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REACCELERATED OVER RELAXATION (ROR) METHOD

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Abstract. In this paper, a method named as 'Reaccelerated over relaxation (ROR) Method' for solving linear system of equations is introduced. The eigenvalues of its iteration matrix are obtained interms of eigenvalues of the Jacobi matrix and the choices of the parameters involved in ROR method are estimated. And also, this method is compared with the well known AOR, SOR, Gauss-Seidal and Jacobi methods through some numerical examples.

1. Introduction

For solving the linear system

$$AX = b \tag{1.1}$$

where A is non-singular matrix with non-vanishing diagonal elements of order n, X and b are unknown and known n-dimensional vectors. We split the coefficient matrix A without any loss of generality, as

$$A = I - L - U \tag{1.2}$$

where I is the unit matrix, $-L$ and $-U$ are strictly lower and upper triangular parts of A respectively ([**3, 4, 5**]).

The Accelerated over Relaxation (AOR) method for solving (1.1) is given by

$$(I - \omega L)X^{(n+1)} = \{(1-r)I + (r-\omega)L + rU\}X^{(n)} + rb \ (n = 0, 1, 2, 3, \cdots) . \tag{1.3}$$

As noted by A. Hadjidimos [**4**], the methods Successive over Relaxation (SOR), Gauss-Seidal (G.S) and Jacobi (J) can be realized from (1.3) for the choices of r and ω as

$$(r, \omega) = (\omega, \omega), (1, 1), (1, 0) \tag{1.4}$$

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The iteration matrices of the above methods are:

$$AOR : (I - \omega L)^{-1}\{(1-r)I + (r-\omega)L + rU\} \tag{1.5}$$

$$SOR : (I - \omega L)^{-1}\{(1-\omega)I + \omega U\} \tag{1.6}$$

$$G.S : (I - L)^{-1}U \tag{1.7}$$

$$J : (L + U) \tag{1.8}$$

and whose spectral radii are given by

(1.9)

$$S(AOR) = \begin{cases} \dfrac{\overline{\mu}^2}{(1+\sqrt{1-\overline{\mu}^2})^2} & \text{if } \underline{\mu} = 0 \ (or) \ \sqrt{1-\overline{\mu}^2} \leqslant 1 - \underline{\mu}^2 \text{ and } 0 < \underline{\mu} < \overline{\mu} \\ 0 & \text{if } \underline{\mu} = \overline{\mu} \\ \dfrac{\underline{\mu}\sqrt{\overline{\mu}^2 - \underline{\mu}^2}}{\sqrt{1-\underline{\mu}^2}(1+\sqrt{1-\overline{\mu}^2})} & \text{if } 0 < \underline{\mu} < \overline{\mu} \text{ and } 1 - \underline{\mu}^2 < \sqrt{1-\overline{\mu}^2} \end{cases}$$

$$S(SOR) = \frac{\overline{\mu}^2}{(1+\sqrt{1-\overline{\mu}^2})^2} \tag{1.10}$$

$$S(G.S) = \overline{\mu}^2 \tag{1.11}$$

$$S(J) = \overline{\mu} \tag{1.12}$$

where $\underline{\mu}$ and $\overline{\mu}$ are the smallest and the largest eigenvalues of Jacobian matrix J in magnitude respectively. It is to note that Reza Behzadi [**8**] studied a new class AOR preconditioned for L-Matrices.

We introduce the ROR method in section 2 and obtain the eigenvalues of its iteration matrix. The choices of the parameters are estimated in section 3 and ROR method is compared with other methods through some numerical examples in Section 4.

2. Reaccelerated over relaxation (ROR) method

Considering two parameters $r \neq 0$ and $\omega \neq 0$ and adding $r\omega(AX - b)$ which is absolutely zero, to the right hand side of AOR methods (1.3), we obtain

$$\begin{aligned} (I - \omega L)X^{(n+1)} &= \{(1-r)I + (r-\omega)L + rU\}X^{(n)} + rb + r\omega(AX^{(n)} - b) \\ &= \{(1-r)I + (r-\omega)L + rU)\}X^{(n)} + rb \\ &\qquad + r\omega[(I - L - U)X^{(n)} - b] \\ &= \{(1 - r + r\omega)I + (r - \omega - r\omega)L + (r - r\omega)U\}X^{(n)} \\ &\qquad + (r - r\omega)b. \end{aligned} \tag{2.1}$$

which is completely consistent one. This method can be called as Reaccelerated over relaxation (ROR) method for solving (1.1) and whose iteration matrix is

$$R_{r,\omega} = (I - \omega L)^{-1}\{(1 - r + r\omega)I + (r - \omega - r\omega)L + (r - r\omega)U\} \tag{2.2}$$

THEOREM 2.1. *If λ be the eigen value of the iteration matrix $R_{r,\omega}$ then the characteristic equation of $R_{r,\omega}$ is*

$$|[\lambda + r(1 - \omega) - 1]I - \{[\omega\lambda + r(1 - \omega) - \omega]L + [r(1 - \omega)]U\}| = 0$$

PROOF. Let
2.1.1

$$|R_{r,\omega} - \lambda I| = 0 \text{ (or) } |\lambda I - R_{r,\omega}| = 0$$

Then $|(\lambda I - (I - \omega L)^{-1}\{((1 - r + r\omega)I + (r - \omega - r\omega)L + (r - r\omega)U)\}| = 0$ $\Rightarrow |\lambda(I - \omega L) - \{(1 - r + r\omega)I + (r - \omega - r\omega)L + (r - r\omega)U\}| = 0$
2.1.2

$$|(\lambda + r(1 - \omega) - 1)I - \{(\lambda\,\omega + r(1 - \omega) - \omega)L + r(1 - \omega)U\}| = 0$$

is the characteristic equation of $R_{r,\omega}$ □

THEOREM 2.2. *If λ be the eigen value of $R_{r,\omega}$ and μ be the eigenvalue of Jacobi matrix J, then λ and μ are connected by the relation*

$$[\lambda + r(1 - \omega) - 1)]^2 = r\omega\mu^2(1 - \omega)\lambda - r\mu^2(1 - \omega) + r^2\mu^2(1 - \omega)^2$$

PROOF. Let λ be the eigenvalue of ROR iteration matrix $R_{r,\omega}$. Then, from Theorem 2.1, we have

$$\begin{aligned} &|\lambda I - (I - \omega L)^{-1}\{(1 - r + r\omega)I + (r - r\omega - \omega)L + (r - r\omega)U\}| = 0 \\ &\Rightarrow |\lambda(I - \omega L) - \{(1 - r + r\omega)I + (r - r\omega - \omega)L + (r - r\omega)U\}| = 0 \\ &\Rightarrow |(\lambda + r(1 - \omega) - 1)I - [(\omega(\lambda - 1) + r(1 - \omega))L + r(1 - \omega)U]| = 0. \end{aligned}$$

By using the theory given in G. Avdelas and A. Hadjidimos [**2**] and D. M. Young [**7**], we obtain

$$\frac{\lambda + r(1 - \omega) - 1}{(\omega(\lambda - 1) + r(1 - \omega))^{\frac{1}{2}} r^{\frac{1}{2}} (1 - \omega)^{\frac{1}{2}}} = \mu.$$

$$\Rightarrow [\lambda + r(1 - \omega) - 1)]^2 = r\omega\mu^2(1 - \omega)\lambda - r\mu^2(1 - \omega)\omega + r^2\mu^2(1 - \omega)^2. \tag{2.2.1}$$

□

THEOREM 2.3. *If λ be the eigenvalue of $R_{r,\omega}$ then*

$$\lambda = \frac{r\omega\mu^2(1 - \omega)}{2} - r(1 - \omega) + 1$$

provoided

$$\omega^2\mu^2 - 4\omega + 4 = 0$$

PROOF. From (2.2.1), we have

$$\lambda^2 - [\omega(r - r\omega)\mu^2 - 2(r - r\omega - 1)]\lambda + (r - r\omega - 1)^2 + \omega(r - r\omega)\mu^2 - (r - r\omega)^2\mu^2 = 0$$

$$\Rightarrow \lambda = \frac{r\omega\mu^2(1-\omega) - 2[r(1-\omega) - 1]}{2} \pm \frac{\sqrt{\Delta}}{2}, \tag{2.3.1}$$

where

$$\begin{aligned}\Delta &= \omega^2(r - r\omega)^2\mu^4 + 4(r - r\omega - 1)^2 - 4(r - r\omega - 1)\omega(r - r\omega)\mu^2 \\ &\quad - 4(r - r\omega - 1)^2 - 4\omega(r - r\omega)\mu^2 + 4(r - r\omega)^2\mu^2 \\ &= (r - r\omega)\mu^2[\omega^2(r - r\omega)\mu^2 - 4\omega(r - r\omega - 1 + 1) + 4(r - r\omega)] \\ &= (r - r\omega)^2\mu^2[\omega^2\mu^2 - 4\omega + 4],\end{aligned}$$

which will be zero if

$$\omega^2\mu^2 - 4\omega + 4 = 0 \tag{2.3.2}$$

i.e.,if

$$\omega = \frac{2}{1 + \sqrt{1 - \mu^2}} \tag{2.3.3}$$

for any μ. Therefore,λ of (2.3.1) becomes

$$\lambda = \frac{r\omega\mu^2(1-\omega)}{2} - r(1-\omega) + 1. \tag{2.3.4}$$

□

3. Choices of the Parameters r and ω

We rewrite λ of (2.3.4) as

$$\lambda = \frac{r\omega\mu^2(1-\omega)}{2} - [r(1-\omega) - 1] \tag{3.1}$$

The two terms appearing in square brackets in the right hand side play a major role in obtaining the minimum of the maximum$'\lambda'$ in magnitude with respect to the Parameters $'r'$ and $'\omega'$.

Let $'\omega'$ be fixed as

$$\omega = \omega^* = \frac{2}{1 + \sqrt{1 - \overline{\mu}^2}} \tag{3.2}$$

And now let the terms in (3.1) be connected by the relation

$$\frac{r\omega^*\mu^2(1-\omega^*)}{2} = \alpha[r(1-\omega^*) - 1] \tag{3.3}$$

where α be any real constant not zero.

Solving the equation (3.3) for $'r'$ in terms of $'\alpha$' and $'\alpha'$ in terms of $'r'$, we obtain

$$\alpha = \frac{r\omega^*\mu^2(1-\omega^*)}{2[r(1-\omega^*) - 1]} \tag{3.4}$$

$$r = \left(\frac{1}{\omega^* - 1}\right)\left(\frac{\alpha}{(1-\alpha) - \sqrt{1-\overline{\mu}^2}}\right) \tag{3.5}$$

From (3.4),we can have

$$\frac{\alpha}{\omega^* - 1} = \frac{-r\omega^*\mu^2}{2[r(1-\omega^*) - 1]} \tag{3.6}$$

From (3.5),we have

$$\frac{\alpha}{\omega^* - 1} = r[(1-\alpha) - \sqrt{1-\overline{\mu}^2}] \tag{3.7}$$

Equating (3.6) and (3.7) we get

$$r[(1-\alpha) - \sqrt{1-\mu^2}] = \left(\frac{-r}{[r(1-\omega^*) - 1]}\right)\left(\frac{\omega^*\mu^2}{2}\right)$$

(or)

$$[r(1-\omega^*) - 1)][\alpha - (1 - \sqrt{1-\mu^2}))] = \frac{\omega^*\mu^2}{2} \tag{3.8}$$

Now, on taking $\mu = \overline{\mu}$ and multiplying and dividing the R.H.S term by $[\omega + \frac{\overline{\mu}^2 - \underline{\mu}^2}{2}]$ we obtain

$$[r(1-\omega^*) - 1)][\alpha - (1 - \sqrt{1-\mu^2})] = \left[\omega^* + \frac{\overline{\mu}^2 - \underline{\mu}^2}{2}\right]\left[\frac{\omega^*\overline{\mu}^2}{2(\omega^* + \frac{\overline{\mu}^2 - \underline{\mu}^2}{2})}\right] \tag{3.9}$$

On equating the respective factors above, we obtain

$$r(1-\omega^*) - 1 = \omega^* + \frac{\overline{\mu}^2 - \underline{\mu}^2}{2} \tag{3.10}$$

$$\alpha - (1 - \sqrt{1-\overline{\mu}^2}) = \frac{\omega^*\overline{\mu}^2}{2\left(\omega^* + \frac{\overline{\mu}^2 - \underline{\mu}^2}{2}\right)} \tag{3.11}$$

From (3.10) and (3.11), we obtain $'r'$ and $'\alpha'$ as

$$r = \frac{1 + \omega^* + \frac{\overline{\mu}^2 - \underline{\mu}^2}{2}}{1 - \omega^*} \tag{3.12}$$

and

$$\alpha = \left(1 - \sqrt{1-\overline{\mu}^2}\right) + \left(\frac{\omega^*\overline{\mu}^2}{2\left(\omega^* + \frac{\overline{\mu}^2 - \underline{\mu}^2}{2}\right)}\right) \tag{3.13}$$

It is observed that r should be taken as $'r'$ as in (3.12) if $\alpha > 1$ and if $\alpha < 1$, then $'r'$ should be taken as $\frac{r}{2}$ of (3.12). When α=1, the choice of r is as given (3.5). Therefore, we summarize these results below.

Type 1: when $\underline{\mu} = \overline{\mu}$ and $\alpha = 1$.

Here, the choices of ω and r are to be taken as

$$\omega = \omega^* = \frac{2}{1+\sqrt{1-\overline{\mu}^2}} \tag{3.14}$$

$$r = r^* = \left(\frac{1}{\sqrt{1-\overline{\mu}^2}}\right)\left(\frac{1}{1-\omega^*}\right) \tag{3.15}$$

Type 2: when $\underline{\mu} \neq \overline{\mu}$ and $\alpha > 1$

In type, ω and r are to be considered as

$$\omega = \omega^* = \frac{2}{1+\sqrt{1-\overline{\mu}^2}} \tag{3.16}$$

$$r = r^* = \frac{1+\omega^* + \frac{\overline{\mu}^2-\underline{\mu}^2}{2}}{1-\omega^*} \tag{3.17}$$

Type 3: when $\underline{\mu} \neq \overline{\mu}$ and $\alpha < 1$

In this type,r and ω are

$$\omega = \omega^* = \frac{2}{1+\sqrt{1-\overline{\mu}^2}} \tag{3.18}$$

$$r = r^* = \frac{1+\omega^* + \frac{\overline{\mu}^2-\underline{\mu}^2}{2}}{2(1-\omega^*)} \tag{3.19}$$

It is to note that $\underline{\mu}$ can also zero in all the above types.

4. Numerical Examples

Let unit vector be a solution of the linear system

$$AX = b \tag{4.1}$$

for a given matrix A and known vector b. The methods discussed in this paper are applied to obtain the solution of (4.1) correct to 9 decimal places taking null vector as an initial guess considering various $A's$ and $b's$ and tabulated the results such as choices of the parameters, number of iterations conceded by each method, spectral radii of the iteration matrices and the error

$$E = \sqrt{\sum_{i=1}^{r} |1 - x_r|}.$$

We now start with the problems satisfied the Type 1 criteria.

Example 4.1. If

$$A = \begin{bmatrix} 1 & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{18}{5} & 6 \\ \frac{24}{5} & \frac{1}{5} & 1 & 0 \\ \frac{12}{5} & \frac{1}{5} & 0 & 1 \end{bmatrix} \text{ and } b = \begin{bmatrix} 1.4 \\ 3.4 \\ 6 \\ 3.6 \end{bmatrix},$$

then the eigenvalues of Jacobian matrix J are $\pm\frac{\sqrt{24}}{5}$ with a multiplicity two. Therefore,

$$\underline{\mu} = \frac{\sqrt{24}}{5} = \overline{\mu}.$$

This matrix is considered by G. Avdelas and A. Hadjidimos [**2**].

Table 1

S.No.	Method	Choices of paramers	Number of iterations	Spectral radius	error
1	AOR	$\omega = 5/3$,$r = 5$	4	0	0.18×10^{-11}
2	ROR	$\omega = 5/3$,$r = -7.5$	4	0	0.14×10^{-11}
3	SOR	$\omega = 5/3$,$r = 5/3$	92	2/3	0.48×10^{-9}
4	G.S.	-	713	0.96	0.486×10^{-9}
5	J	-	1431	0.9798	0.496×10^{-9}

Example 4.2. If $A = \begin{bmatrix} 3 & -4 \\ 2 & -3 \end{bmatrix}$ considered by A. Hadjidimos [**4**] and if $b = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ of (4.1) then the Jacobian matrix eigen values are given by $\pm\frac{2\sqrt{2}}{3}$ and hence

$$\underline{\mu} = \frac{2\sqrt{2}}{3} = \overline{\mu}.$$

Table 2

S.No.	Method	Choices of paramers	Number of iterations	Spectral radius	error
1	AOR	ω=1.5,$r = 3$	2	0	0
2	ROR	$\omega = 1.5$,$r = -6$	2	0	0
3	SOR	$\omega = 1.5$,$r = 1.5$	38	0.5	0.31×10^{-9}
4	G.S.	-	187	0.88889	0.49×10^{-9}
5	J	-	370	0.94281	0.49×10^{-9}

EXAMPLE 4.3. If

$$A = \begin{bmatrix} 1 & 0 & \frac{1}{5} & \frac{1}{5} \\ 0 & 1 & -\frac{71}{10} & \frac{113}{10} \\ \frac{16}{5} & \frac{1}{5} & 1 & 0 \\ 2 & \frac{1}{5} & 0 & 1 \end{bmatrix}$$

which is considered by G. Avdelas and Hadjidimos [**2**] and $b = \begin{bmatrix} 1.4 \\ 5.2 \\ 4.4 \\ 3.2 \end{bmatrix}$ then the eigenvalues of the Jacobi matrix are $\pm\frac{\sqrt{23}}{5}$ and $\pm\frac{\sqrt{24}}{5}$ and hence

$$\underline{\mu} = \frac{\sqrt{23}}{5}, \ \overline{\mu} = \frac{\sqrt{24}}{5}/$$

It can be seen that the conditions given by G.Avdelas and A. Hadjidimos [**2**] i.e.,

$$0 < \underline{\mu} < \overline{\mu} \text{ and } 1 - \underline{\mu}^2 < \sqrt{1 - \overline{\mu}^2}$$

are satisfied. As given by them, the spectral radius of AOR matrix calculated from the formula (1.9) is $\frac{\sqrt{46}}{12}$ and mentioned that which is less than the spectral radius of SOR i.e.,$\frac{2}{3}$.

It is calculated that the AOR iteration matrix $M_{r,\omega}$ is

$$M_{r,\omega} = \begin{bmatrix} \frac{9}{4} & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{9}{4} & -\frac{71}{8} & \frac{113}{8} \\ -\frac{8}{3} & -\frac{1}{6} & \frac{93}{24} & -\frac{145}{24} \\ -\frac{5}{3} & -\frac{1}{6} & \frac{17}{8} & -\frac{79}{24} \end{bmatrix}$$

whose spectral radius happens to be $\frac{\sqrt{246}}{12}$ but not $\frac{\sqrt{46}}{12}$ as mentioned by Avdelas and A. Hadjidimos [**2**] and G. Avdelas and A. Hadjimos [**7**].

TABLE 3

S.No.	Method	Choices of paramers	Number of iterations	Spectral radius	error
1	AOR	ω=5/3,$r = -5/4$	Diverging	1.30703262	-
2	AOR	ω=5/3,$r = 14/3$	91	0.75129518	0.184×10^{-9}
3	ROR	$\omega = 5/3$,$r = -4.03$	44	0.56892569	0.426×10^{-9}
4	SOR	$\omega = 5/3$,$r = 5/3$	76	2/3	0.348×10^{-9}
5	G.S.	-	614	0.96	0.491×10^{-9}
6	J	-	1203	0.97979590	0.4902×10^{-9}

It is to note that AOR method converged for different choices of the parameters as shown in S. No. 2 in Table 3 given down.

EXAMPLE 4.4. If

$$A = \begin{bmatrix} 35 & -2 & -3 & -1 & 0 & -2 & -3 & -1 \\ -5 & 27 & -3 & -4 & -4 & -1 & -2 & 0 \\ -7 & -4 & 71 & -9 & -2 & -6 & 0 & -3 \\ -1 & -1 & -2 & 20 & -4 & -3 & -2 & -4 \\ -2 & -2 & -3 & 0 & 71 & -2 & -1 & -1 \\ 0 & -2 & -5 & -4 & -3 & 53 & -5 & -4 \\ -3 & -2 & -1 & -3 & -5 & -4 & 32 & -3 \\ -4 & -3 & -2 & -5 & -1 & -2 & -7 & 31 \end{bmatrix} \text{ and } b = \begin{bmatrix} 23 \\ 8 \\ 40 \\ 3 \\ 60 \\ 30 \\ 11 \\ 7 \end{bmatrix}$$

Then, we have $\underline{\mu} = 0.050245255$ and $\overline{\mu} = 0.533454460$.

In this case,

$$1 - \underline{\mu}^2 \not< \sqrt{1 - \overline{\mu}^2}$$

and hence the rate of convergence of AOR will be same as that of SOR method. The choices of ω and r are estimated using (3.17) and (3.18) as α in this case happens to be less than one.

TABLE 4

S.No.	Method	Choices of paramers	Number of iterations	Spectral radius	error
1	AOR	$\omega = 1.08352411$, $r = 1.08352411$	15	0.08352411	0.442×10^{-9}
2	ROR	$\omega = 1.08352411$, $r = -13.316805$	14	0.059206388	0.491×10^{-9}
3	SOR	$\omega = 1.08352411$, $r = 1.08352411$	15	0.08352411	0.442×10^{-9}
4	G.S.	–	21	0.284573661	0.184×10^{-9}
5	J	–	36	0.533454460	0.435×10^{-9}

EXAMPLE 4.5. If

$$A = \begin{bmatrix} 4 & 0 & 0 & 0 & -1 & -1 \\ 0 & 4 & 0 & 0 & -1 & -1 \\ 0 & 0 & 4 & -1 & -1 & 0 \\ 0 & 0 & -1 & 4 & 0 & 0 \\ -1 & -1 & -1 & 0 & 4 & 0 \\ -1 & -1 & 0 & 0 & 0 & 4 \end{bmatrix} \text{ and } b = (2, 2, 2, 3, 1, 2)^T.$$

Considered by I. K. Youssef and A. A. Taha [**8**] of (4.1), then

$$\underline{\mu} = 0 \text{ and } \overline{\mu} = 0.543831937 \text{ and } 1 - \underline{\mu}^2 \not< \sqrt{1 - \overline{\mu}^2}.$$

TABLE 5

S.No.	Method	Choices of paramers	Number of iterations	Spectral radius	error
1	AOR	$\omega = 1.08743277$, $r = 1.08743277$	11	0.087432775	0.194×10^{-9}
2	ROR	$\omega = 1.08743277$, $r = -12.7830174$	10	0.062070778	0.392×10^{-9}
3	SOR	$\omega = 1.08743277$, $r = 1.08743277$	11	0.087432775	0.194×10^{-9}
4	G.S.	–	19	0.295753176	0.293×10^{-9}
5	J	–	37	0.543831937	0.3799×10^{-9}

Conclusion: It can be clearly seen from all the examples considered that the rate of convergence of ROR method is a bit faster compared to the other methods discussed in this paper in all the cases irrespective of the values of $\underline{\mu}$ and $\overline{\mu}$.

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