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Γ-INCLINES

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Abstract. In this paper, we introduce the notion of complemented $\Gamma-$incline. In addition,we study the properties of structures of the semigroup $(M,+)$ and the $\Gamma-$semigroup M of a $\Gamma-$incline M and a complemented $\Gamma-$incline M.

1. Introduction

In 1995, M.M. Krishna Rao [**14, 15, 17, 18**] introduced the notion of Γ-semiring as a generalization of Γ-ring, ternary semiring and semiring. As a generalization of ring, the notion of a Γ-ring was introduced by Nobusawa [**13**] in 1964. In 1981, Sen [**34**] introduced the notion of a Γ-semigroup as a generalization of semigroup. The notion of a ternary algebraic system was introduced by Lehmer [**10**] in 1932. Dutta and Sardar introduced the notion of operator semirings of $\Gamma-$semiring. M. M. Krishna Rao and B. Venkateswarlu [**26, 27, 28**] introduced the notion of $\Gamma-$incline, generalized right derivation of $\Gamma-$ incline and right derivation of ordered $\Gamma-$semiring. The set of all negative integers $\mathbb{Z}$ is not a semiring with respect to usual addition and multiplication but $\mathbb{Z}$ forms a $\Gamma-$semiring where $\Gamma = \mathbb{Z}$. The important reason for the development of Γ-semiring is a generalization of results of rings, Γ-rings, semirings, semigroups and ternary semirings. The notion of a semiring was introduced by H. S. Vandiver [**36**] in 1934, but semirings had appeared in earlier studies on the theory of ideals of rings. Semirings play an important role in studying matrices and determinants. Semirings are useful in the areas of theoretical computer science as well as in the solution of graph theory, optimization theory, in particular for studying automata, coding theory and formal languages. Semiring theory has many applications in other branches of mathematics. The notion of bi-ideals in rings and semigroups were introduced by S. Lajos and F. A. Szasz [**8, 9**].

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Bi-ideal is a special case of $(m-n)$ ideal. In 1976, the concept of interior ideals was introduced by S. Lajos [**7**] for semigroups.

M. Satyanarayana [**13**] studied the additive semigroup of ordered semirings. J. Hanumanthachari and K. Venuraju [**5**] studied the additive semigroup structure of semiring. T. Vasanthi et al. [**35, 36**] studied semiring satisfying the identity and properties of semirings. The concept of incline was first introduced by Z. Q. Cao in 1984. Z. Q. Cao et al. [**4**] studied the incline and its applications. K. H. Kim and F. W. Rowsh [**6**] have studied matrices over an incline. Inclines are additively idempotent semiring in which products are less than or equal to either factor. Recently idempotent semirings and Kleene Algebras have been established as fundamental structures in computer sciences. Ahn et al. [**1, 2**] studied ideals and r-ideals in inclines. A. R. Meenakshi and S. Anbalagan [**11**] studied regular elements in an incline and proved that regular commutative incline is a distributive lattice. An incline is a more general algebraic structure than a distributive lattice. In an incline every ideal is a k-ideal. A. R. Meenakshi and N. Jayabalan [**12**] have proved that every prime ideal in an incline M is irreducible ideal of M and every maximal ideal in an incline M is an irreducible ideal. W. Yao and S. C. Han [**37**] studied the relations between ideals, filters and congruences in inclines and it is shown that there is a one to one correspondence between the set of ideals and the set of all regular congruences. M. M. Krishna Rao et al. [**16, 19, 20, 21, 22, 23, 24, 25, 29, 30, 31, 32**] studied ideals, derivations and properties of Γ-incline and Γ-semiring.

In this paper, we study the properties of additive semigroup structure and Γ-semigroup structure of a Γ-incline and a complemented Γ-incline and study their properties.

2. Preliminaries

In this section, we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

Definition 2.1 ([**1**]). *A set M together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively will be called semiring provided*

(i) *addition is a commutative operation.*
(ii) *multiplication distributes over addition both from the left and from the right.*
(iii) *there exists $0 \in M$ such that $x+0=x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in M$.*

Definition 2.2. *Let $(M,+)$ and $(\Gamma,+)$ be commutative semigroups. Then we call M a Γ-semiring, if there exists a mapping $M \times \Gamma \times M \to M$ (images of (x,α,y) will be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) such that it satisfies the following axioms for all $x,y,z \in M$ and $\alpha,\beta \in \Gamma$*

(i) $x\alpha(y+z) = x\alpha y + x\alpha z$
(ii) $(x+y)\alpha z = x\alpha z + y\alpha z$
(iii) $x(\alpha+\beta)y = x\alpha y + x\beta y$

(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Definition 2.3. *Let M and Γ be non-empty sets. Then we call M a Γ- semigroup, if there exists a mapping $M \times \Gamma \times M \to M$ (images of (x, α, y) will be denoted by $x\alpha y, x, y \in M, \alpha \in \Gamma$) such that it satisfies $x\alpha(y\beta z) = (x\alpha y)\beta z$. for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$.*

Definition 2.4. *Let M be a Γ- semigroup. M is is said to be left (right) singular if $a\alpha b = a$,($a\alpha b = b$,) for some $\alpha \in \Gamma$.*

Definition 2.5. *A Γ-semiring M is said to be commutative Γ-semiring if $x\alpha y = y\alpha x$, for all $x, y \in M$ and $\alpha \in \Gamma$.*

Definition 2.6. *Let M be a Γ-semiring. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.*

Definition 2.7. *In a Γ-semiring M with unity 1, an element $a \in M$ is said to be left invertible (right invertible) if there exist $b \in M, \alpha \in \Gamma$ such that $b\alpha a = 1(a\alpha b = 1)$.*

Definition 2.8. *In a Γ-semiring M with unity 1, an element $a \in M$ is said to be invertible if there exist $b \in M, \alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 1$.*

Definition 2.9. *A Γ-semiring M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x$ and $0\alpha x = x\alpha 0 = 0$, for all $x \in M, \alpha \in \Gamma$.*

Definition 2.10. *An element a in a Γ-semiring M is said to be idempotent if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$.*

Definition 2.11. *Every element of M, is an idempotent of M then M is said to be idempotent Γ-semiring M.*

Definition 2.12. *A Γ-semiring M is called a division Γ-semiring if for each non-zero element of M has multiplication inverse.*

Definition 2.13. *A non-empty subset A of a Γ-semiring M is called*

(i) *a Γ-subsemiring of M if $(A, +)$ is a subsemigroup of $(M, +)$ and $A\Gamma A \subseteq A$.*
(ii) *a quasi ideal of M if A is a Γ-subsemiring of M and $A\Gamma M \cap M\Gamma A \subseteq A$.*
(iii) *a bi-ideal of M if A is a Γ-subsemiring of M and $A\Gamma M\Gamma A \subseteq A$.*
(iv) *an interior ideal of M if A is a Γ-subsemiring of M and $M\Gamma A\Gamma M \subseteq A$.*
(v) *a left (right) ideal of M if A is a Γ-subsemiring of M and $M\Gamma A \subseteq A$ ($A\Gamma M \subseteq A$).*
(vi) *an ideal if A is a Γ-subsemiring of $M, A\Gamma M \subseteq A$ and $M\Gamma A \subseteq A$.*
(vii) *a $k-$ideal if A is a Γ-subsemiring of $M, A\Gamma M \subseteq A, M\Gamma A \subseteq A$ and $x \in M$, $x + y \in A, y \in A$ then $x \in A$.*
(viii) *a bi-interior ideal of M if A is a Γ-subsemiring of M and*

$$M\Gamma B\Gamma M \cap B\Gamma M\Gamma B \subseteq B.$$

(ix) *a left bi-quasi ideal (right bi-quasi ideal) of M if A is a subsemigroup of* $(M, +)$ *and* $M\Gamma A \cap A\Gamma M\Gamma A \subseteq A$ $(A\Gamma M \cap A\Gamma M\Gamma A \subseteq A)$.

(x) *a bi-quasi ideal of M if B is a $\Gamma-$subsemiring of M and $B$is a left bi-quasi ideal and a right bi-quasi ideal of M.*

(xi) *a left quasi-interior ideal (right quasi-interior ideal) of M if A is a Γ-subsemiring of M and* $M\Gamma A\Gamma M\Gamma A \subseteq A$ $(A\Gamma M\Gamma A\Gamma M \subseteq A)$.

(xii) *a quasi-interior of M if B is a $\Gamma-$subsemiring of M and $B$is a left quasi-interior ideal and a right quasi-interior ideal of M.*

(xiii) *a bi-quasi-interior ideal of M if A is a Γ-subsemiring of M and*

$$B\Gamma M\Gamma B\Gamma M\Gamma B \subseteq B.$$

(xiv) *a left tri- ideal (right tri- ideal) of M if A is a Γ-subsemiring of M and* $A\Gamma M\Gamma A\Gamma A \subseteq A$ $(A\Gamma A\Gamma M\Gamma A \subseteq A)$.

(xv) *a tri- ideal of M if A is a Γ-subsemiring of M and* $A\Gamma M\Gamma A\Gamma A \subseteq A$ *and* $A\Gamma A\Gamma M\Gamma A \subseteq A$.

(xvi) *a left(right) weak-interior ideal of M if B is a $\Gamma-$subsemiring of M and* $M\Gamma B\Gamma B \subseteq B$ $(B\Gamma B\Gamma M \subseteq B)$.

(xvii) *a weak-interior ideal of M if B is a $\Gamma-$subsemiring of M and $B$is a left weak-interior ideal and a right weak-interior ideal)of M.*

Definition 2.14 ([**4**]). *A commutative incline M with additive identity* 0 *and multiplicative identity* 1 *is a non-empty set M with operations addition* $(+)$ *and multiplication* $(.)$ *defined on* $M \times M \to M$ *such that satisfying the following conditions for all* $x, y, z \in M$

(i) $x + y = y + x$
(ii) $x + x = x$
(iii) $x + xy = x$
(iv) $y + xy = y$
(v) $x + (y + z) = (x + y) + z$
(vi) $x(yz) = x(yz)$
(vii) $x(y + z) = xy + xz$
(viii) $(x + y)z = xz + yz$
(ix) $x1 = 1x = x$
(x) $x + 0 = 0 + x = x$
(xi) $xy = yx$

Definition 2.15. *Let* $(M, +)$ *and* $(\Gamma, +)$ *be commutative semigroups. If there exists a mapping* $M \times \Gamma \times M \to M\big((x, \alpha, y) \to x\alpha y\big)$ *such that it satisfies the following axioms for all* $x, y, z \in M$ *and* $\alpha, \beta \in \Gamma$

(i) $x\alpha(y + z) = x\alpha y + x\alpha z$
(ii) $(x + y)\alpha z = x\alpha z + y\alpha z$
(iii) $x(\alpha + \beta)y = x\alpha y + x\beta y$
(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$
(v) $x + x = x$

(vi) $x + x\alpha y = x$
(vii) $y + x\alpha y = y$.

Then M is called a Γ- incline.

Every incline M is a Γ-incline with $\Gamma = M$ and ternary operation $x\gamma y$ as the usual incline multiplication. In an Γ-incline define the order relation such that for all $x, y \in M, y \leqslant x$ if and only if $y + x = x$. Obviously $\leqslant$ is a partially order relation over M.

Definition 2.16. *In a Γ-incline M*

(i) *semigroup $(M, +)$ is said to be positively ordered (negatively ordered) if $a + b \geqslant a$, $b(a + b \leqslant a, b)$ for all $\alpha \in \Gamma$, $a, b \in M$.*
(ii) *Γ-semigroup M is said to be positively ordered (negatively ordered) if $a\alpha b \geqslant a$, $b(a\alpha b \leqslant a, b)$ for all $\alpha \in \Gamma$, $a, b \in M$.*

Definition 2.17. *A Γ-incline M is said to have zero element if there exists an element $0 \in M$ such that $0 + x = x = x + 0$ and $0\alpha x = x\alpha 0 = 0$ for all $\alpha \in \Gamma$, $x \in M$.*

Definition 2.18. *A Γ- incline M is said to be commutative Γ- incline if $x\alpha y = y\alpha x$ for all $x, y \in M$ for all $\alpha \in \Gamma$, $x, y \in M$.*

Definition 2.19. *A Γ-subincline I of a Γ-incline M is a non-empty subset of M which is closed under the Γ-incline operations addition and ternary operation.*

Definition 2.20. *Let M be a Γ-incline. An element $1 \in M$ is said to be unity if for each $x \in M$ there exists $\alpha \in \Gamma$ such that $x\alpha 1 = 1\alpha x = x$.*

Definition 2.21. *In a Γ-incline M with unity 1, an element $a \in M$ is said to be invertible if there exist $b \in M$, $\alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 1$.*

Definition 2.22. *A non zero element a in a Γ-incline M is said to be a zero divisor if there exist a non zero element $b \in M$, $\alpha \in \Gamma$ such that $a\alpha b = b\alpha a = 0$.*

Definition 2.23. *A Γ-incline M with zero element 0 is said to be hold cancellation laws if $a \neq 0$, $a\alpha b = a\alpha c$, $b\alpha a = c\alpha a$, where $a, b, c \in M$, $\alpha \in \Gamma$ then $b = c$.*

3. Γ-inclines

In this section, we introduce the notion of complemented Γ-incline and study the properties of additive semigroup structure and Γ-semigroup structure of Γ-incline and complemented Γ-incline and study their properties.

Definition 3.1. *An element a of a Γ-incline M is said to be complemented if there exists an element $b \in M$ such that $a + b = 1$, $b\alpha a = 0$ and $a\alpha b = 0$, for all $\alpha \in \Gamma$.*

Definition 3.2. *Let M be a Γ-incline. An element $a \in M$ is said to be idempotent of M if there exists $\alpha \in \Gamma$ such that $a = a\alpha a$ and a is also said to be α idempotent.*

Definition 3.3. *Let M be a Γ-incline. Every element of M is an idempotent of M then M is said to be idempotent Γ-incline M.*

Definition 3.4. *Let M be a Γ-incline. If $x \leqslant y$ for all $y \in M$ then x is called the least element of M and denoted as 0. If $x \geqslant y$ for all $y \in M$ then x is called the greatest element of M and denoted as 1.*

Definition 3.5. *A Γ-incline M is said to be linearly ordered if $x, y \in M$ then either $x \leqslant y$ or $y \leqslant x$, where $\leqslant$ is an incline order relation.*

Definition 3.6. *A Γ-incline M is said to be mono Γ-incline if for every pair $a, b \in M$ there exists $\gamma \in \Gamma$ such that $a\gamma b = a + b$.*

Definition 3.7. *A Γ-incline M is said to be complemented if for every element of M is complemented.*

Theorem 3.1. *Let M be a Γ-incline. Then Γ- semigroup M is non-positively ordered.*

Proof. Let M be a Γ-incline. We have $a\alpha b + a = a$, for all $a \in M$, $\alpha \in \Gamma$. Now $(a\alpha b) \leqslant a$. Therefore, Γ-semigroup M is non-positively ordered. □

Theorem 3.2. *Let M be a Γ-incline. Then additive semigroup $(M, +)$ is positively ordered.*

Proof. Let M be a Γ-incline with unity 1 and $x, y \in M$. Then $x\alpha y \leqslant x$, for all $\alpha \in \Gamma$. Thus $x\alpha y + y \leqslant x + y$, for all $\alpha \in \Gamma$ and $y \leqslant x + y$. Hence additive semigroup $(M, +)$ is positively ordered. □

Theorem 3.3. *Let M be a Γ- incline. Then 1 is the maximal element.*

Proof. Let M be a Γ-incline. Then semigroup $(M, +)$ is positively ordered of Γ-incline. Thus $1 + 1\alpha a = 1$ for all $a \in M$, $\alpha \in \Gamma$ and $1 + a = 1$.. Hence $1 = 1 + a \geqslant a$, for all $a \in M$. Hence 1 is the maximal element. □

Theorem 3.4. *Let M be a Γ-incline. Then 1 is the unity.*

Proof. Suppose M is a Γ-incline. Then $a + 1 = 1$ for all $a \in M$ and there exists $\alpha \in \Gamma$ such that $a\alpha a = a$. We have $a + 1 = 1$ and $a\alpha(a + 1) = a\alpha 1$ and $a\alpha a + a\alpha 1 = a\alpha 1$. Thus $a + a\alpha 1 = a\alpha 1$ and $a = a\alpha 1$. Therefore $a = a\alpha 1$. Similarly, we can prove $1\alpha a = a$. Hence 1 is the unity element of M. □

Theorem 3.5. *Let M be a linearly ordered Γ- incline. Then there exists $\alpha \in \Gamma$ such that $a = a\alpha b = b\alpha a = a\alpha a$.*

Proof. Let $a, b \in M$ and $a \leqslant b$. Since M is an idempotent Γ-incline, there exists $\alpha \in \Gamma$ such that $a\alpha a = a$. Then $a \leqslant b \Rightarrow a + b = b$ and $a\alpha(a + b) = a\alpha b$. Now $a\alpha a + a\alpha b = a\alpha b$ and $a + a\alpha b = a\alpha b$. So, $a = a\alpha b$ and $a \leqslant b \Rightarrow a\alpha a \leqslant b\alpha a$. Thus $a \leqslant b\alpha a \leqslant a$ and finally $a = b\alpha a$. Hence $a = a\alpha b = b\alpha a = a\alpha a$. □

Theorem 3.6. *Let M be a Γ-incline. Then 0 is the minimal element of M.*

PROOF. Let M be a Γ-incline. Now $a + 0 = a \Rightarrow (a + 0) \geqslant 0$ and $a \geqslant 0$. Therefore 0 is the minimal element. Hence the Theorem. □

THEOREM 3.7. *If M is a Γ-incline and $(M, +)$ is right cancellative then $|M| = 1$.*

PROOF. Let $a \in M$. Then $a + 1 = 1$ and $a + 1 = 1 + 1$. Thus $a = 1$. Hence $|M| = 1$. □

THEOREM 3.8. *Let M be a Γ-incline and for every pair $a, b \in M$, there exists $\gamma \in \Gamma$ such that $1\gamma b = b$ and $a\gamma b = a$. Then Γ-incline M is linearly ordered.*

PROOF. Let M be a Γ-incline and $a, b \in M$. Then $a + 1 = 1$ for all $a \in M$. Since $a, b \in M$ there exists $\gamma \in \Gamma$ such that $1\gamma b = b$ and $a\gamma b = a$. Now

$$a + b = a + 1\gamma b = a\gamma b + 1\gamma b = (a + 1)\gamma b = 1\gamma b = b.$$

Then $a \leqslant b$. Hence the theorem. □

THEOREM 3.9. *Let M be a Γ- incline. If $a \leqslant b$, then $a + a\gamma b + b = b$, for all $a, b \in M$, for some $\gamma \in \Gamma$.*

PROOF. Let M be a Γ-incline and $a, b \in M$. Then $a + 1 = 1$. for all $a \in M$. Since $b \in M$ there exists $\gamma \in \Gamma$ such that $1\gamma b = b$. Now

$$a + a\gamma b + b = a + a\gamma b + 1\gamma b = a + (a + 1)\gamma b = a + 1\gamma b = a + b = b.$$

Hence $a + a\gamma b + b = b$, for all $a, b \in M$. □

THEOREM 3.10. *Let M be a linearly ordered mono Γ-incline. Then Γ-semigroup M of a Γ-incline M is a left singular.*

PROOF. Let M be a linearly ordered mono Γ-incline. Suppose $a, b \in M$ and $b \leqslant a$. Then there exists $\gamma \in \Gamma$ such that $a\gamma b = a + b$. Now $a + b = a$, and $a\gamma b = a$. Hence Γ-semigroup M of the Γ-incline M is a left singular. □

THEOREM 3.11. *Let M be a linearly ordered Γ-incline and $a, b \in M$, $a \leqslant b$. Then there exists $\gamma \in \Gamma$ such that $a\gamma a = a$ and $a\gamma b = a$. Then M is mono Γ-incline.*

PROOF. Let M be a linearly ordered Γ-incline and $a, b \in M$. Then there exists $\gamma \in \Gamma$ such that $a\gamma b = b$ and $a\gamma a = a$. Thus $a+b = a\gamma a + a\gamma b$ and $a+b = a\gamma(a+b)$. Now $a + b = a\gamma b$. Hence the theorem. □

THEOREM 3.12. *Let M be a Γ-incline. If b is a complement of a and $c \in M$ $a + c = 1$ then there exists $\alpha \in \Gamma$ such that $b\alpha c = c\alpha b = b\alpha b = b$.*

PROOF. Let b be the complement of a and $c \in M$ such that $a + c = 1$. Then there exists a $\alpha \in \Gamma$ such that $b = b\alpha 1 = 1\alpha b$, $b + a = 1$ and $b\gamma a = a\gamma b = 0$, for all $\gamma \in \Gamma$. Thus

$$b = b\alpha(1) = b\alpha(a + c) = b\alpha a + b\alpha c = 0 + b\alpha c = b\alpha c.$$

$$b = 1\alpha b = (a + c)\alpha b = a\alpha b + c\alpha b = 0 + c\alpha b = c\alpha b.$$

and $b + a = 1$. Now, $b\gamma(b + a) = b\gamma 1$, for all $\gamma \in \Gamma$. From here it follows $b\gamma b + b\gamma a = b\gamma 1$, for all $\gamma \in \Gamma$ and $b\alpha b + b\alpha a = b\alpha 1$. Hence $b\alpha b + 0 = b\alpha 1$. Therefore $b\alpha b = b$. Hence $b\alpha c = c\alpha b = b\alpha b = b$. □

Theorem 3.13. *Let M be a zero sum free Γ-incline. If $a, b \in M$ are complemented elements of M, then $a\alpha b\beta c = 0$, for all $\beta \in \Gamma$, for some $\alpha \in \Gamma$, where c is the complement of a.*

Proof. Let M be a zero sum free Γ-incline, c, d be complements of a and b respectively and $\beta \in \Gamma$. Since 1 is the unity, there exists $\alpha \in \Gamma$ such that $a\alpha 1 = a$. Then

$$a\alpha b\beta c + a\alpha d\beta c = a\alpha(b + d)\beta c = a\alpha 1\beta c = a\beta c = 0.$$

Hence $a\alpha b\beta c = 0$. □

Theorem 3.14. *If Γ- semigroup M of a complemented Γ- incline M holds cancellation law then every element in M has a unique complement.*

Proof. Let Γ-semigroup M of the complemented Γ-incline M holds cancellation law. Suppose b and c are complements of a then $a + b = 1$, $a + c = 1$, $a\alpha b = 0$, $a\alpha c = 0$, for all $\alpha \in \Gamma$. Then $a\alpha b = 0 = a\alpha c$ and $a\alpha b = a\alpha c$. Thus $b = c$. Hence the Theorem. □

Theorem 3.15. *If M is a complemented Γ-incline then M is an idempotent Γ-incline.*

Proof. Suppose M is a complemented Γ-incline and $a \in M$. Then there exists $\alpha \in \Gamma$ such that $a\alpha 1 = a$, $a + b = 1$ and $a\gamma b = 0$, for all $\gamma \in \Gamma$. Then $a + b = 1$. Thus $a\gamma(a + b) = a\gamma 1$, for all $\gamma \in \Gamma$. Now, we have $a\alpha(a + b) = a\alpha 1$ and $a\alpha a + a\alpha b = a\alpha 1$. So, it follows $a\alpha a + 0 = a\alpha 1$ and finally $a\alpha a = a$. Therefore a is an $\alpha-$idempotent. Hence M is an idempotent Γ-incline. □

4. Conclusion:

We studied the properties of structures of the semigroup $(M, +)$ and the Γ-semigroup M of Γ-incline M, and complemented Γ-incline M. We conclude that structures of the semigroup $(M, +)$ and the Γ-semigroup M of Γ-incline M are dependent.

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