BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Vol. **10**(2)(2020), 283-290
DOI: 10.7251/BIMVI2002283P

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

F-INDEX AND ITS COINDEX OF BLOCK-EDGE TRANSFORMATION GRAPHS

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Abstract. The F-index of a connected graph G denoted by $F(G)$ is defined as the sum of cubes of the degrees of vertices of the graph. The total π-electron energy depends on the degree based sum $M_1(G) = \sum\limits_{u\in V(G)} d_G(u)^2$(first Zagreb index) and $F(G) = \sum\limits_{u\in V(G)} d_G(u)^3$, it was shown in the study of structure-dependency of total π-electron energy in 1972. The inverse degree of a graph G with no isolated vertices is defined as $ID(G) = \sum\limits_{v\in V(G)} \frac{1}{d_G(v)}$. In this paper, we compute the exact expression for F-indices and its coindices of block-edge transformation graph G^{ab} and its complement.

1. Introduction

A *topological index* for a (chemical) graph G is a numerical quantity invariant under automorphisms of G and it does not depend on the labeling or pictorial representation of the graph. It has been used for examining quantitative structure-property relationships (QSPR) and quantitative structure-activity relationships (QSAR) extensively in which the biological activity or other properties of molecules are correlated with their chemical structures, in [**6**]. In the current chemical literature, a large number of graph-based structure descriptors (topological indices) have been put forward, that all depend only on the degrees of the vertices of the molecular graph.

The first and second *Zagreb indices* are defined as

$$M_1(G) = \sum_{u\in V(G)} d_G^2(u) = \sum_{uv\in E(G)} (d_G(u) + d_G(v))$$

2010 *Mathematics Subject Classification.* 05C12, 05C76.
Key words and phrases. F-index, F-coindex, block-edge transformation.

and

$$M_2(G) = \sum_{uv \in E(G)} d_G(u)d_G(v),$$

respectively. Another topological index, defined as sum of cubes of degrees of all the vertices was also introduced in the same paper, where the first and second Zagreb indices were introduced in [**11**]. Furtula and Gutman in [**9**] recently investigated this index and named this index as *forgotten topological index* (or) F-index and showed that the predictive ability of this index is almost similar to that of first Zagreb index and for the entropy and acetic factor, both of them yield correlation coefficients greater than 0.95. The F-index of a graph G is defined as

$$F(G) = \sum_{u \in V(G)} d_G^3(u) = \sum_{uv \in E(G)} (d_G^2(u) + d_G^2(v)).$$

In this sequence, the *forgotten topological coindex* (or) *F-coindex* is defined as

$$\overline{F}(G) = \sum_{uv \notin E(G)} (d_G^2(u) + d_G^2(v)).$$

The exact expressions for the F-index and F-polynomial of six infinite classes of nanostar dendrimers are presented in [**5**]. The Findex for the four new sums of special well-known graphs is discussed in [**10**]. Che and Chen [**4**] provided the new lower and upper bounds of the forgotten topological index $F(G)$ in terms of graph irregularity, Zagreb indices, graph size, and maximum/minimum vertex degrees. Also they characterized all graphs that attain the new bounds of $F(G)$ and show that the new bounds are better than the bounds given in [**9**] for all benzenoid systems with more than one hexagon. The values of F-indices and its coindices of bridge graphs, chain graphs and some product of graphs are given in [**16, 15**]. In this paper, we compute the exact expression for F-indices and its coindices of block-edge transformation graph G^{ab} and its complement.

The *degree* denoted by $d_G(e)$, is defined by $d_G(e) = d_G(u) + d_G(v) - 2$. The *line graph* $L(G)$ of G is the graph whose vertex set is $E(G)$ in which two vertices are adjacent if and only if they are adjacent in G. The *jump graph* $J(G)$ of G is the graph whose vertex set is $E(G)$ in which two vertices are adjacent if and only if they are not adjacent in G [**3**]. A *block* of a graph is connected nontrivial graph having no cut vertices.

1.1. Main results. Let G be a graph with block set

$$U(G) = \{B_i; B_i \text{ is a block of } G, 1 \leqslant i \leqslant r\}.$$

If a block $B \in U(G)$ with the edge set $\{e_1, e_2, \ldots, e_s; s \geqslant 1\}$, then we say that the edge e_i and block B are incident with each other, where $1 \leqslant i \leqslant s$. Basavanagoud and patil [**1**], we introduced the block-edge transformation graphs G^{ab} and defined as follows:

Let a, b be two variables taking values $+$ or $-$. The *block-edge transformation graph* G^{ab} is a graph whose vertex set is $E(G) \cup U(G)$, and two vertices x and y ofG^{ab} are joined by an edge if and only if one of the following holds:

(i) $x, y \in E(G)$. x and y are adjacent in G if $a = +$; x and y are not adjacent in G if $a = -$.

(ii) $x \in E(G), y \in U(G)$, x and y are incident with each other in G if $b = +$; x and y are not incident with each other in G if $b = -$.

Hence the four kinds of block-edge transformation graphs $G^{++}, G^{+-}, G^{-+}$and G^{--}. The degree of a block $B \in G$, denoted by $d_G(B)$, is the number of edges incident with B in G. We denote $\sum\limits_{i=r}^{r} d_G^k(B_i)$ by $\overline{\eta}_k(G)$. The vertex $e^{'}(B^{'})$ of G^{ab} corresponding to edge e (resp., block B) of G and is referred as edge (resp.,block)-vertex.

From the structure of the graph G^{ab}, we have the following lemma.

Lemma 1.1 ([**2**]). *Let G be a graph with n vertices, m edges and r blocks. Then the degree of edge-vertex $e^{'}(e = uv \in G)$ and block-vertex $B^{'}$ in G^{ab} are*

(i) $d_{G^{++}}(e^{'}) = d_G(u) + d_G(v) - 1$ *and* $d_{G^{++}}(B^{'}) = d_G(B)$.

(ii) $d_{G^{+-}}(e^{'}) = d_G(u) + d_G(v) + r - 3$ *and* $d_{G^{+-}}(B^{'}) = m - d_G(B)$.

(iii) $d_{G^{-+}}(e^{'}) = m + 2 - d_G(u) - d_G(v)$ *and* $d_{G^{-+}}(B^{'}) = d_G(B)$.

(iv) $d_{G^{--}}(e^{'}) = m + r - d_G(u) - d_G(v)$ *and* $d_{G^{--}}(B^{'}) = m - d_G(B)$.

1.2. F-index of G^{ab}. The (a, b)-*Zagreb index* of G is defined as follows

$$Z'_{a,b}(G) = \frac{1}{2} \sum_{uv \in E(G)} \Big(d_G(u)^a d_G(v)^b + d_G(u)^b d_G(v)^a \Big).$$

It is not hard to see that $Z'_{1,1} = M_2(G)$. Recently, Shirdel et al. [**17**] introduced a variant of the first Zagreb index called hyper-Zagreb index. The *hyper-Zagreb index* of G is denoted by $HM(G)$ and defined as

$$HM(G) = \sum_{uv \in E(G)} (d(u) + d(v))^2.$$

Define $M_3(G) = \sum\limits_{uv \in E(G)} (d(u) + d(v))^3$. In this section, we obtain the F-index of the graph G^{ab} and its complement.

Lemma 1.2 ([**15**]). *Let G be a connected graph with n vertices and m edges. Then*

$$F(\overline{G}) = n(n-1)^3 - F(G) - 6(n-1)^2 m + 3(n-1) M_1(G).$$

Theorem 1.1. *Let G be a connected graph with n vertices, m edges and r blocks. Then*

$$F(G^{++}) = M_3(G) + 6Z_{2,1}(G) - m + 3M_1(G) - 3HM(G) + \overline{\eta_3}(G).$$

PROOF. One can observe that the number of vertices and edges of G^{++} are $m+r$ and $\frac{1}{2}M_1(G)$, respectively. By the definition of F-index,

$$\begin{aligned} F(G^{++}) &= \sum_{x\in V(G^{++})} d_{G^{++}}(x)^3 \\ &= \sum_{e'\in V(G^{++})\cap E(G)} d_{G^{++}}(e^{'})^3 + \sum_{B'\in V(G^{++})\cap U(G)} d_{G^{++}}(B^{'})^3. \end{aligned}$$

From Lemma 1.1, we obtain

$$\begin{aligned} F(G^{++}) &= \sum_{uv\in E(G)} [d_G(u)+d_G(v)-1]^3 + \sum_{B\in U(G)} d_G(B)^3 \\ &= \sum_{uv\in E(G)} \Big[(d_G(u)+d_G(v))^3 + 3(d_G(u)+d_G(v)) \\ &- 3(d_G(u)+d_G(v))^2 - 1\Big] + \sum_{B\in U(G)} d_G(B)^3 \\ &= M_3(G) + 6Z_{2,1}(G) - m + 3M_1(G) - 3HM(G) + \overline{\eta_3}(G). \end{aligned}$$

□

Using Lemma 1.1, Theorem 1.1 and the expression

$$M_1(G^{++}) = F(G) + 2M_2(G) + m - 2M_1(G) + \overline{\eta_2}(G),$$

we have the following corollary.

COROLLARY 1.1. *Let G be a connected graph with n vertices, m edges and r blocks. Then*
$F(\overline{G^{++}}) = m^3(m+4r-3) + 3m^2(r^2-2r+2-M_1(G)) + m(4r^3-9r^2-3r-6M_1(G)(r+1)-4) + 3r^2(1-M_(G)) + 3M_1(G)(2m+4r-3) + 3(m+r-1)(F(G)+2M_2(G)+\overline{\eta_2}(G)) - M_3(G) - 6Z_{2,1}(G) + 3HM(G) - \overline{\eta_3}(G)$.

THEOREM 1.2. *Let G be a connected graph with n vertices, m edges and r blocks. Then*
$F(G^{+-}) = M_3(G) + 6Z_{2,1}(G) + m(r-3)^3 + 3(r-3)HM(G) + 3(r+3)^2M_1(G) + rm^3 - \overline{\eta_3}(G) + 3m\overline{\eta_2}(G) - 3m^3$.

PROOF. Note that

$$\left|V(G^{+-})\right| = m+r \text{ and } \left|E(G^{+-})\right| = \frac{1}{2}M_1(G) + m(r-2).$$

By the definition of F-index,

$$\begin{aligned} F(G^{+-}) &= \sum_{x\in V(G^{+-})} d_{G^{+-}}(x)^3 \\ &= \sum_{e'\in V(G^{+-})\cap E(G)} d_{G^{+-}}(e^{'})^3 + \sum_{B'\in V(G^{+-})\cap U(G)} d_{G^{+-}}(B^{'})^3. \end{aligned}$$

From Lemma 1.1, we obtain

$$F(G^{+-}) = \sum_{uv\in E(G)} [d_G(u)+d_G(v)+r-3]^3 + \sum_{B\in U(G)} [m-d_G(B)]^3 \tag{1.1}$$

and

$$\begin{aligned} F(G^{+-}) &= \sum_{uv\in E(G)} \Big[(d_G(u)+d_G(v))^3 + (r-3)^3 + 3(r-3)(d_G(u)+d_G(v))^2 \\ &+ 3(r-3)^2(d_G(u)+d_G(v))\Big] + \sum_{B\in U(G)} [m-d_G(B)]^3 \\ &= M_3(G) + 6Z_{2,1}(G) + m(r-3)^3 + 3(r-3)HM(G) \\ &+ 3(r-3)^2 M_1(G) + rm^3 - \overline{\eta_3}(G) + 3m\overline{\eta_2}(G) - 3m^3. \end{aligned}$$

□

The proof of the following corollary is follows from Lemma 1.1, Theorem 1.2 and the expression

$$M_1(G^{+-}) = 2M_2(G) + F(G) + m(r-3)^2 + 2(r-3)M_1(G) + rm^2 + \overline{\eta_2}(G) - 2m^2.$$

Corollary 1.2. *Let G be a connected graph with n vertices, m edges and r blocks. Then*
$F(\overline{G^{+-}}) = m^3(m+r+6) + 3m^2(3r^2-16r-M_1(G)+20) + m(-2r^3-9r^2+12r-12M_1(G)+6M_2(G)+3F(G)+48) - 3(r-3)HM(G) + 3r^2(1-M_1(G)) + \overline{\eta_3}(G) + r^4-3r^3-3M_1(G)(r^2+8r+10) + 3(r-1)(2M_2(G)+F(G)-2(r-3)M_1(G))$.

Theorem 1.3. *Let G be a connected graph with n vertices, m edges and r blocks. Then*
$F(G^{-+}) = m(m+2) - M_3(G) + 6Z_{2,1}(G) + 3(m+2)HM(G) - 3(m+2)^2M_1(G) + \overline{\eta_3}(G)$.

Proof. From the structure of the graph G^{-+}, we have $|V(G^{+-})| = m+r$ and

$$\left|E(G^{+-})\right| = \frac{m(m+3)}{2} - \frac{M_1(G)}{2}.$$

By the definition of F-index,

$$\begin{aligned} F(G^{-+}) &= \sum_{x\in V(G^{-+})} d_{G^{-+}}(x)^3 \\ &= \sum_{e'\in V(G^{-+})\cap E(G)} d_{G^{-+}}(e^{'})^3 + \sum_{B'\in V(G^{-+})\cap U(G)} d_{G^{-+}}(B^{'})^3. \end{aligned}$$

By Lemma 1.1, we have

$$F(G^{-+}) = \sum_{uv\in E(G)} [m+2-d_G(u)-d_G(v)]^3 + \sum_{B\in U(G)} d_G(B)^3 \tag{1.2}$$

and

$$\begin{aligned} F(G^{-+}) &= \sum_{uv\in E(G)} \Big[(m+2)^3 - (d_G(u)+d_G(v))^3 + 3(m+2)(d_G(u)+d_G(v))^2 \\ &- 3(m+2)^2(d_G(u)+d_G(v))\Big] + \sum_{B\in U(G)} d_G(B)^3 \\ &= m(m+2)^3 - M_3(G) - 6Z_{2,1}(G) + 3(m+2)HM(G) \\ &- 3(m+2)^2 M_1(G) + \overline{\eta_3}(G). \end{aligned}$$

□

The proof of the following corollary is follows from Lemma 1.1, Theorem 1.3 and the expression

$$M_1(G^{-+}) = 2M_2(G) + F(G) + m(r+2)^2 - 2(m+2)M_1(G) + \overline{\eta_2}(G).$$

COROLLARY 1.3. *Let G be a connected graph with n vertices, m edges and r blocks. Then*
$F(\overline{G^{-+}}) = m^3(r-9) + 18m^2(5r-23) + m(4r^3 - 18r^2 - 12r - 29) + M_1(G)(r^2 - 6r - 8m + 9) + r^4 - 3r^3 - 3r^2 + M_3(G)6Z_{2,1}(G) - 3(m+2)HM(G) - \overline{\eta_3}(G)$.

The number of vertices and edges of the graph G^{--} are $m+r$ and $\frac{m(m+2r-1)}{2} - \frac{M_1(G)}{2}$, respectively. The proof of the following theorem is similar to Theorem 1.3.

THEOREM 1.4. *Let G be a connected graph with n vertices, m edges and r blocks. Then*
$F(G^{--}) = m(m+r)^3 - M_3(G) + 6Z_{2,1}(G) + 3(m+r)HM(G) - 3(m+r)^2 M_1(G) + rm^3 - \overline{\eta_3}(G) + 3m\overline{\eta_2}(G) - 3m^3$.

COROLLARY 1.4. *Let G be a connected graph with n vertices, m edges and r blocks. Then*
$F(\overline{G^{-+}}) = m^3(r-9) + 18m^2(5r-23) + m(4r^3 - 18r^2 - 12r - 29) + M_1(G)(r^2 - 6r - 8m + 9) + r^4 - 3r^3 - 3r^2 + M_3(G)6Z_{2,1}(G) - 3(m+2)HM(G) - \overline{\eta_3}(G)$.

PROOF. The proof is follows from Lemma 1.1, Theorem 1.4 and the expression
$M_1(G^{--}) = 2M_2(G) + F(G) + m(m+r)^2 - 2(m+r)M_1(G) + rm^2 + \overline{\eta_2}(G) + 2m^2$.

□

2. F-coindices of G^{ab}

In this section, we compute the F-coindices of G^{ab} and its complements.

LEMMA 2.1 ([**15**]). *Let G be a connected graph with n vertices and m edges. Then*

(i) $\overline{F}(G) = (n-1)M_1(G) - F(G)$.

(ii) $\overline{F}(\overline{G}) = F(G) - 2(n-1)M_1(G) + 2(n-1)^2 m$.

Using Lemma 2.1, Theorems 1.1 to 1.4, we obtain the following corollaries.

COROLLARY 2.1. *Let G be a connected graph with n vertices, m edges and r blocks. Then*

(i) $\overline{F}(G^{++}) = (m+r-1)(F(G)+2M_2(G)+\overline{\eta_2}(G)) - M_3(G) - 6Z_{2,1}(G) + 3HM(G) - \overline{\eta_3}(G) + m(m+r) - M_1(G)(2m+2r+1)$.

(ii) $\overline{F}(\overline{G^{++}}) = M_3(G)+6Z_{2,1}(G)-m-3HM(G)+\overline{\eta_3}(G)-2(m+r-1)(F(G)+2M_2(G)+m-2M_1(G)+\overline{\eta_2}(G)) + M_1(G)(M^2+r^2+2mr-2m-2r+4)$.

COROLLARY 2.2. *Let G be a connected graph with n vertices, m edges and r blocks. Then*

(i) $\overline{F}(G^{+-}) = (m+r-1)(2M_2(G)+F(G)+\overline{\eta_2}(G)) - m(2r^2m+9m-6rm+42r-16r^2+r^3+3\overline{\eta_2}(G)-36) - M_1(G)(r^2-2mr+6m+16r+21) - M_3(G) - 6Z_{2,1}(G) - 3(r-3)HM(G) - M^3(r^2-2r-1) + \overline{\eta_3}(G)$.

(ii) $\overline{F}(\overline{G^{+-}}) = (m+r-1)^2(M_1(G)+2m(r-2)) - 2(m+r-1)(2M_2(G)+F(G)-\overline{\eta_2}(G)) - m(2m(r^2-6r+9)+2r^3+22r^2+3r+45) - M_1(G)(r^2+17r+3m+4rm-30) + m^3(1-r) + M_3(G) + 6Z_{2,1}(G) + 3(r-3)HM(G) - \overline{\eta_3}(G) - 2r^2+6r-4$.

COROLLARY 2.3. *Let G be a connected graph with n vertices, m edges and r blocks. Then*

(i) $\overline{F}(G^{-+}) = m(2m^3+7m^2+14m+r(M^2+4-2m)+4) + M_1(G)(M^2-2mr-8m-4r-16) + (m+r-1)(2M_2(G)+F(G)+\overline{\eta_2}(G)) + M_3(G) - 6z_{2,1}(G)3(m+2)HM(G) - \overline{\eta_3}(G)$.

(ii) $\overline{F}(\overline{G^{-+}}) = m(17M^2-9m+mr^2+12mr+3r^2-14r+19) - M_1(G)(6m^2+r^2+4mr-12m+2r+9) - 2(m+r-1)(2M_2(G)+F(G)+\overline{\eta_2}(G)) - M_3(G) + 6Z_{2,1}(G) + 3(m+2)HM(G) - \overline{\eta_3}(G)$.

COROLLARY 2.4. *Let G be a connected graph with n vertices, m edges and r blocks. Then*

(i) $\overline{F}(G^{--}) = m^3(-(r+4)) + m^2(2rm-r^2-r+2) + m(r^2m-r^2+3\overline{\eta_2}(G)) + M_1(G)(m^2+2m-r^2+2r-2rm) + (m+r-1)(2M_2(G)+F(G)+\overline{\eta_3}(G)+M_3(G)-6Z_{2,1}(G)-3(m+r)HM(G)-\overline{\eta_3}(G))$.

(ii) $\overline{F}(\overline{G^{--}}) = m^3(7m-2mr-2r^2+11r-6) + M^2(17r^2-20mr+32r) - m(2r^2m+r^3+2r^2) - 2(m+r-1)(2M_2(G)+F(G)+\overline{\eta_2}(G)) - M_1(G)(7m^2+14mr-4m+7r^2+4r) - M_3(G) + 6Z_{2,1}(G) + 3(m+r)HM(G) + 3m\overline{\eta_2}(G) - \overline{\eta_3}(G) + (m+r-1)(m(m+2r-1)-M_1(G)))$.

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Received by editors 16.02.2018; Revised version 12.03.2019 and 05.11.2019; Available online 18.11.2019.

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