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GENERALIZED RICCI SOLITONS ON (ε, δ)-TRANS SASAKIAN MANIFOLD

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ABSTRACT. The purpose of the present research is to shows that a (ε, δ) trans-Sasakian manifold, which also satisfies the Ricci soliton and generalized Ricci soliton equation, satisfying some conditions, is necessarily the Einstein manifold. Generalized Ricci solitons for 3-dimensional (ε, δ)-trans-Sasakian manifolds are worked out. Also an example of Ricci solitons in 3-dimensional (ε, δ)-trans-Sasakian manifold is provided in the region where trans-Sasakian manifold is expanding (shrinking) the Lorentzian trans-Sasakian manifold is shrinking (expanding).

1. Introduction

In [**3**], Bejancu-Duggal introduced (ε)-Sasakian manifolds. Later, these manifolds were studied by Xufeng and Xiaoli [**21**] from real hypersurfaces of indefinite Kahlerian manifolds. Kumar et al. [**9**] studied the curvature conditions of these manifolds. De and Sarkar [**7**] also introduced (ε)- Kenmotsu manifolds with indefinite metric. The notion of (ε)- trans-Sasakian manifolds with indefinite metric, which are natural generalization of both (ε)-Sasakian and (ε)-Kenmotsu manifolds was introduced by Shukla and Sing [**16**]. Nagaraja et. al. [**12**] and authors Rahman et. al. [**14**] studied (ε, δ)-trans-Sasakian manifolds and CR submanifolds of nearly (ε, δ)-trans-Sasakian manifolds, which are extensions of (ε)-trans-Sasakian manifolds.

There are stationary points of the Ricci flow given by

$$\frac{\partial g}{\partial t} = -2Ric(g), \qquad \text{for } g(0) = g_0. \tag{1.1}$$

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Ricci solitons move under the Ricci flow initiated by Hamilton [**8**] simply by diffeomorphisms of the initial metric. A Ricci soliton (g, V, λ) on a Riemannian manifold is defined by

$$\mathcal{L}_V g + 2S + 2\lambda g = 0, \tag{1.2}$$

where S is the Ricci tensor, $\mathcal{L}_V$ is the Lie derivative along the vector field V on M and λ is a real scalar. Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$. If the vector field V is the gradient of a potential function $-\psi$, then g is called a gradient Ricci soliton and equation (1.2) assumes the form $Hess\psi = S + \lambda g$.

A metric g_0 on a smooth manifold M is a Ricci soliton if there exist a function $\sigma(t)$ and a family of diffeomorphisms $\{\eta(t)\} \subset Diff(M)$ such that

$$g(t) = \sigma(t)\eta(t)^* g_0,$$

is a solution of the Ricci flow. In this expression, $\eta(t)^* g_0$ indicates to pullback of the metric g_0 by the diffeomorphism $\eta(t)$. Equivalently, a metric g_0 is a Ricci soliton if and only if it satisfies equation (1.2), which is a generalization of the Einstein condition for the metrics

$$Ric(g_0) = \lambda g_0.$$

Some generalizations, like, gradient Ricci solitons [**4**], quasi Einstein manifolds [**5**], and generalized quasi Einstein manifolds [**6**], play an important role in solutions of geometric flows and describe the local structure of certain manifolds. Nurowski and Randall [**13**] introduced the concept of generalized Ricci soliton as a class of over determined system of equations

$$\mathcal{L}_X g = -2aX^{\#} \odot X^{\#} + 2bS + 2\lambda g, \tag{1.3}$$

where $\mathcal{L}_X g$ and $X^{\#}$ denote, respectively, the Lie derivative of the metric g in the directions of vector field X and the canonical one-form associated to X, and some real constants a, b, and λ. Levy [**10**] acquired the necessary and sufficient conditions for the existence of such tensors. Sharma [**15**] initiated the study of Ricci solitons in almost contact Riemannian geometry. Followed by Tripathi [**19**], Nagaraja et al. [**12**], Turan [**20**], and others extensively studied Ricci solitons in almost contact metric manifold. Recently [**2, 1, 17, 18**], the authors extensively studied Ricci solitons in almost (ε)-contact metric manifolds.

2. Preliminaries

If $\bar{M}$ is an almost contact metric manifold of dimension n equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$-tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0. \tag{2.1}$$

An almost contact metric manifold $\bar{M}$ is called an (ε)-*almost contact metric* manifold if there exists a semi Riemannian metric g such that

$$\eta(X) = \varepsilon g(X, \xi), \quad g(\xi, \xi) = \varepsilon,$$

$$g(\phi X, \phi Y) = g(X, Y) - \varepsilon\eta(X)\eta(Y), \quad \forall X, Y \in TM, \tag{2.2}$$

where $\varepsilon = g(\xi, \xi) = \pm 1$.

An (ε)-almost contact metric manifold is called an (ε, δ)-*trans-Sasakian* manifold if it follows,

$$(\bar{\nabla}_X \phi)Y = \alpha\{g(X, Y)\xi - \varepsilon\eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \delta\eta(Y)\phi X\} \tag{2.3}$$

$$\bar{\nabla}_X \xi = -\varepsilon\alpha\phi X - \beta\delta\phi^2 X, \tag{2.4}$$

and

$$(\nabla_X \eta)Y = \beta\delta[\varepsilon g(X, Y) - \eta(X)\eta(Y)] - \alpha g(\phi X, Y), \tag{2.5}$$

holds for some smooth functions α and β on $\bar{M}$ and $\varepsilon = \pm 1$, $\delta = \pm 1$. For $\beta = 0$, $\alpha = 1$, an (ε, δ)-trans-Sasakian manifold reduces to an (ε)-Sasakian and for $\alpha = 0$, $\beta = 1$, it reduces to a (δ)-Kenmotsu manifold.

The Riemannian curvature tensor R with respect to LeviCivita connections ∇ and the Ricci tensor S of a Riemannian manifold M are defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.6}$$

$$S(X, Y) = \sum_{i=1}^{n} g(R(X, e_i)e_i, Y), \tag{2.7}$$

for $X, Y, Z\Gamma(TM)$, where ∇ is with respect to the Riemannian metric g and , $\{e_1, e_2, ..., e_i\}$ where $1 \leqslant i \leqslant n$ is the orthonormal frame.

Given a smooth function ψ on M , the gradient of ψ is defined by

$$g(grad\psi, X) = X(\psi), \tag{2.8}$$

and the $Hessian$ of ψ is defined by

$$(Hess\psi)(X, Y) = g(\nabla_X grad\psi, Y), \tag{2.9}$$

where $X, Y \in \Gamma(TM)$. For $X \in \Gamma(TM)$, we define $X^{\#} \in \Gamma(TM)$ by

$$X^{\#}(Y) = g(X, Y). \tag{2.10}$$

The generalized Ricci soliton equation in Riemannian manifold M is defined in [**15**] by

$$\mathcal{L}_X g = -2aX^{\#} \odot X^{\#} + 2bS + 2\lambda g, \tag{2.11}$$

where $X \in \Gamma(TM)$ and $\mathcal{L}_X g$ is the Lie-derivative of g along X given by

$$\mathcal{L}_X g(Y, Z) = g(\nabla_Y X, Z) + g(\nabla_Z X, Y) \tag{2.12}$$

for all $Y, Z \in \Gamma(TM)$, and $a, b, \lambda, \in R$.

The Lie-derivative of g along X is said to be (see [**4, 13, 15**])

(1) Killings equation if $a = b = \lambda = 0$,

(2) equation for homotheties if $a = b = 0$,

(3) Ricci soliton if $a = 0, b = -1$,

(4) case of EinsteinWeyl if $a = 1, b = \frac{-1}{n-2}$,

(5) metric projective structures with skew-symmetric Ricci tensor in projective class if $a = 1, b = \frac{-1}{n-2}, \lambda = 0$, and

(6) vacuum near-horizon geometry equation if $a = 1, b = \frac{1}{2}$.

The Lie-derivative of g along X is also a generalization of Einstein manifolds [**10**]. Note that, if $X = grad\psi$, where $\psi \in C^\infty(M)$, the generalized Ricci soliton equation is given by

$$Hess\psi = -ad\psi \odot d\psi + bS + \lambda g. \tag{2.13}$$

Remark 2.1. From (2.3), we have the following Remarks:

(1) $\varepsilon = \delta$, (ε, δ)-trans-Sasakian manifold of type (α, β) reduces to (ε)- trans-Sasakian manifold of type (α, β).

(2) $\varepsilon = \delta = 1$, (ε, δ)-trans-Sasakian manifold of type (α, β) reduces to trans-Sasakian manifold of type (α, β).

(3) $\alpha \neq 0, \beta \neq 0$, and $\varepsilon = -1, \delta = -1$, (ε, δ)-trans-Sasakian manifold of type (α, β) reduces to the form Lorentzian trans-Sasakian manifold of type (α, β).

(4) $\alpha \neq 0, \beta \neq 0$, and $\varepsilon = 1, \delta = -1$, (ε, δ)-trans-Sasakian manifold of type (α, β) reduces in the form α-Sasakian Lorentzian β- Kenmostu manifold of type (α, β).

(5) $\alpha \neq 0, \beta \neq 0$, and $\varepsilon = -1, \delta = 1$, (ε, δ)-trans-Sasakian manifold of type (α, β) reduces in the form Lorentzian α-Sasakian β- Kenmostu manifold of type (α, β).

(6) $\alpha \neq 0, \beta = 0$, and $\varepsilon = 1$, or $\varepsilon = -1$, the (ε, δ)-trans-Sasakian manifold reduces to α-Sasakian manifold or Lorentzian α-Sasakian manifold respectively.

(7) $\alpha = 0$, $\beta \neq 0$, and $\delta = 1$, or $\delta = -1$, the (ε, δ)-trans-Sasakian manifold reduces to β-Kenmotsu manifold or Lorentzian β-Kenmotsu manifold respectively.

(8) If α and β are scalars and $\alpha = 1$ and $\beta = 0$ or $\alpha = 0$ and $\beta = 1$ then the (ε, δ)-trans-Sasakian manifold reduces to to (ε)-Sasakian manifolds and (δ)-Kenmostu manifolds.

(a) Again, if in (ε)-Sasakian manifolds ε is 1 or -1 then the (ε)-Sasakian manifolds reduces to Sasakian manifolds or Lorentzian Sasakian manifolds.

(b) Further, if in (δ)-Kenmostu manifolds δ is 1 or -1 then the (δ)-Kenmostu manifolds reduces to Kenmotsu manifold or Lorentzian Kenmotsu manifold.

3. Main results

In an n-dimensional (ε, δ)- trans-Sasakian manifold M, we have the following relations:

$$\begin{aligned} R(X,Y)\xi &= \varepsilon[(Y\alpha)\phi X - (X\alpha)\phi Y] + \delta[(Y\beta)\phi^2 X - (X\beta)\phi^2 Y], \\ &+2\alpha\beta(\delta - \varepsilon)g(\phi X, Y)\xi + 2\varepsilon\alpha\beta\delta[\eta(Y)\phi X - \eta(X)\phi Y] \\ &+(\alpha^2 - \beta^2)[\eta(Y)X - \eta(X)Y], \end{aligned} \tag{3.1}$$

(3.2) $$S(X,\xi)=[(n-1)(\varepsilon\alpha^2-\delta\beta^2)-(\xi\beta)]\eta(X)-\varepsilon((\phi X)\alpha)-(n-2)\varepsilon(X\beta),$$

(3.3) $$Q\xi=[(n-1)(\varepsilon\alpha^2-\delta\beta^2)-(\xi\beta)]\xi+\varepsilon(grad\alpha)-2n\varepsilon(grad\beta),$$

where R is curvature tensor, while Q is the Ricci operator given by $S(X,Y)=g(QX,Y)$.

Further, in 3-dimensional (ε,δ)-trans-Sasakian manifold, we have

(3.4) $$\phi(grad\alpha)=grad\beta,$$

and

(3.5) $$\varepsilon(\xi\alpha)+2\varepsilon\alpha\beta\delta=0.$$

Using (3.4) and (3.5), for constants α and β, we have

(3.6)
$$\begin{aligned}R(\xi,Y)X=\varepsilon[(grad\alpha)g(\phi X,Y)+(X\alpha)\phi Y]+\delta[(grad\beta)g(\phi^2X,Y)-(X\beta)\phi^2Y]\\+2\alpha\beta\varepsilon(\delta-\varepsilon)\eta(Y)\phi X+2\varepsilon\alpha\beta\delta[\varepsilon g(\phi X,Y)\xi+\eta(X)\phi Y]\\+(\alpha^2-\beta^2)[\varepsilon g(X,Y)\xi-\eta(X)Y],\end{aligned}$$

(3.7) $$R(X,Y)\xi=(\alpha^2-\beta^2)[\eta(Y)X-\eta(X)Y].$$

An important consequence of (2.4) is that ξ is a geodesic vector field; that is,

(3.8) $$\nabla_\xi\xi=0.$$

For an arbitrary vector field X, we have that

(3.9) $$d\eta(\xi,X)=0.$$

The ξ -sectional curvature K_ξ of M is the sectional curvature of the plane spanned by ξ and a unit vector field X. From (3.7), we have

(3.10) $$K_\xi=g(R(\xi,X),\xi,X)=(\alpha^2-\beta^2).$$

It follows from (3.10) that the ξ-sectional curvature does not depend on X.

THEOREM 3.1. *If M is a (ε,δ)-trans-Sasakian manifold of dimension n and it satisfy the generalized Ricci soliton (2.13) with $a[\lambda+(n+1)b(\beta^2-\alpha^2)]\neq-1$, then ψ is a constant function. In such case, if $b\neq0$, then M is an Einstein manifold.*

Next we state following remarks.

REMARK 3.1. If M is a (ε,δ)- trans-Sasakian manifold which satisfies the gradient Ricci soliton equation $Hess\psi=S+\lambda g$; then ψ is a constant function and M is an Einstein manifold.

REMARK 3.2. In a (ε,δ)-trans-Sasakian manifold M, there is no nonconstant smooth function ψ such that $Hess\psi=\lambda g$ for some constant λ .

For the proof of Theorem 3.1, first we need to prove the following lemmas.

LEMMA 3.1. *If M is a (ε,δ)- trans-Sasakian manifold, then*

(3.11) $$\begin{aligned}(\mathcal{L}_\xi(\mathcal{L}_X))g(Y,\xi)=-2\{\varepsilon^2\beta(\xi-\alpha)+2\varepsilon\alpha\beta\delta\}g(X,\phi Y)\\+(\alpha^2-\beta^2-\varepsilon\delta\beta(\xi-\beta))g(X,Y)+g(\nabla_\xi\nabla_\xi X,Y)+Yg(\nabla_\xi X,\xi).\end{aligned}$$

Proof. Using the property of Lie-derivative, we infer

$$(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) = \xi((\mathcal{L}_X g)(Y,\xi)) - (\mathcal{L}_X g)(\mathcal{L}_\xi Y,\xi) - (\mathcal{L}_X g)(Y,\mathcal{L}_\xi \xi), \tag{3.12}$$

since $(\mathcal{L}_\xi Y = [\xi,Y], (\mathcal{L}_\xi \xi = [\xi,\xi]$ by using (2.12) and (3.12), we have

$$\begin{aligned}(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) &= \xi g(\nabla_Y X,\xi) + \xi g(\nabla_\xi X,Y) - g(\nabla_{[\xi,Y]}X,\xi) \\ &\quad - g(\nabla_\xi X,[\xi,Y]) \\ &= g(\nabla_\xi \nabla_Y X,\xi) + g(\nabla_Y X,\nabla_\xi \xi) + g(\nabla_\xi \nabla_\xi X,Y) + g(\nabla_\xi X,\nabla_\xi Y \\ &\quad - g(\nabla_\xi X,\nabla_\xi Y) - g(\nabla_{[\xi,Y]}X,\xi) + g(\nabla_\xi X,\nabla_Y \xi).\end{aligned} \tag{3.13}$$

From (2.4), we get $\nabla_\xi \xi = \phi\xi = 0$; so that we deduce

$$(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) = g(\nabla_\xi \nabla_Y X,\xi) + g(\nabla_\xi \nabla_\xi X,Y) - g(\nabla_{[\xi,Y]}X,\xi) + Yg(\nabla_\xi X,\xi) - g(\nabla_Y \nabla_\xi X,\xi), \tag{3.14}$$

using (3.6) and (3.14), we infer

$$(\mathcal{L}_\xi(\mathcal{L}_X g))(Y,\xi) = g(R(\xi,Y)X,\xi) + g(\nabla_\xi \nabla_\xi X,Y) + Yg(\nabla_\xi X,\xi). \tag{3.15}$$

Now from (3.6), with $g(Y,\xi) = 0$, we infer

$$g(R(\xi,Y)X,\xi) = g(R(Y,\xi)\xi,X) = -2\{\varepsilon^2\beta(\xi-\alpha) + 2\varepsilon\alpha\beta\delta\}g(X,\phi Y) + (\alpha^2 - \beta^2 - \varepsilon\delta\beta(\xi-\beta))g(X,Y), \tag{3.16}$$

the lemma follows from (3.14) and (3.15). □

Now, we have another useful lemma.

Lemma 3.2. *If M is a Riemannian manifold with $\psi \in C^\infty(M)$, then*

$$(\mathcal{L}_\xi(d\psi \odot d\psi))(Y,\xi) = Y(\xi(\psi))\xi(\psi) + Y(\psi)\xi(\xi(\psi)), \tag{3.17}$$

for $\xi, Y \in \Gamma(TM)$.

Proof. It is easy to see that

$$\begin{aligned}(\mathcal{L}_\xi(d\psi \odot d\psi))(Y,\xi) &= \xi(Y(\psi))\xi(\psi) - [\xi,Y](\psi)\xi(\psi) - Y(\psi)[\xi,\xi](\psi) \\ &= \xi(Y(\psi))\xi(\psi) + Y(\psi)\xi(\xi(\psi)) - [\xi,Y](\psi)\xi(\psi),\end{aligned}$$

since $[\xi,Y](\psi) = \xi(Y(\psi)) - Y(\xi(\psi))$, we get

$$\begin{aligned}(\mathcal{L}_\xi(d\psi \odot d\psi))(Y,\xi) &= [\xi,Y](\psi)\xi(\psi) + Y(\xi(\psi))\xi(\psi) \\ &\quad + Y(\psi)\xi(\xi(\psi)) - [\xi,Y](\psi)\xi(\psi) \\ &= Y(\xi(\psi))\xi(\psi) + Y(\psi)\xi(\xi(\psi)).\end{aligned}$$

□

Lemma 3.3. *If M is a (ε,δ)-trans-Sasakian manifold of dimension n, which satisfies the generalized Ricci soliton equation (2.13), then*

$$\nabla_\xi grad\psi = [\lambda + b(n+1)(\beta^2 - \alpha^2)]\xi - a\xi(\psi)grad\psi. \tag{3.18}$$

PROOF. Because $Y \in \Gamma(TM)$, using the definition of Ricci curvature S (2.7) and the curvature condition (3.7), we infer

$$\begin{aligned} S(\xi, Y) = g(R(\xi, e_i)e_i, Y) &= g(R(e_i, Y)\xi,)e_i) \\ &= (\beta^2 - \alpha^2)(g(Y, e_i) + \eta(Y)g(e_i, e_i) \\ &= (\beta^2 - \alpha^2)(\eta(Y) + n\eta(Y)) = (\beta^2 - \alpha^2)(n+1)\eta(Y) \\ &= (\beta^2 - \alpha^2)(n+1)g(\xi, Y), \end{aligned}$$

where $\{e_1, e_2, ..., e_i\}$, and $1 \leqslant i \leqslant n$ is an orthonormal frame on M implies that

$$\begin{aligned} \lambda g(\xi, Y) + bS(\xi, Y) &= \lambda g(\xi, Y) + b(\beta^2 - \alpha^2)(n+1)g(\xi, Y) \\ &= [\lambda + b(n+1)(\beta^2 - \alpha^2)]g(\xi, Y). \end{aligned} \tag{3.19}$$

From (2.13) and (3.19), we obtain

$$\begin{aligned} (Hess\psi)(\xi, Y) &= -a\xi(\psi)(Y)(\psi) + [\lambda + b(n+1)(\beta^2 - \alpha^2)]g(\xi, Y) \\ &= -a\xi(\psi)g(grad\psi, Y) + [\lambda + b(n+1)(\beta^2 - \alpha^2)]g(\xi, Y), \end{aligned} \tag{3.20}$$

the lemma follows from equation (3.20) and the definition of $Hessian$ (2.7). □

Now, with help of Lemmas 3.1, 3.2, and 3.3, we can prove Theorem 3.1.

PROOF. (**Proof of Theorem 3.1.**) If $Y \in \Gamma(TM)$ is such that $g(\xi, Y) = 0$, then from Lemma 3.1, with $X = grad\psi$, we infer

$$2(\mathcal{L}_\xi(Hess\psi))(Y, \xi) = Y(\psi) + g(\nabla_\xi \nabla_\xi grad\psi, Y) + Yg(\nabla_\xi grad\psi, \xi), \tag{3.21}$$

from Lemma 3.3 and equation (3.21), we deduce

$$\begin{aligned} 2(\mathcal{L}_\xi(Hess\psi))(Y, \xi) &= Y(\psi) + [\lambda + b(n+1)(\beta^2 - \alpha^2)]g(\nabla_\xi \xi, Y) \\ -ag(\nabla_\xi(\xi(\psi)grad\psi), Y) &- aY(\xi(\psi^2)) + [\lambda + b(n+1)(\beta^2 - \alpha^2)Yg(\xi, \xi). \end{aligned} \tag{3.22}$$

Since $\nabla_\xi \xi = 0$ and $g(\xi, \xi) = 1$, from equation (3.22), we deduce

$$\begin{aligned} 2(\mathcal{L}_\xi(Hess\psi))(Y, \xi) &= Y(\psi) - a\xi(\xi(\psi))Y(\psi) - a\xi(\psi)g(\nabla_\xi grad\psi, Y) \\ &\quad -2a\xi(\psi)Y(\xi(\psi)). \end{aligned} \tag{3.23}$$

From Lemma 3.3 and equation (3.23) and since $g(\xi, Y) = 0$, we deduce

$$\begin{aligned} 2(\mathcal{L}_\xi(Hess\psi))(Y, \xi) &= Y(\psi) - a\xi(\xi(\psi))Y(\psi) + a^2\xi(\psi)^2Y(\psi) \\ &\quad -2a\xi(\psi)Y(\xi(\psi)). \end{aligned} \tag{3.24}$$

Note that, from (2.11) and (2.12), we have $\mathcal{L}_\xi g = 0$, which is a Killing vector field; it implies that $\mathcal{L}_\xi S = 0$; taking the Lie derivative of the generalized Ricci soliton equation (2.13) yields

$$\begin{aligned} (\alpha^2 - \beta^2)Y(\psi) - a\xi(\xi(\psi))Y(\psi) + a^2\xi(\psi)^2Y(\psi) - 2a\xi(\psi)Y(\xi(\psi)) \\ = -2aY(\xi(\psi))\xi(\psi) - 2aY(\psi)\xi(\xi(\psi)), \end{aligned} \tag{3.25}$$

which is equivalent to

$$Y(\psi)[(\alpha^2 - \beta^2) + a\xi(\xi(\psi)) + a^2\xi(\psi)^2] = 0, \tag{3.26}$$

according to Lemma 3.3, we infer

$$a\xi(\xi(\psi)) = a\xi g(\xi, grad\psi) = ag(\xi, \nabla_\xi grad\psi) \tag{3.27}$$

$$= a[\lambda + b(n+1)(\beta^2 - \alpha^2)] - a^2\xi(\psi)^2,$$

by equations (3.26) and (3.27), we deduce

$$Y(\psi)[1 + a(\lambda + b(n+1)(\beta^2 - \alpha^2))] = 0,$$

since $a[\lambda + b(n+1)(\beta^2 - \alpha^2)] \neq -1$, we find that $Y(\psi) = 0$; that is, $grad\psi$ is parallel to ξ . Hence $grad\psi = 0$ as $D = ker\eta$ is not integrable any where, which means ψ is a constant function. □

For particular values of α and β, there arise possible cases:

Case (i) For $\alpha = 0$ or $(\beta = 1)$, we infer

Corollary 3.1. *If M is a δ -Kenmotsu (or Kenmotsu) manifold of dimension n, and it satisfies the generalized Ricci soliton (2.13) with condition $a(\lambda+b(n+1)) \neq -1$, then ψ is a constant function. In such case, if $b \neq 0$, then M is an Einstein manifold.*

Case (ii) For $\beta = 0$, or $(\alpha = 1)$, we infer

Corollary 3.2. *If M is a ε -Sasakian (or Sasakian) manifold of dimension n, and it satisfies the generalized Ricci soliton (2.13) with $a(\lambda - b(n+1)) \neq -1$, then ψ is a constant function. In such case, if $b \neq 0$, then M is an Einstein manifold.*

Corollary 3.3. *There exist no steady Ricci soliton (g, ξ, λ) indefinite (ε, δ) -trans-Sasakian manifold.*

Proof. From Linear Algebra either the vector field $V \epsilon Span\xi$ or $V \perp \xi$. However the second case seems to be complex to analyse in practice. For this reason we investigate for the case $V = \xi$. A simple computation of $\mathcal{L}_\xi g + 2S$ gives

$$(\mathcal{L}_\xi g)(X,Y) = 2\beta\delta[g(X,Y) - \varepsilon\eta(X)\eta(Y)]. \tag{3.28}$$

From equation (1.1), we have $h(X,Y) = -2\lambda g(X,Y)$ and then putting $X = Y = \xi$, we infer

$$h(\xi,\xi) = -2\lambda\varepsilon, \tag{3.29}$$

where $h(X,Y) = (\mathcal{L}_\xi g)(X,Y) + 2S(X,Y)$ and then if we put $X = Y = \xi$ and again by using (3.31) and (3.2), we deduce

$$h(\xi,\xi) = 2\beta\delta[g(\xi,\xi) - \varepsilon\eta(\xi)\eta(\xi)] + 2\{\varepsilon[(n-1)(\varepsilon\alpha^2 - \delta\beta^2) - (\xi\beta)]\eta(\xi)$$
$$-\varepsilon((\phi\xi)\alpha) - (n-2)\varepsilon(\xi\beta)\}.$$

By using (2.1), (2.2) and (3.4) in the above equation, we infer

$$h(\xi,\xi) = 2(n-1)\varepsilon(\varepsilon\alpha^2 - \delta\beta^2). \tag{3.30}$$

Equating (3.29) and (3.30), we deduce

$$\lambda = -(n-1)(\varepsilon\alpha^2 - \delta\beta^2). \tag{3.31}$$

Since from (3.31), we have $\lambda \neq 0$. The proof is complete. □

4. Ricci solitons in 3-dimensional (ε,δ)-trans-Sasakian manifold

Corollary 4.1. *There exist no steady Ricci soliton (g,ξ,λ) indefinite 3-dimensional (ε,δ)-trans-Sasakian manifold with varying scalar curvature where*

$$\lambda = -2(\varepsilon\alpha^2 - \delta\beta^2).$$

Proof. The Riemannian curvature tensor R of M with respect to the 3-dimensional (ε,δ)-trans-Sasakian manifold is defined by

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y]. \tag{4.1}$$

Putting $Z=\xi$ in (4.1) and by using (2.12) and (2.15) for 3-dimensional (ε,δ)-trans-Sasakian manifold, we infer

$$\begin{aligned} &\varepsilon[(Y\alpha)\phi X - (X\alpha)\phi Y] + \delta[(Y\beta)\phi^2 X - (X\beta)\phi^2 Y] \\ &+2\alpha\beta(\delta-\varepsilon)g(\phi X,Y)\xi + 2\varepsilon\alpha\beta\delta[\eta(Y)\phi X - \eta(X)\phi Y] \\ &+(\alpha^2-\beta^2)[\eta(Y)X - \eta(X)Y] = \varepsilon\eta(Y)QX - \varepsilon\eta(X)QY \\ &+\varepsilon[(\varepsilon\alpha^2 - \delta\beta^2) - (\xi\beta)](\eta(Y)X - \eta(X)Y) \\ &-\varepsilon[((\phi Y)\alpha)X + (Y\beta)X] + \varepsilon[((\phi X)\alpha)Y + (X\beta)Y]. \end{aligned} \tag{4.2}$$

Again, putting $Y=\xi$ in the above equation and by using (2.1) and (2.17), we deduce

$$\begin{aligned} QX = {}&[\frac{r}{2} - 2(\varepsilon\alpha^2-\delta\beta^2) + \varepsilon(\alpha^2-\beta^2)]X \\ &+[4(\varepsilon\alpha^2-\delta\beta^2) - \frac{r}{2} - \varepsilon(\alpha^2-\beta^2)]\eta(X)\xi. \end{aligned} \tag{4.3}$$

From (4.3), we deduce

$$\begin{aligned} S(X,Y) = {}&[\frac{r}{2} - 2(\varepsilon\alpha^2-\delta\beta^2) + \varepsilon(\alpha^2-\beta^2)]g(X,Y) \\ &+[4(\varepsilon\alpha^2-\delta\beta^2) - \frac{r}{2} - \varepsilon(\alpha^2-\beta^2)]\varepsilon\eta(X)\eta(Y). \end{aligned} \tag{4.4}$$

Equation (4.4) shows that a 3-dimensional (ε,δ)-trans-Sasakian manifold is η -Einstein. Now we show that the scalar curvature r is not a constant that is r is varying. Now

$$h(X,Y) = (\mathcal{L}_\xi g)(X,Y) + 2S(X,Y). \tag{4.5}$$

By using (3.13) and (4.4) in (4.5), we deduce

$$\begin{aligned} h(X,Y) = {}&[r - 4(\varepsilon\alpha^2-\delta\beta^2) + 2\varepsilon(\alpha^2-\beta^2) + 2\beta\delta]g(X,Y) \\ &+[8(\varepsilon\alpha^2-\delta\beta^2) - 2\beta\delta - 2\varepsilon(\alpha^2-\beta^2) - r]\varepsilon\eta(X)\eta(Y). \end{aligned} \tag{4.6}$$

Differentiating (4.6) covariantly with respect to Z, we infer

$$\begin{aligned} (\nabla_Z h)(X,Y) = {}&[\nabla_Z r - 4(2\varepsilon\alpha(Z\alpha) - 2\delta\beta(Z\beta)) + 2\varepsilon(2\alpha(Z\alpha) - 2\beta(Z\beta)) \\ &+2\delta(Z\beta)]g(X,Y) + [8(2\varepsilon\alpha(Z\alpha) - 2\delta\beta(Z\beta)) - 2\delta(Z\beta) - 2\varepsilon(2\alpha(Z\alpha) \\ &-2\beta(Z\beta) - \nabla_Z r]\varepsilon\eta(X)\eta(Y) + [8(\varepsilon\alpha^2-\delta\beta^2) - 2\beta\delta \\ &-2\varepsilon(\alpha^2-\beta^2) - r][g(X,\nabla_Z\xi)\eta(Y) + g(Y,\nabla_Z\xi)\eta(X)]. \end{aligned} \tag{4.7}$$

Substituting $Z = \xi, X = Y \in (Span\xi)^{\perp}$ in (4.7) and by virtue of $\nabla h = 0$ and (2.17), we infer

$$\nabla_{\xi} r - 4\varepsilon\alpha(\xi\alpha) = 0.$$

By using (2.16) in the above equation, we deduce

$$\nabla_{\xi} r = -8\varepsilon\alpha^2\beta\delta. \tag{4.8}$$

Thus, r is not a constant. Now we have to check the nature of the soliton that is Ricci soliton (g, ξ, λ) where $\lambda = -2(\varepsilon\alpha^2 - \delta\beta^2)$ in 3-dimensional (ε, δ)-trans-Sasakian manifold:

From (1.1), we have $h(X, Y) = -2\lambda g(X, Y)$ and then putting $X = Y = \xi$, we infer

$$h(\xi, \xi) = -2\lambda\varepsilon. \tag{4.9}$$

If $X = Y = \xi$, in (4.6), we deduce

$$h(\xi, \xi) = 4\varepsilon(\varepsilon\alpha^2 - \delta\beta^2). \tag{4.10}$$

Equating (4.9) and (4.10), we deduce

$$\lambda = -2(\varepsilon\alpha^2 - \delta\beta^2). \tag{4.11}$$

Since from (4.11), we have $\lambda \neq 0$. There exist no steady Ricci soliton (g, ξ, λ) of 3-dimensional (ε, δ)-trans-Sasakian manifold. □

EXAMPLE 4.1. ([**2**]) We consider the 3-dimensional manifold $M = \{(x, y, z) : (x, y, z) \in R^3, z \neq 0\}$. Let $\{E_1, E_2, E_3\}$ be linearly independent global frame field on M given by

$$E_1 = z(\frac{\partial}{\partial z} + \delta y\frac{\partial}{\partial z}), E_2 = \delta z\frac{\partial}{\partial y}, E_3 = \frac{\partial}{\partial z}. \tag{4.12}$$

Let g be the Riemannian metric defined by

$$g(E_1, E_2) = g(E_2, E_3) = g(E_1, E_3) = 0,\ g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = \varepsilon,$$

where g is given by

$$g = \frac{\varepsilon}{z^2}[(1 - y^2z^2)dx \otimes dx + dy \otimes dy + z^2 dz \otimes dz].$$

The (ϕ, ξ, η) is given by

$$\eta = dz - \delta y dx,\ \xi = E_3 = \tfrac{\partial}{\partial z},\ \phi E_1 = E_2,\ \phi E_2 = -E_1,\ \phi E_3 = 0.$$

Clearly (ϕ, ξ, η, g) structure is an indefinite (ε, δ)-trans-Sasakian structure and satisfy,

$$(\bar{\nabla}_X\phi)Y = \alpha\{g(X, Y)\xi - \varepsilon\eta(Y)X\} + \beta\{g(\phi X, Y)\xi - \delta\eta(Y)\phi X\}, \tag{4.13}$$

$$\bar{\nabla}_X\xi = -\varepsilon\alpha\phi X - \beta\delta\phi^2 X, \tag{4.14}$$

where $\alpha = -\frac{z^2\delta}{2\varepsilon} \neq 0$ and $\beta = -\frac{1}{z\delta} \neq 0$. Hence (ϕ, ξ, η, g) structure defines indefinite (ε, δ)-trans-Sasakian structure. Thus M equipped with indefinite (ε, δ)-trans-Sasakian structure is a (ε, δ)-trans- Sasakian manifold.

Using the above α and β in (4.8), we infer

$$\nabla_\xi r = -8\varepsilon\alpha^2\beta\delta = -8\varepsilon\delta(\frac{z^4\delta^2}{4\varepsilon^2})(\frac{1}{z\delta}) = 2\varepsilon z^3 \tag{4.14}$$

$$r = \frac{z^4\varepsilon}{2}. \tag{4.15}$$

Using the above α and β in (4.11), we infer

$$\lambda = \frac{4\delta - \varepsilon z^6}{2z^2}. \tag{4.16}$$

Hence Ricci soliton (g,ξ,λ) is given by (4.16) with varying scalar curvature (4.15).

(1) If $\varepsilon = \delta = 1$, in (4.16), then $\lambda = \frac{4-Z^6}{2Z^2} = \frac{(2-Z^3)(2+Z^3)}{2Z^2}$.

(a) $\lambda > 0$ in $\{Z : -2^{\frac{1}{3}} < Z < 2^{\frac{1}{3}}\}$:

Hence by Remark (2.1), Ricci soliton of the given 3-dimensional trans-Sasakian manifold is expanding in the region

$$\{(x,y,z) \in R^3 : -2^{\frac{1}{3}} < Z < 2^{\frac{1}{3}}\}. \tag{4.17}$$

(b) Also $\lambda < 0$ in $\{Z : -2^{\frac{1}{3}} > Z > 2^{\frac{1}{3}}\}$:

Hence by Remark (2.1), Ricci soliton of the given 3-dimensional trans-Sasakian manifold is shrinking in the region

$$\{(x,y,z) \in R^3 : -2^{\frac{1}{3}} > Z > 2^{\frac{1}{3}}\}. \tag{4.18}$$

Hence the regions (4.17) and (4.18) are complementary to one another that is

$$M = \{(x,y,z) \in R^3 : -2^{\frac{1}{3}} < Z < 2^{\frac{1}{3}}\} \mp \{-2^{\frac{1}{3}} > Z > 2^{\frac{1}{3}}\}. \tag{4.19}$$

(2) If $\varepsilon = \delta = -1$, in (4.16), then $\lambda = \frac{Z^6-4}{2Z^2} = \frac{(Z^3-2)(2+Z^3)}{2Z^2}$.

(a) $\lambda > 0$ in $\{Z : -2^{\frac{1}{3}} > Z > 2^{\frac{1}{3}}\}$:

Hence by Remark (2.1), Ricci soliton in Lorentzian trans-Sasakian manifold is expanding in the region

$$\{(x,y,z) \in R^3 : -2^{\frac{1}{3}} > Z > 2^{\frac{1}{3}}\}. \tag{4.20}$$

(b) Also $\lambda < 0$ in $\{Z : -2^{\frac{1}{3}} < Z < 2^{\frac{1}{3}}\}$:

Hence by Remark (2.1), Ricci soliton in Lorentzian trans-Sasakian manifold is shrinking in the region

$$\{(x,y,z) \in R^3 : -2^{\frac{1}{3}} < Z < 2^{\frac{1}{3}}\}. \tag{4.18}$$

Hence the regions (4.17) and (4.18) are complementary to one another that is

$$M = \{(x,y,z) \in R^3 : -2^{\frac{1}{3}} > Z > 2^{\frac{1}{3}}\} \mp \{-2^{\frac{1}{3}} < Z < 2^{\frac{1}{3}}\}. \tag{4.19}$$

Thus from cases (1) and (2) one can conclude that in a region where the trans-Sasakian manifold is shrinking the Lorentzian trans-Sasakian manifold is expanding and in a region where the trans-Sasakian manifold is expanding the Lorentzian trans-Sasakian manifold is shrinking. Hence in given example trans-Sasakian and Lorentzian trans-Sasakian manifolds are complementary to each other.

(3) If $\varepsilon = -1, \delta = 1$, in (4.16), then $\lambda = \frac{Z^6+4}{2Z^2} > 0$. By Remark (2.1), Ricci soliton in Lorentzian α-Sasakian β -Kenmotsu manifold is expanding.

(4) If $\varepsilon = 1, \delta = -1$, in (4.16), then $\lambda = -\frac{Z^6+4}{2Z^2} < 0$. By Remark (2.1), Ricci soliton in α-Sasakian Lorentzian β -Kenmotsu manifold is shrinking.

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