \ldots, z_n/r) \mid z_i \in \mathbb{Z}\}.$$

An adjacency relation κ on $\mathbb{Z}^n$ naturally induces an adjacency relation (which we also call κ) on $\mathbb{Z}_r^n$ as follows: $(z_1/r, \ldots, z_n/r) \leftrightarrow_\kappa (z_1'/r, \ldots, z_n'/r)$ in $\mathbb{Z}_r^n$ if and only if $(z_1, \ldots, z_n) \leftrightarrow_\kappa (z_1', \ldots, z_n')$ in $\mathbb{Z}^n$.

Given a digital image $(X, \kappa) \subset (\mathbb{Z}^n, \kappa)$, the *r-th subdivision* of X is

$$S(X, r) = \{(x_1, \ldots, x_n) \in \mathbb{Z}_r^n \mid (\lfloor x_1 \rfloor, \ldots, \lfloor x_n \rfloor) \in X\}.$$

Let $E_r : S(X, r) \to X$ be the natural map sending $(x_1, \ldots, x_n) \in S(X, r)$ to $(\lfloor x_1 \rfloor, \ldots, \lfloor x_n \rfloor)$.

For a digital image $(X, \kappa) \subset (\mathbb{Z}^n, \kappa)$, a function $f : S(X, r) \to Y$ *induces a multivalued function* $F : X \multimap Y$ as follows:

$$F(x) = \bigcup_{x' \in E_r^{-1}(x)} \{f(x')\}.$$

A multivalued function $F : X \multimap Y$ is called (κ, λ)-*continuous* when there is some r such that F is induced by some single-valued (κ, λ)-continuous function $f : S(X, r) \to Y$.

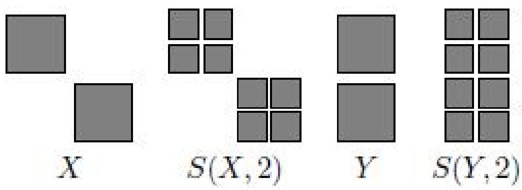

FIGURE 1. Two images X and Y with their second subdivisions [**6**]

EXAMPLE 2.1. ([**6**]) An example of two digital images and their subdivisions is given in Figure 1. Note that the subdivision construction (and thus the notion of continuity) depends on the particular embedding of X as a subset of $\mathbb{Z}^n$. In particular we may have $X, Y \subset \mathbb{Z}^n$ with X isomorphic to Y but $S(X,r)$ not isomorphic to $S(Y,r)$. This is the case for the two images in Figure 1, when we use 8-adjacency for all images: X and Y in the figure are isomorphic, each being a set of two adjacent points, but $S(X,2)$ and $S(Y,2)$ are not isomorphic since $S(X,2)$ can be disconnected by removing a single point, while this is impossible in $S(Y,2)$. $\square$

It is known [**10**] that a composition of digitally continuous multivalued functions need not be continuous. However, we have the following.

THEOREM 2.3 ([**10**]). *Let $X \subset \mathbb{Z}^v$, $Y \subset \mathbb{Z}^n$, $Z \subset \mathbb{Z}^p$. Let $F : X \multimap Y$ be a (c_v, κ)-continuous multivalued function and let $G : Y \multimap Z$ be a (κ, λ)-continuous multivalued function. Then $G \circ F : X \multimap Z$ is (c_v, λ)-continuous.*

Another way in which composition preserves continuity is the following.

PROPOSITION 2.1. *Let $F : X \multimap Y$ be a (κ, λ)-continuous multivalued function and let $g : Y \to Z$ be a (λ, μ)-continuous single-valued function. Then $g \circ F : X \multimap Z$ is a (κ, μ)-continuous multivalued function.*

PROOF. Let $f : S(X,r) \to Y$ be a (κ, λ)-continuous function that induces F. Then $g \circ f : S(X,r) \to Z$ is a (κ, μ)-continuous single-valued function that induces $g \circ F$. $\square$

2.4. Approximate fixed points. Let $f \in C(X,\kappa)$, let $F : X \multimap X$ be a κ-continuous multivalued function, and let $x \in X$. We say

- x is a *fixed point* of f if $f(x) = x$; x is a *fixed point* of F if $x \in F(x)$.
- If $f(x) \leftrightarrows_\kappa x$, then x is an *almost fixed point* [**12, 14**] or *approximate fixed point* [**5**] of (f, κ).
- If there exists $x' \in F(x)$ such that $x \leftrightarrows_\kappa x'$, then x is an *approximate fixed point* [**7**] of (F, κ).
- A digital image (X, κ) has the *approximate fixed point property with respect to continuous single-valued functions ($AFPP_S$)* [**5**] if for every $g \in C(X, \kappa)$ there is an approximate fixed point of g.
- A digital image (X, κ) has the *approximate fixed point property with respect to continuous multivalued functions ($AFPP_M$)* [**7**] if for every (κ, κ)-continuous multivalued function $G : X \multimap X$ there is an approximate fixed point of G.

THEOREM 2.4 ([**5**]). *Let X and Y be digital images such that (X, κ) and (Y, λ) are isomorphic. If (X, κ) has the $AFPP_S$, then (Y, λ) has the $AFPP_S$.*

THEOREM 2.5 ([**5**]). *Let X and Y be digital images such that Y is a κ-retract of X. If (X, κ) has the $AFPP_S$, then (Y, κ) has the $AFPP_S$.*

The latter inspired the following.

Theorem 2.6 ([**7**]). *Let $X \subset \mathbb{Z}^v$ and let Y be a c_v-multivalued retract of X. If (X, c_v) has the $AFPP_M$, then (Y, c_v) has the $AFPP_M$.*

Proposition 2.2. *If (X, κ) has the $AFPP_M$, then (X, κ) has the $AFPP_S$.*

Proof. The assertion follows from the observation that a continuous single-valued function between digital images is a continuous multivalued function. □

3. Results for digital cubes

In this section, we consider approximate fixed point properties for digital cubes. We start with the special case of dimension 1.

Theorem 3.3 of [**12**], which proves that a digital interval has the $AFPP_S$, is extended as follows.

Theorem 3.1. *The digital image $([a, b]_{\mathbb{Z}}, c_1)$ has the $AFPP_M$.*

Proof. We modify the proof given for Theorem 3.3 of [**12**]. Let $F : [a, b]_{\mathbb{Z}} \multimap [a, b]_{\mathbb{Z}}$ be c_1-continuous. If $a \in F(a)$ or $b \in F(b)$, we are done. Otherwise, let $f : (S(X, r), c_1) \to ([a, b]_{\mathbb{Z}}, c_1)$ generate F. Then $f(a) > a$ and $f(b) < b$, so

(3.1) $g(t) = f(t) - \lceil t \rceil$ is positive at $t = a$ and negative at $t = b$.

Since f is continuous, we must have $f(t - 1/r) \in \{f(t) - 1, f(t), f(t) + 1\}$ for all $t \in S(X, r) \smallsetminus \{a\}$. Let $z \in \mathbb{Z}$. Since $\lceil t \rceil$ is constant for $z \leqslant t < z + 1$, and changes by 1 as t increases from $z + (r - 1)/r$ to $z + 1$, it follows that for $a \leqslant t < b$, an increase of $1/r$ in the value of t causes the expression $|g(t)|$ to change by at most 1 for $z \leqslant t < z + 1$, and by at most 2 for $t = z + (r - 1)/r$. It follows from (3.1) that there exists c such that $g(c) \in \{-1, 0, 1\}$, i.e., c is an approximate fixed point for F. □

A. Rosenfeld's paper [**12**] states the following as its Theorem 4.1 (quoted verbatim).

> Let I be a digital picture, and let f be a continuous function from I into I; then there exists a point $P \in I$ such that $f(P) = P$ or is a neighbor or diagonal neighbor of P.

Several subsequent papers have incorrectly concluded that this result implies that I with some c_u adjacency has the $AFPP_S$. By *digital picture* Rosenfeld means a digital cube, $I = [0, n]_{\mathbb{Z}}^v$. By a "continuous function" he means a (c_1, c_1)-continuous function; by "a neighbor or diagonal neighbor of P" he means a c_v-adjacent point. Thus, if we generalize our definition of the $AFPP_S$ as Definition 3.1 below, what Rosenfeld's theorem shows in terms of an AFPP is stated as Theorem 3.2 below.

Definition 3.1. Let κ, λ, μ be adjacencies for a digital image X. Then X has the *approximate fixed point property for single-valued functions and* (κ, λ, μ), denoted $AFPP_S(\kappa, \lambda, \mu)$, if for every (κ, λ)-continuous single-valued function $f : X \to X$ there exists $x \in X$ such that $x \leftrightarroweq_\mu f(x)$. X has the *approximate fixed point property for multivalued functions and* (κ, λ, μ), denoted $AFPP_M(\kappa, \lambda, \mu)$, if for every (κ, λ)-continuous multivalued function $F : X \multimap X$ there exist $x \in X$ and $y \in F(x)$ such that $x \leftrightarroweq_\mu y$.

Thus, (X, κ) has the $AFPP_S$ if and only if X has the $AFPP_S(\kappa, \kappa, \kappa)$.

Theorem 3.2 ([**12**]). *Let $X = [0, n]_{\mathbb{Z}}^v \subset \mathbb{Z}^v$ for some $v \in \mathbb{N}$. Then X has the $AFPP_S(c_1, c_1, c_v)$.*

Since Rosenfeld's proof can be easily modified to any digital cube $\Pi_{i=1}^v [a_i, b_i]_{\mathbb{Z}}$, and since for $1 \leqslant u \leqslant v$, a (c_u, c_1)-continuous $f : X \to X$ is (c_1, c_1)-continuous [**3**], we have the following.

Corollary 3.1. *Let n and v be positive integers. Let $X = \Pi_{i=1}^v [a_i, b_i]_{\mathbb{Z}} \subset \mathbb{Z}^v$. Let $u \in [1, v]_{\mathbb{Z}}$. Then X has the $AFPP_S(c_u, c_1, c_v)$.*

Theorem 3.3. *Let $X = [0, n]_{\mathbb{Z}}^v \subset \mathbb{Z}^v$. Then X has the $AFPP_M(c_v, c_v, c_v^{\lceil n/2 \rceil})$.*

Proof. Let x be a point of X such that each coordinate of x is a member of $\{\lfloor n/2 \rfloor, \lceil n/2 \rceil\}$. Let $F : X \multimap X$ be (c_v, c_v)-continuous. Then for some (indeed, every) $y \in F(x)$, there is a c_v-path from x to y of length at most $\lceil n/2 \rceil$. □

As in Proposition 2.2, we have the following.

Corollary 3.2. *Let $X = [0, n]_{\mathbb{Z}}^v \subset \mathbb{Z}^v$. Then X has the $AFPP_S(c_v, c_v, c_v^{\lceil n/2 \rceil})$.*

The following (restated here in our terminology) is Theorem 1 of Han's paper [**11**].

Theorem 3.4. *Let $X = [-1, 1]_{\mathbb{Z}}^v$ and $1 \leqslant u \leqslant v$. Then (X, c_u) has the $AFPP_S$ if and only if $u = v$.*

We show below that Theorem 3.4 is correct. This is necessary, since Han fails to give a correct proof for either implication of this theorem.

- Han offers two "proofs" of the assertion that (X, c_v) has the $AFPP_S$. Both of Han's arguments are incorrect.
  (1) Han's first "proof" of the assertion that (X, c_v) has the $AFPP_S$ is based on the false assertion that $f \in C(X, c_u)$ implies $f \in C(X, c_v)$. The latter assertion is incorrect, as shown in Example 3.1 below.
  (2) Han's second "proof" argues that assuming otherwise yields a contradiction. He gives an example, for $v = 2$, of a self-map on X. He claims without explanation that if (X, c_v) fails to have the $AFPP_S$, then this map shows that all self maps on (X, c_v) are discontinuous; further, he offers no argument that this example generalizes to all self maps on X for all dimensions v.

  That (X, c_v) has the $AFPP_S$ is proven below at Example 3.2(3). Han's focus on the point $(0, 0)$ raises the possibility that he had such an example in mind for his second "proof," but this is not clear.
- Han's argument for the assertion that (X, c_u) does not have the $AFPP_S$ for $u < v$ is incorrect, as it discusses only particular self-maps on the digital images $([-1, 1]_{\mathbb{Z}}^2, c_1)$ and $([-1, 1]_{\mathbb{Z}}^3, c_2)$, with no indication that these examples generalize. Theorem 3.5, below, correctly shows this assertion.

EXAMPLE 3.1. Let $X = [-1,1]_{\mathbb{Z}}^2$. Let $f : X \to X$ be defined by

$$f(x,y) = \begin{cases} (\min\{1, x+1\}, y) & \text{if } y \in \{-1,0\}; \\ (x,0) & \text{if } y = 1. \end{cases}$$

See Figure 2. Then it is easily seen that $f \in C(X, c_1)$. However, $f \notin C(X, c_2)$, since $(-1,1) \leftrightarrow_{c_2} (0,0)$, but $f(-1,1) = (-1,0)$ and $f(0,0) = (1,0)$ are not c_2-adjacent. □

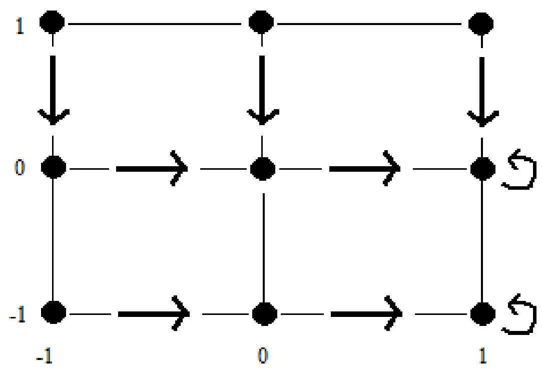

FIGURE 2. The function f of Example 3.1. For each $x \in X$, an arrow indicates $f(x)$. This function belongs to $C(X, c_1)$, but not to $C(X, c_2)$.

PROPOSITION 3.1. *Let v be a positive integer, $v > 1$. Let $X = [0,1]_{\mathbb{Z}}^v \subset \mathbb{Z}^v$. Let $u \in [1, v-1]_{\mathbb{Z}}$. Then (X, c_u) does not have the $AFPP_S$.*

PROOF. Let $f : [0,1]_{\mathbb{Z}} \to [0,1]_{\mathbb{Z}}$ be defined by $f(z) = 1 - z$. Let $F : X \to X$ be defined by

$$F(x_1, \ldots, x_v) = (f(x_1), \ldots, f(x_v)).$$

It is easily seen that if $x, x' \in X$, then $F(x)$ and $F(x')$ differ in as many coordinates as x and x', and therefore $F \in C(X, c_u)$. However, for each $x \in X$, x and $F(x)$ differ in all v coordinates, so F has no c_u-approximate fixed point. □

THEOREM 3.5. *Let $X \subset \mathbb{Z}^v$ be such that X has a subset $Y = \Pi_{i=1}^v [a_i, b_i]_{\mathbb{Z}}$, where $v > 1$; for all indices i, $b_i \in \{a_i, a_i + 1\}$; and, for at least 2 indices i, $b_i = a_i + 1$. Then (X, c_u) fails to have the $AFPP_S$ for $1 \leqslant u < v$. (This contains one of the assertions of Theorem 3.4.)*

PROOF. Let $r : X \to Y$ be defined by its coordinate functions,

$$p_i(r(x)) = \begin{cases} a_i & \text{if } p_i(x) \leqslant a_i; \\ b_i & \text{if } p_i(x) \geqslant b_i. \end{cases}$$

It is easily seen that r is a c_u-retraction of X to Y. The assertion follows from Proposition 3.1 and Theorem 2.5. □

We expand on one of the assertions of Theorem 3.4 as follows.

Corollary 3.3. *Let $X = \Pi_{i=1}^{v}[a_i, b_i]_{\mathbb{Z}}$ such that $b_i > a_i$ for at least 2 indices i. Let $u \in \mathbb{Z}$, $1 \leqslant u < v$. Then (X, c_u) does not have the $AFPP_S$.*

Proof. It follows easily from Theorem 3.5 that (X, c_u) does not have the $AFPP_S$. □

Lemma 3.1. *Let $X = [0,1]_{\mathbb{Z}}^{v} \subset \mathbb{Z}^v$. Then given $u \in [1, v]_{\mathbb{Z}}$ and a (c_u, c_v)-continuous multivalued function $F : X \multimap X$, every $x \in X$ satisfies $\{x\} \leftrightarrows_{c_v} F(x)$.*

Proof. The assertion follows from the observation that (X, c_v) is a complete graph. □

Theorem 3.5 states a severe limitation on the $AFPP_S$ and on the $AFPP_M$ for digital images $X \subset \mathbb{Z}^v$ and the c_u adjacency, where $1 \leqslant u < v$. We now consider the case $u = v$.

The paper [**7**] introduces the $AFPP_M$ but provides no nontrivial examples of digital images with this property. Simple examples are given in the following.

Example 3.2. The following digital images have the $AFPP_M$.

(1) A singleton.
(2) $([0,1]_{\mathbb{Z}}^{v}, c_v)$.
(3) $([-1,1]_{\mathbb{Z}}^{v}, c_v)$. (By Proposition 2.2, this contains one of the assertions of Theorem 3.4.)

Proof. (1) The case of a singleton is trivial.

(2) The assertion for $([0,1]_{\mathbb{Z}}^{v}, c_v)$ follows from Lemma 3.1.

(3) The assertion for $([-1,1]_{\mathbb{Z}}^{v}, c_v)$ follows from the observation that the point $(0, 0, \ldots, 0)$ is c_v-adjacent to every other point of the image, hence must be an approximate fixed point for every (c_v, c_v)-continuous multivalued self-map on this image. □

4. Retraction and preservation of $AFPP_M$

Retraction preserves the $AFPP_S$ [**5**]. The paper [**7**] claims the following analog for the $AFPP_M$ (restated here in our terminology) as its Theorem 4.4.

> Let $X \subset \mathbb{Z}^v$ such that (X, c_v) has the $AFPP_M$. Let $Y \subset X$ be a (c_v, c_v)-continuous multivalued retract of X. Then (Y, c_v) has the $AFPP_M$.

However, there are errors in the argument offered as proof of this claim, so the assertion must be regarded as unproven. The authors argue as follows. Given a (c_v, c_v)-continuous multivalued $F : Y \multimap Y$, let $I : Y \to X$ be the inclusion and $R : X \multimap Y$ a (c_v, c_v)-continuous multivalued retraction. Then $G = I \circ F \circ R : X \multimap X$ is shown to be (c_v, c_v)-continuous and therefore has an approximate fixed point x_0. Thus, there exists $x_1 \in G(x_0)$ such that $x_1 \leftrightarrows_{c_v} x_0$. By continuity of G we have

$$x_1 \in G(x_0) \leftrightarrows_{c_v} G(x_1) = I \circ F \circ R(x_1).$$

- There follows the claim that the latter is equal to $I \circ F(x_1)$; this is unjustified since we do not know whether x_1 belongs to Y. Indeed, we do not know if $F(x_1)$ is defined.
- Further, after observing that we would have $I \circ F(x_1) = F(x_1)$, it is claimed that $x_1 \leftrightarrows_{c_v} F(x_1)$, but this is unjustified since we can not assume that $G(x_0)$ and $G(x_1)$ are singletons.

5. Universal and weakly universal multivalued functions

Universal and weakly universal single-valued functions for digital images were introduced in [**5**] and [**4**], respectively. The paper [**7**] seeks to obtain analogous results for multivalued functions. In this section, we correct a small error of [**7**] in its treatment of universal multivalued functions. The error of concern parallels an error of [**5**], and our corrections parallel those in [**4**]. The error is due to using *universal* multivalued functions rather than *weak universal* multivalued functions (see Definition 5.1, below); this error propagates through multiple assertions of [**7**].

Definition 5.1. Let (X, κ) and (Y, λ) be digital images. Let $F : X \multimap Y$ be a (κ, λ)-continuous multivalued function.

- F is a *universal* for (X, Y) [**7**] if given a (κ, λ)-continuous multivalued function $G : X \multimap Y$, there exist $x \in X$ and $y \in F(x)$, $y' \in G(x)$ such that $y \leftrightarrow_\lambda y'$.
- F is a *weak universal* for (X, Y) if given a (κ, λ)-continuous multivalued function $G : X \multimap Y$, there exist $x \in X$ and $y \in F(x)$, $y' \in G(x)$ such that $y \leftrightarrows_\lambda y'$.

Proposition 3.1 of [**7**] asserts the following (quoted verbatim).

> Let X and Y be digital images. Suppose Y is finite. Then the multivalued function $F : X \multimap Y$ defined by $F(x) = Y$ for all $x \in X$ is universal.

This assertion is incorrect, as shown by the following. Let $(X, \kappa) = (Y, \kappa)$ be a digital image with a single point x_0. Since there is no point in X adjacent to x_0, no universal for (X, X) exists, contrary to the assertion of Proposition 3.1 of [**7**]. However, we have the following (note we do not need to assume that Y is finite).

Proposition 5.1. *Let X and Y be digital images. Then the multivalued function $F : X \multimap Y$ defined by $F(x) = Y$ for all $x \in X$, is a weak universal.*

Proof. This follows from Definition 5.1. □

Definition 5.2. ([**7**]) A multivalued function $F : (X, \kappa) \multimap (Y, \lambda)$ is *injective* if $F(x) = F(y)$ implies $x = y$. A injective multivalued function $G : X \multimap X$ is an *identity* if $x \in G(x)$ for all $x \in X$.

Proposition 4.1 of [**7**] asserts the following (restated here in our terminology).

> Let X be a digital image. Then (X, κ) has the $AFPP_M$ if and only if an identity multivalued function is universal.

This assertion is incorrect, as shown by the example of a digital image X with a single point x_0; X trivially has the $AFPP_M$, but no multivalued identity is universal since x_0 has no adjacent point y. However, we have the following (notice we show that we can take our identity multivalued function to be the unique single-valued identity function 1_X).

Proposition 5.2. *Let X be a digital image. Then (X,κ) has the $AFPP_M$ if and only if the identity function 1_X is a weak universal.*

Proof. Our argument requires only minor changes in the argument given for its analog in [**7**].

Suppose (X,κ) has the $AFPP_M$. Then given a (κ,κ)-continuous multivalued function $F : X \multimap X$, there exist $x \in X$ and $y \in F(x)$ such that $1_X(x) = x \leftrightarrows y \in F(x)$. Therefore, 1_X is a weak universal.

Suppose 1_X is a weak universal. Then for any (κ,κ)-continuous multivalued function $F : X \multimap X$, there exists $x \in X$ such that $x = 1_X(x) \leftrightarrows_\kappa y$ for some $y \in F(x)$. Thus, (X,κ) has the $AFPP_M$. $\square$

6. Further remarks

We have studied approximate fixed point properties for both single-valued and multivalued digitally continuous functions on digital images.

The question of whether the converse of Proposition 2.2 is valid appears to be a difficult problem. I.e., we have the following.

Question 6.1. If (X,κ) has the $AFPP_S$, does (X,κ) has the $AFPP_M$?

We also have not answered the following.

Question 6.2. Let $X = \Pi_{i=1}^{v}[a_i,b_i]_{\mathbb{Z}}$, where for at least 2 indices i we have $b_i > a_i$. Does (X,c_v) have the $AFPP_S$?

We saw in Section 4 that the following question remains unanswered.

Question 6.3. Let $X \subset \mathbb{Z}^v$ such that (X,c_v) has the $AFPP_M$. Let $Y \subset X$ be a (c_v,c_v)-continuous multivalued retract of X. Does (Y,c_v) have the $AFPP_M$?

Since [**4**] 1_X is a weak universal function for single-valued continuous self-maps on (X,κ) if and only if (X,κ) has the $AFPP_S$, in view of Proposition 5.2, a positive solution to the following question would yield a positive solution to Question 6.1.

Question 6.4. If 1_X is a weak universal function for $C(X,\kappa)$, is 1_X is a weak universal for continuous multivalued functions on (X,κ)?

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