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PSEUDO-UP ALGEBRAS, AN INTRODUCTION

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ABSTRACT. As a generalization of UP-algebras, the notion of pseudo-UP algebras is introduced, and some of their properties are investigated. Some characterizations of pseudo-UP algebras are established. In addition, the concept of pseudo-KU algebras was introduced and it is shown that each pseudo-KU algebra is a pseudo-UP algebra.

1. Introduction

The concept of pseudo-BCK algebras is introduce in [**4**] by G. Georgescu and A. Iorgulescu as an extension of BCK-algebras. The notion of pseudo-BCI algebras was introduced and analyzed in [**3**] by W. A. Dudek and Y. B. Jun as a generalization of BCI-algebras. These algebraic structures has been in the focus of many authors (for example, see [**5, 7, 8, 15, 16**])

Iampan [**6**] introduced a new algebraic structure which is called UP-algebras as a generalization of KU-algebras. (About these algebras a reader can look in [**11**].) He studied ideals and congruences in UP-algebras. He also introduced the concept of homomorphism of UP-algebras and investigated some related properties. Moreover, he derived some straightforward consequences of the relations between quotient UP-algebras and isomorphism. In the study of this algebraic structure, this author took part also (for example: [**12, 13, 14**]).

In this paper we introduced the concept of pseudo-UP algebras and some types properties of pseudo-UP algebras are studied. In addition, the concept of pseudo-KU algebras was introduced and it is shown that each pseudo-KU algebra is a pseudo-UP algebra.

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2. Preliminaries

In this section we will describe some elements of UP-algebras from the literature [**6**] necessary for our intentions in this text.

DEFINITION 2.1. ([**6**]) An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a UP-algebra where A is a nonempty set, $' \cdot '$ is a binary operation on A, and 0 is a fixed element of A (i.e. a nullary operation) if it satisfies the following axioms:
(UP-1) $(\forall x, y \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP-2) $(\forall x \in A)(0 \cdot x = x)$,
(UP-3) $(\forall x \in A)(x \cdot 0 = 0)$, and
(UP-4) $(\forall x, y \in A)((x \cdot y = 0 \ \wedge \ y \cdot x = 0) \implies x = y)$.

The following statement is important to us in further developing the idea of pseudo-UP algebras.

PROPOSITION 2.1 ([**6**], Theorem 1.15). *An algebra* $A = (A, \cdot, 0)$ *of type* $(2, 0)$ *is a UP-algebra if and only if it satisfies the following conditions:*
(UP-1) $(\forall x, y \in A)((y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0)$,
(UP-4) $(\forall x, y \in A)((x \cdot y = 0 \ \wedge \ y \cdot x = 0) \implies x = y)$ *and*
(UP-5) $(\forall x, y \in A)((y \cdot 0) \cdot x = x)$.

On a UP-algebra $A = (A, \cdot, 0)$, we define the UP-ordering $\leqslant$ on A as follows:

$$(\forall x, y \in A)(x \leqslant y \iff x \cdot y = 0).$$

PROPOSITION 2.2 ([**6**], Proposition 1.8). *In a UP-algebra A, the following properties hold:*
(1) $(\forall x \in A)(x \leqslant x)$,
(2) $(\forall x, y \in A)((x \leqslant y \ \wedge \ y \leqslant x), \implies x = y)$,
(3) $(\forall x, y, z \in A)((x \leqslant y \ \wedge \ y \leqslant z) \implies x \leqslant z)$,
(4) $(\forall x, y, z \in A)(x \leqslant y \implies z \cdot x \leqslant z \cdot y)$,
(5) $(\forall x, y, z \in A)(x \leqslant y \implies y \cdot z \leqslant x \cdot z)$,
(6) $(\forall x, y \in A)(x \leqslant y \cdot x)$ *and*
(7) $(\forall x \in A)(x \leqslant 0)$.

3. On pseudo-UP algebras

3.1. Concept of pseudo-UP algebras. The notion of pseudo-UP algebras is introduced by the following definition

DEFINITION 3.1. A pseudo-UP algebra is a structure $\mathfrak{A} = ((A, \leqslant), \cdot, *, 0)$, where $' \leqslant '$ is a binary relation on a set A, $' \cdot '$ and $' * '$ are internal binary operations on A and $'0'$ is an element of A, verifying the following axioms:
(pUP-1) $(\forall x, yz \in A)(y \cdot z \leqslant (x \cdot y) * (x \cdot z) \ \wedge \ y * z \leqslant (x * y) \cdot (x * z))$;
(pUP-4) $(\forall x, y \in A)((x \leqslant y \ \wedge \ y \leqslant x) \implies x = y)$;
(pUP-5) $(\forall x, y \in A)((y \cdot 0) * x = x \ \wedge \ (y * 0) \cdot x = x)$ and
(pUP-6) $(\forall x, y \in A)((x \leqslant y \iff x \cdot y = 0) \ \wedge \ (x \leqslant y \iff x * y = 0))$.

From the previous definition, it immediately follows

LEMMA 3.1. *In a pseudo-UP algebra* $\mathfrak{A}$ *the following holds:*
(8) $(\forall x \in A)(x \cdot 0 = 0 \wedge x * 0 = 0)$;
(9) $(\forall x \in A)(0 \cdot x = x \wedge 0 * x = x)$; *and*
(10) $(\forall x \in A)(x \cdot x = 0 \wedge x * x = 0)$.

PROOF. If we put $y = 0$ and $z = 0$ in (pUP-1) we get

$$0 = (0 \cdot 0) * ((x \cdot 0) * (x \cdot 0))$$

by (pUP-6). Since $(x \cdot 0) * (x \cdot 0) = (x \cdot 0)$ according to (pUP-5), we obtain $0 = (0 \cdot 0) * (x \cdot 0) = (x \cdot 0)$ from the previous equality. From here we get $0 = x \cdot 0$ applying again (pUP-5). Further, we have

$$0 = x \cdot 0 \Longleftrightarrow x \leqslant 0 \Longleftrightarrow x * 0 = 0$$

according to (pUP-6).

Combining (8) with (pUP-5), we obtain

$$0 * x = (x \cdot 0) * x = x$$

showing $0 * x = x$. Both parts of formula (8) are proven. To show that it values $0 \cdot x = x$, it is enough to put $y = 0$ in the right side of the formula (pUP-5). We obtain $(0 * 0) \cdot = x$. From here we get $0 \cdot x = x$ by applying (8). Both parts of formula (9) have been proved.

If we put $x = 0$, $y = 0$ and $z = x$ in (pUP-1), we get

$$0 = (0 \cdot x) * ((0 \cdot 0) * (0 \cdot x)).$$

Since $(0 \cdot 0) * (0 \cdot x) = 0 \cdot x$ by (pUP-5), from the previous equation follows

$$0 = (0 \cdot x) * (0 \cdot x) = x * x$$

with respect (9). From here, on, by (pUP-6), we have

$$0 = x * x \Longleftrightarrow x \leqslant x \Longleftrightarrow x \cdot x = 0.$$

□

COROLLARY 3.1. *Every pseudo-UP algebra A satisfying $x * y = x \cdot y$ for all $x, y \in A$ is a UP-algebra.*

The previous corollary says that the concept of pseudo-UP algebras is a generalization of the concept of UP algebras.

EXAMPLE 3.1. Take the UP-algebra $A = \{0, 1, 2, 3\}$ with internal operation $' \cdot '$ in Example 1.6 in the text [**6**] and the UP-subalgebra $A = \{0, 1, 2, 3\}$ with the opertion $' * '$ of UP-algebra in the Example 1.12. in the same text [**6**]. Then $((A, \leqslant), \cdot, *, 0)$ is a pseudo-UP algebra.

3.2. Some fundamental properties. The following lemma is important

LEMMA 3.2. *In a pseudo-UP algebra $\mathfrak{A}$ the following holds:*
(p7) $(\forall x \in A)(x \leqslant 0)$ *and*
(p1) $(\forall x \in A)(x \leqslant x)$.

PROOF. The claims of this lemma are direct consequences of the claims (9) and (10) of Lemma 3.1. □

Some of the fundamental properties of pseudo-UP algebras are given in the following theorem

THEOREM 3.1. *In a pseudo-UP algebra $\mathfrak{A}$ the following holds:*
(11) $(\forall x, y, z \in A)((x \leqslant y \wedge y \leqslant z) \implies x \leqslant z)$;
(12.1) $(\forall x, y, z \in A)(x \leqslant y \implies z \cdot x \leqslant z \cdot y)$;
(12.2) $(\forall x, y, z \in A)(x \leqslant y \implies z * x \leqslant z * y)$;
(13.1) $(\forall x, y, z \in A)(x \leqslant y \implies y \cdot z \leqslant x \cdot z)$; *and*
(13.2) $(\forall x, y, z \in A)(x \leqslant y \implies y * z \leqslant x * z)$.

PROOF. Let $x, y, z \in A$ be arbitrary elements such that $x \leqslant y$ and $y \leqslant z$. Then $x \cdot z = 0 * (0 * (x \cdot z))$ by (9). From this it is follows $(y \cdot z) * ((x \cdot y) \star (x \cdot z)) = 0$ by (pUP-5) and (pUP-1) using substitution and hypotheses. This completes the proof of (11).

Let $x, y, z \in A$ be arbitrary elements such that $x \leqslant y$. Then $x \cdot y = 0$ by (pUP-6). On the other hand, by (9) we have

$$(z \cdot x) * (z \cdot y) = 0 * ((z \cdot x) * (z \cdot y)).$$

From here, using the substitution $x \cdot y = 0$ and (pUP-1), we obtain

$$(z \cdot x) * (z \cdot y) = (x \cdot y) * ((z \cdot x) * (z \cdot y)) = 0.$$

So, the inequality $x \cdot x \leqslant z \cdot y$ is proven.

Using the right side of the formula (pUP-1), the implication (12.2) can be proved analogously.

Let $x, y, z \in A$ be arbitrary elements such that $x \leqslant y$. Then $x \cdot y = 0$ by (pUP-6). On the other hand, by (9) and (pUP-1) we have

$$(y \cdot z) * (x \cdot z) = (y \cdot z) * (0 * (x \cdot z)) = (y \cdot z) * ((x \cdot y) * (x \cdot z)) = 0.$$

From here we get (13.1).

The claim (13.2) can be proven by analogy with the preceding evidence. □

COROLLARY 3.2. *The relation $' \leqslant '$ in a pseudo-UP-algebra $\mathfrak{A}$ is a partial order left associate and right anti-associate with the internal binary operations $' \cdot '$ and $' * '$ in $\mathfrak{A}$.*

PROPOSITION 3.1. *In a pseudo-UP algebra $\mathfrak{A}$ the following holds:*
(p6.1) $(\forall x, y \in A)(x \leqslant y \cdot x)$ *and*
(p6.2) $(\forall x, y \in A)(x \leqslant y * x)$.

Proof. Obviously, the following equations

$$x * (y \cdot x) = (0 \cdot x) * ((y \cdot 0) * (y \cdot x)) = 0$$

are valid. Thus $x \leqslant y \cdot x$.

Similar to the previous case, we have

$$x \cdot (y * x) = (0 * x) \cdot ((y * 0) \cdot (y * x)) = 0.$$

□

4. Pseudo-KU algebra

In 2009, the notion of a KU-algebra was first introduced by Prabpayak and Leerawat [**11**] as follows: An algebra $A = (A, \cdot, 0)$ of type $(2, 0)$ is called a KU-algebra if it satisfies the following axioms:

(KU-1): $(\forall x, y, z \in A)((y \cdot x) \cdot ((x \cdot z) \cdot (y \cdot z)) = 0)$,
(KU-2): $(\forall x \in A)(0 \cdot x = x)$,
(KU-3): $(\forall x \in A)(x \cdot 0 = 0)$, and
(KU-4): $(\forall x, y \in A)((x \cdot y = 0 \wedge y \cdot x = 0) \implies x = y)$.

If in a pseudo-UP algebra $\mathfrak{A}$ the following formula

(pKU) $(\forall x, y, z \in A)(x * (y \cdot z) = y * (x \cdot z) \;\wedge\; x \cdot (y * z) = y \cdot (x * z))$

is valid, then we can transform the axiom (pUP-1) into the following formula

(pKU-1) $(\forall x, y, z \in A)((y \cdot x) \leqslant ((x \cdot z) * (y \cdot z)) \wedge (y * x) \leqslant ((x * z) \cdot (y * z)))$.

Indeed, let the right side $x \cdot (y * z) = y \cdot (x * z)$ of the formula (pKU) be a valid formula. Then

$$\begin{aligned}
(y \cdot z) \leqslant (x \cdot y) * (x \cdot z) &\iff (y \cdot z) \cdot ((x \cdot y) * (x \cdot z)) = 0 \qquad \text{(by (pUP-6))}\\
&\iff (x \cdot y) \cdot ((y \cdot z) * (x \cdot z)) = 0 \qquad \text{(by (pKU))}\\
&\iff x \cdot y \leqslant (y \cdot z) * (x \cdot z) \qquad \text{(putting: x = y and y = x)}\\
&\iff y \cdot x \leqslant (x \cdot z) * (y \cdot z).
\end{aligned}$$

The possibility of transforming the right side of the formula (pUP-1) can be proved analogously to the previous transformation.

These transformations open up the possibility for us to determine the concept of pseudo-KU algebras in the sense in which it is used in this text and in all articles cited in the literature except [**9**]. However, the term 'pseudo KU-algebra' and mark 'PKU' has already been used in [**9**] for different purposes. This name has been renamed in the articles [**1, 2**] in the term 'JU-algebra'. Although introducing the term 'pseudo-KU algebra' as a name for a structure constructed in the manner described here and using the abbreviation 'pKU' for this algebra could lead to confusion, we will do it for needs of this text.

Definition 4.1. An algebra $A = ((A, \leqslant), \cdot, *, 0)$ of type $(2, 2, 0)$ is called a pseudo-KU algebra if it satisfies the following axioms:

(pKU-1): $(\forall x, y, z \in A)((y \cdot x) \leqslant ((x \cdot z) * (y \cdot z)) \wedge (y * x) \leqslant ((x * z) \cdot (y * z)))$,
(pKU-2): $(\forall x \in A)((0 \cdot x = x) \wedge (0 * x = x))$,
(pKU-3): $(\forall x \in A)(x \leqslant 0)$,

(pKU-4): $(\forall x, y \in A)((x \leqslant y \wedge y \leqslant x) \implies x = y)$, and
(pKU-5): $(\forall x, y \in A)((x \leqslant y \iff x \cdot y = 0) \wedge (x \leqslant y \iff x * y = 0))$.

Lemma 4.1. *If A is a pseudo-KU algebra, then holds*
(pKU-6) $(\forall x \in A)((x \cdot x = 0) \wedge (x * x = 0))$.

Proof. If we put $x = 0$, $y = 0$, and $z = x$ in the formula (pKU-1), we get

$$(0 \cdot 0) * ((0 \cdot x) \star (0 \cdot x)) = 0 \ \wedge \ (0 * 0) \cdot ((0 * x) \cdot (0 * x)) = 0.$$

From where we get

$$x \cdot x = 0 \ \wedge \ x * x = 0$$

with respect to (pKU-2). □

Proposition 4.1. *If A is a pseudo-KU algebra, then holds*
(11) $(\forall x, y, z \in A)(x \leqslant y \implies ((y \cdot z \leqslant x \cdot z) \wedge (y * z \leqslant x * z))$ *and*
(12) $(\forall x, y, z \in A)(x \leqslant y \implies ((z \cdot x \leqslant z \cdot y) \wedge (z * x \leqslant z * y))$.

Proof. Let $x, y, z \in A$ such that $x \leqslant y$. Then $x \cdot y = 0 = x * y$. If we put $x = y$ and $y = x$ in (pKU-1), we get

$$0 = (x \cdot y) * ((y \cdot z) * (x \cdot z)) = 0 * ((y \cdot z) * (x \cdot z)) = (y \cdot z) * (x \cdot z).$$

So, we have $y \cdot z \leqslant x \cdot z$. Similarly, we have

$$0 = (x * y) \cdot ((y * z) \cdot (x * z)) = 0 \cdot ((y * z) \cdot (x * z)) = (y * z) \cdot (x * z)$$

and $y * z \leqslant z * x$.

On the other hand, if we put $z = y$ and $y = z$ in (pKU-1), we have

$$0 = (z \cdot x) * ((x \cdot y) * (z \cdot y))) = (z \cdot x) * ((0 * (z \cdot y)) = (z \cdot x) * (z \cdot y).$$

This means $z \cdot x \leqslant z \cdot y$. It can be similarly proved that it is $z * x \leqslant z * y$. □

2011, Mostafa, Naby and Yousef showed in [**10**], Lemma 2.6 that the following

$$(\forall x, y, z \in A)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$$

is valid in KU-algebras. In the following Proposition, we show that analogous equality is also valid in pseudo-KU algebras.

Proposition 4.2. *In pseudo-KU algebra $\mathfrak{A}$, the formula* (pKU) *is valid formula.*

Proof. On the one hand, if we put $y = 0$ in (pKU-1), we have

$$0 \cdot x \leqslant (x \cdot z) * (0 \cdot z).$$

So, we have $x \leqslant (x \cdot z) * z$. From here it follows

$$((x \cdot z) * z) \cdot (y * z) \leqslant x \cdot (y * z)$$

by (11). On the other hand, if we put $x = x \cdot z$ in (pKU-1), we get

$$y * (x \cdot z) \leqslant ((x \cdot z) * z) \cdot (y * z) \leqslant x \cdot (y * z).$$

Since the variables $x, y, z \in A$ are free variables, if we put $x = y$ and $y = x$, we get an inverse inequality. From here it follows (pKU) by (pKU-4).

The other equality can be proved in an analogous way. □

The following theorem is an important result of pseudo-KU algebras for study in the connections between pseudo-UP algebras and pseudo-KU algebras.

Theorem 4.1. *Any pseudo-KU algebra is a pseudo-UP algebra.*

Proof. It only needs to show (pUP-1). By Proposition 4.2, we have that any pseudo-KU algebra satisfies (pUP-1). □

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