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ECCENTRICITY RELATED INDICES OF CARTESIAN PRODUCT OF GRAPHS

K. Pattabiraman and T. Suganya

Abstract. Topological indices are useful tools provided by graph theory for theoretical study of chemical compounds. Topological indices play an important role in studying certain topological properties of chemical compounds especially organic materials. In this paper, we obtain upper bounds for eccentricity based topological indices such as adjacent eccentric distance, eccentric distance index, inverse degree, inverse total eccentricity and eccentric connective polynomial for Cartesian product of two graphs.

1. Introduction

All graphs considered in this paper are simple and connected. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The degree $d_G(u)$ of the vertex u is the number of edges incidence to u. The distance between the vertex u and vertex v in G denoted by $d_G(u,v)$ and $D_G(u) = \sum\limits_{u \in V(G)} d_G(u,v)$ is the sum of all distances from the vertex u in G. The *eccentricity* of a vertex v, denoted by $\epsilon_G(u)$, is the largest distance from v to any other vertex u of G.

A topological index is a numerical value associated with chemical constitution for correlation of chemical structure with various physical properties, chemical reactivity or biological activity. Quantitative structure- activity and structure-property relationships of chemical networks require expressions for the topological properties of these networks. Topological indices provide those expressions of topological properties. Quantitative structure-activity relationship models (QSAR models) are regression or classification models used in the chemical and biological science and

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control system engineering. One of the first historical QSAR chemical applications was to predict boiling points. Those numerical quantities which transform a chemical structure to a numerical number are called the topological indices.

The Wiener index is the first distance-based topological index which was introduced by Harold Wiener [**21**] in 1947. Wiener used his index for the calculation of boiling points of alkanes. The Wiener index of a graph G is denoted by $W(G)$ and defined as the sum of distances between all pairs of vertices in graph G, that is,

$$W(G) = \sum_{\{u,v\}\subseteq V(G)} d_G(u,v).$$

The Harary index of a connected graph G, denoted by $H(G)$, has been introduced independently by Plavšić et al. [**14**] and by Ivančić et al. [**11**] in 1993. It has been named in honor of Professor Frank Harary on the occasion of his 70th birthday. The Harary index of a connected graph G is defined as

$$H(G) = \sum_{\{u,v\}\subseteq V(G)} \frac{1}{d_G(u,v)}.$$

Sardana and Madan [**16**] introduced a novel topological descriptor called *adjacent eccentric distance sum index,* which is defined for a connected graph G as

$$\xi^{sv}(G) = \sum_{u\in V(G)} \frac{\epsilon_G(u)D_G(u)}{d_G(u)}.$$

To verify that this index has a vast potential for SAR, Sardana and Madan [**16, 18**] selected different data sets to investigate some properties of adjacent eccentric distance sum index.

Motivated with the above indices Malik [**12**] proposed other topological descriptors related to eccentricity as follows

(i) *The inverse total eccentricity* index of connected graph G defined as

$$\zeta^{-1}(G) = \sum_{u\in V(G)} \frac{1}{\epsilon_G(u)}.$$

(ii) *The inverse connective eccentricity index* of G, denoted by $\xi_{ce}^{-1}(G)$, is defined as

$$\xi_{ce}^{-1}(G) = \sum_{u\in V(G)} \frac{\epsilon_G(u)}{d_G(u)}.$$

(iii) The *inverse degree distance* $d^{-1}(G)$ of G is defined as

$$d^{-1}(G) = \sum_{u\in V(G)} \frac{D_G(u)}{d_G(u)}.$$

The eccentric connectivity polynomial is the polynomial version of the eccentric-connectivity index which was proposed by Ghorbani and Hemmasi [**8**] as

$$ECP(G,x) = \sum_{u\in V(G)} d_G(u)x^{\epsilon_G(u)}.$$

The total eccentricity polynomial [**1**] is the polynomial version of the total-eccentricity index which is defined as

$$TECP(G,x) = \sum_{u\in V(G)} x^{\epsilon_G(u)}.$$

It is easy to see that the total-eccentricity index can be obtained from the corresponding polynomial by evaluating its first derivative at $x = 1$.

Recently, a novel graph invariant for predicting biological and physical properties eccentric distance sum was introduced by Gupta, Singh and Madan [**10**]. The authors in [**10**] have shown that some structure activity and quantitative structure property studies using called eccentric distance sum were better than the corresponding values obtained using the Wiener index. *The eccentric distance sum* of G is defined as

$$\xi^d(G) = \sum_{u\in V(G)} \epsilon_G(u)D_G(u).$$

The eccentric distance sum of several graph products are obtain in [**2**].

A number of topological indices based on vertex eccentricity are already subject to various studies [**3, 4**]. The total eccentricity index of G is defined as

$$\zeta(G) = \sum_{u\in V(G)} \epsilon_G(u).$$

Similar to this index, Dankelmann et.al. [**5**] and Tang et al. [**20**] studied average eccentricity of graphs. Fathalikhani et al. in [**7**], studied total eccentricity of some graph operations.

The eccentric connectivity index [**19, 15, 9, 17**] of a graph G, denoted by $\xi_c(G)$, is defined as

$$\xi^c(G) = \sum_{u\in V(G)} \epsilon_G(u)d_G(u),$$

Inverse degree $ID(G)$ of G is defined as

$$ID(G) = \sum_{u\in V(G)} \frac{1}{d_G(u)}.$$

The inverse degree (also known as the sum of reciprocals of degrees) first attracted attention through numerous conjectures generated by the computer programme Graffiti [**6**].

In this sequence, we need to define some new indices which are modified forms of previously defined indices. *The eccentric distance index* $\xi^{ed}(G)$, *the degree eccentric index*ξ^{de} and *the distance degree index*$\zeta^d(G)$ of a graph G are defined as;

$$\xi^{ed}(G) = \sum_{u\in V(G)} \frac{\epsilon_G(u)d_G(u)}{D_G(u)},\ \ \xi^{de} = \sum_{u\in V(G)} \frac{\epsilon_G(u)}{D_G(u)},\ \ \zeta^d(G) = \sum_{u\in V(G)} \frac{d_G(u)}{D_G(u)}.$$

In this paper, we obtain upper bounds for eccentricity based topological indices such as adjacent eccentric distance, inverse degree, inverse total eccentricity and eccentric connective polynomial for Cartesian product of two graphs.

2. Main Results

Let G_1 and G_2 be two connected graphs. *The Cartesian product* of G_1 and G_2, denoted by $G_1\Box G_2$, is the graph with vertex set $V(G_1)\times V(G_2)$, and any two vertices (u_p, v_r) and (u_q, v_s) are adjacent if and only if

$$\Big[u_p = u_q \in V(G_1) \text{ and } v_r v_s \in E(G_2)\Big] \text{ or } \Big[v_r = v_s \in V(G_2) \text{ and } u_p u_q \in E(G_1)\Big]$$

and $p, q = 1, 2, ..., |V(G_1)|; r, s = 1, 2, ..., |V(G_2)|$. The following Lemma is appeared in [**13**] which can be used in the proof of our results.

LEMMA 2.1. *Let a and b be real number. Then $\frac{1}{a+b} \leqslant \frac{1}{4}\Big(\frac{1}{a}+\frac{1}{b}\Big)$ with equality if and only if $a = b$.*

From the structure of a Cartesian product of G_1 and G_2, we have the following Lemma.

LEMMA 2.2. *Let G_1 and G_2 be two connected graph. Then*
(i) $|V(G_1\Box G_2)| = |V(G_1)| \times |V(G_2)|$ and

$$|E(G_1\Box G_2)| = |V(G_1)|\,, \quad |E(G_2)| + |V(G_2)|\,|E(G_1)|\,.$$

(ii) The eccentricity of a vertex (u_1, u_2) in $G_1\Box G_2$ is

$$\epsilon_{G_1\Box G_2}((u_1, u_2)) = \epsilon_{G_1}(u_1) + \epsilon_{G_2}(u_2).$$

(iii) The degree of a vertex (u_1, u_2) in $G_1\Box G_2$ is

$$d_{G_1\Box G_2}((u_1, u_2)) = d_{G_1}(u_1) + d_{G_2}(u_2).$$

(iv) The distance sum of a vertex (u_1, u_2) in $G_1\Box G_2$ is

$$D_{G_1\Box G_2}((u_1, u_2)) = D_{G_1}(u_1) + D_{G_2}(u_2).$$

First we obtain the adjacent eccentric distance sum index of Cartesian product of two connected graphs G_1 and G_2.

THEOREM 2.1. *Let G_1 and G_2 be connected graphs with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$\xi^{sv}(G_1\Box G_2) \leqslant \frac{1}{4}\Big(\xi^{sv}(G_1)n_2 + \xi^{d}(G_1)ID(G_2) + 2W(G_2)\xi^{-1}_{ce}(G_1) + d^{-1}(G_2)\zeta(G_1) + d^{-1}(G_1)\zeta(G_2) + 2W(G_1)\xi^{-1}_{ce}(G_2) + \xi^{sv}(G_2)n_1 + \xi^{d}(G_2)ID(G_1)\Big).$$

PROOF. By the Definition of adjacent eccentric distance sum index,

$$\xi^{sv}(G_1\Box G_2) = \sum_{(u_1,u_2)\in V(G_1\Box G_2)} \frac{\epsilon_{G_1\Box G_2}((u_1, u_2))D_{G_1\Box G_2}((u_1, u_2))}{d_{G_1\Box G_2}(u_1, u_2)}$$

$$= \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)} \frac{(\epsilon_{G_1}(u_1) + \epsilon_{G_2}(u_2))(D_{G_1}(u_1) + D_{G_2}(u_2))}{(d_{G_1}(u_1) + d_{G_2}(u_2))},$$

by Lemma 2.2

$$\xi^{sv}(G_1\Box G_2) = \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\Big(\frac{\epsilon_{G_1}(u_1)D_{G_1}(u_1)}{d_{G_1}(u_1)+d_{G_2}(u_2)} + \frac{\epsilon_{G_1}(u_1)D_{G_2}(u_2)}{d_{G_1}(u_1)+d_{G_2}(u_2)}\Big)$$
$$+ \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\Big(\frac{\epsilon_{G_2}(u_2)D_{G_1}(u_1)}{d_{G_1}(u_1)+d_{G_2}(u_2)} + \frac{\epsilon_{G_2}(u_2)D_{G_2}(u_2)}{d_{G_1}(u_1)+d_{G_2}(u_2)}\Big).$$

By Lemma 2.1, we have $\frac{1}{d_{G_1}(u_1)+d_{G_{12}}(u_2)} \leqslant \frac{1}{4}\Big(\frac{1}{d_{G_1}(u_1)} + \frac{1}{d_{G_2}(u_2)}\Big)$ with equality if and only if $d_{G_1}(u_1) = d_{G_2}(u_2)$. Hence

$$\xi^{sv}(G_1\Box G_2) \leqslant \frac{1}{4}\Big[\sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_1}(u_1)D_{G_1}(u_1)}{d_{G_1}(u_1)}$$
$$+ \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_1}(u_1)D_{G_1}(u_1)}{d_{G_2}(u_2)} + \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_1}(u_1)D_{G_2}(u_2)}{d_{G_1}(u_1)}$$
$$+ \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_1}(u_1)D_{G_2}(u_2)}{d_{G_2}(u_2)} + \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_2}(u_2)D_{G_1}(u_1)}{d_{G_1}(u_1)}$$
$$+ \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_2}(u_2)D_{G_1}(u_1)}{d_{G_2}(u_2)} + \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_2}(u_2)D_{G_2}(u_2)}{d_{G_1}(u_1)}$$
$$+ \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_2}(u_2)D_{G_2}(u_2)}{d_{G_2}(u_2)}\Big].$$

and

$$\xi^{sv}(G_1\Box G_2) \tag{2.1}$$
$$\leqslant \frac{1}{4}\Big[n_2\sum_{u_1\in V(G_1)}\frac{\epsilon_{G_1}(u_1)D_{G_1}(u_1)}{d_{G_1}(u_1)} + \Big(\sum_{u_1\in V(G_1)}\epsilon_{G_1}(u_1)D_{G_1}(u_1)\Big)$$
$$\Big(\sum_{u_2\in V(G_2)}\frac{1}{d_{G_2}(u_2)}\Big) + \Big(\sum_{u_1\in V(G_1)}\frac{\epsilon_{G_1}(u_1)}{d_{G_1}(u_1)}\Big)\Big(\sum_{u_2\in V(G_2)}D_{G_2}(u_2)\Big)$$
$$+ \Big(\sum_{u_2\in V(G_2)}\frac{D_{G_2}(u_2)}{d_{G_2}(u_2)}\Big)\Big(\sum_{u_1\in V(G_1)}\epsilon_{G_1}(u_1)\Big) + \Big(\sum_{u_1\in V(G_1)}\frac{D_{G_1}(u_1)}{d_{G_1}(u_1)}\Big)$$
$$\Big(\sum_{u_2\in V(G_2)}\epsilon_{G_2}(u_2)\Big) + \Big(\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_2}(u_2)}{d_{G_2}(u_2)}\Big)\Big(\sum_{u_1\in V(G_1)}D_{G_1}(u_1)\Big)$$
$$+ \Big(\sum_{u_2\in V(G_2)}\epsilon_{G_2}(u_2)D_{G_2}(u_2)\Big)\Big(\sum_{u_1\in V(G_1)}\frac{1}{d_{G_1}(u_1)}\Big)$$
$$+ n_1\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_2}(u_2)D_{G_2}(u_2)}{d_{G_2}(u_2)}\Big].$$

From the definition of other graph indices, we obtain

$$\begin{aligned}\xi^{sv}(G\Box H) &\leqslant \frac{1}{4}\Big(\xi^{sv}(G_1)n_2+\xi^{d}(G_1)ID(G_2)+2W(G_2)\xi^{-1}_{ce}(G_1)\\ &+ d^{-1}(G_2)\zeta(G_1)+d^{-1}(G_1)\zeta(G_2)+2W(G_1)\xi^{-1}_{ce}(G_2)+\xi^{sv}(G_2)n_1\\ &+ \xi^{d}(G_2)ID(G_1)\Big).\end{aligned}$$

□

Next we compute the eccentric distance index of Cartesian product of G_1 and G_2.

THEOREM 2.2. *Let G_1 and G_2 be two connected graphs with $n_1, n_2 \geqslant 3$ vertices and m_1, m_2 edges, respectively. Then*
$\xi^{ed}(G_1\Box G_2) \leqslant \frac{1}{4}\Big(\xi^{ed}(G_1)n_2+2\xi^{c}(G_1)H(G_2)$
$+\ 2m_2\xi^{de}(G_1)+\zeta(G_1)\zeta^{d}(G_2)+\zeta(G_2)\zeta^{d}(G_1)+2m_1\xi^{de}(G_2)+2\xi^{c}(G_2)H(G_1)+$
$\xi^{ed}(G_2)n_1\Big)$.

PROOF. By the definition of eccentric distance index

$$\begin{aligned}\xi^{ed}(G_1\Box G_2) &= \sum_{(u_1,u_2)\in V(G_1\Box G_2)}\frac{\epsilon_{G_1\Box G_2}((u_1,u_2))d_{G_1\Box G_2}((u_1,u_2))}{D_{G_1\Box G_2}((u_1,u_2))}\\ &= \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{(\epsilon_{G_1}(u_1)+\epsilon_{G_2}(u_2))(d_{G_1}(u_1)+d_{G_2}(u_2))}{D_{G_1}(u_1)+D_{G_2}(u_2)},\end{aligned}$$

by Lemma 2.2 and

$$\begin{aligned}\xi^{ed}(G_1\Box G_2) &= \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_1}(u_1)d_{G_1}(u_1)}{D_{G_1}(u_1)+D_{G_2}(u_2)}\\ &+ \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_1}(u_1)d_{G_2}(u_2)}{D_{G_1}(u_1)+D_{G_2}(u_2)}\\ &+ \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_2}(u_2)d_{G_1}(u_1)}{D_{G_1}(u_1)+D_{G_2}(u_2)}\\ &+ \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_2}(u_2)d_{G_2}(u_2)}{D_{G_1}(u_1)+D_{G_2}(u_2)}.\end{aligned}$$

By Lemma 2.1, we have

$$\begin{aligned}
\xi^{ed}(G_1\Box G_2) &\leqslant \frac{1}{4}\Big(\sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_1}(u_1)d_{G_1}(u_1)}{D_{G_1}(u_1)}\\
&+ \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_1}(u_1)d_{G_1}(u_1)}{D_{G_2}(u_2)}\\
&+ \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_1}(u_1)d_{G_2}(u_2)}{D_{G_1}(u_1)}\\
&+ \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_1}(u_1)d_{G_2}(u_2)}{D_{G_2}(u_2)}\\
&+ \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_2}(u_2)d_{G_1}(u_1)}{D_{G_1}(u_1)}\\
&+ \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_2}(u_2)d_{G_1}(u_1)}{D_{G_2}(u_2)}\\
&+ \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_2}(u_2)d_{G_2}(u_2)}{D_{G_1}(u_1)}\\
&+ \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_2}(u_2)d_{G_2}(u_2)}{D_{G_2}(u_2)}\Big)
\end{aligned}$$

$$\begin{aligned}
\xi^{ed}(G_1\Box G_2) &\leqslant \frac{1}{4}\Big[n_2\sum_{u_1\in V(G_1)}\frac{\epsilon_{G_1}(u_1)d_{G_1}(u_1)}{D_{G_1}(u_1)} + \Big(\sum_{u_1\in V(G_1)}\epsilon_{G_1}(u_1)d_{G_1}(u_1)\Big)\\
&\Big(\sum_{u_2\in V(G_2)}\frac{1}{D_{G_2}(u_2)}\Big) + \Big(\sum_{u_1\in V(G_1)}\frac{\epsilon_{G_1}(u_1)}{D_{G_1}(u_1)}\Big)\Big(\sum_{u_2\in V(G_2)}d_{G_2}(u_2)\Big)\\
&+ \Big(\sum_{u_2\in V(G_2)}\frac{d_{G_2}(u_2)}{D_{G_2}(u_2)}\Big)\Big(\sum_{u_1\in V(G_1)}\epsilon_{G_1}(u_1)\Big) + \Big(\sum_{u_1\in V(G_1)}\frac{d_{G_1}(u_1)}{D_{G_1}(u_1)}\Big)\\
&\Big(\sum_{u_2\in V(G_2)}\epsilon_{G_2}(u_2)\Big) + \Big(\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_2}(u_2)}{D_{G_2}(u_2)}\Big)\Big(\sum_{u_1\in V(G_1)}d_{G_1}(u_1)\Big)\\
&+ \Big(\sum_{u_2\in V(G_2)}\epsilon_{G_2}(u_2)d_{G_2}(u_2)\Big)\Big(\sum_{u_1\in V(G_1)}\frac{1}{D_{G_1}(u_1)}\Big)\\
&+ n_1\sum_{u_2\in V(G_2)}\frac{\epsilon_{G_2}(u_2)d_{G_2}(u_2)}{D_{G_2}(u_2)}\Big]
\end{aligned}$$

and finally

$$\begin{aligned}\xi^{ed}(G_1\Box G_2) &\leqslant \frac{1}{4}\Big(\xi^{ed}(G_1)n_2+2\xi^{c}(G_1)H(G_2)+2m_2\xi^{de}(G_1)+\zeta(G_1)\zeta^{d}(G_2)\\ &+ \zeta(G_2)\zeta^{d}(G_1)+2m_1\xi^{de}(G_2)+2\xi^{c}(G_2)H(G_1)+\xi^{ed}(G_2)n_1\Big).\end{aligned}$$

□

Now, we find the inverse degree distance of Cartesian product of G_1 and G_2.

THEOREM 2.3. *Let G_1 and G_2 be connected graph with n_1, n_2 vertices and m_1, m_2 edges, respectively. Then*

$$d^{-1}(G_1\Box G_2)\leqslant \frac{1}{4}\Big(d^{-1}(G_1)n_2+d^{-1}(G_2)n_1\Big)+\frac{1}{2}\Big(W(G_2)ID(G_1)+W(G_1)ID(G_2)\Big).$$

PROOF. By the definition of inverse degree distance index,

$$\begin{aligned}d^{-1}(G_1\Box G_2) &= \sum_{(u_1,u_2)\in V(G_1\Box G_2)}\frac{D_{G_1\Box G_2}((u_1,u_2))}{d_{G_1\Box G_2}((u_1,u_2))}\\ &= \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\Big(\frac{D_{G_1}(u_1)+D_{G_2}(u_2)}{d_{G_1}(u_1)+d_{G_2}(u_2)}\Big),\end{aligned}$$

by Lemma 2.2. By Lemma 2.1, we have

$$\begin{aligned}d^{-1}(G_1\Box G_2) &\leqslant \frac{1}{4}\Big(\sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\Big(\frac{D_{G_1}(u_1)+D_{G_2}(u_2)}{d_{G_1}(u_1)}\\ &+ \frac{D_{G_1}(u_1)+D_{G_2}(u_2)}{d_{G_2}(u_2)}\Big)\Big)\\ &\leqslant \frac{1}{4}\Big(d^{-1}(G_1)n_2+d^{-1}(G_2)n_1\Big)\\ &+ \frac{1}{2}\Big(W(G_2)ID(G_1)+W(G_1)ID(G_2)\Big).\end{aligned}$$

□

Next we obtain an upper bound of inverse connective eccentricity index of Cartesian product of G_1 and G_2.

THEOREM 2.4. *Let G_1 and G_2 be connected graph with n_1 and n_2 vertices, respectively. Then*

$$\xi_{ce}^{-1}(G_1\Box G_2)\leqslant \frac{1}{4}\Big(n_2\xi_{ce}^{-1}(G_1)+\zeta(G_2)ID(G_1)+\zeta(G_1)ID(G_2)+n_1\xi_{ce}^{-1}(G_2)\Big).$$

PROOF. By the definition of inverse connective eccentricity index,

$$\begin{aligned}\xi_{ce}^{-1}(G_1\Box G_2) &= \sum_{(u_1,u_2)\in V(G_1\Box G_2)}\frac{\epsilon_{G_1\Box G_2}((u_1,u_2))}{d_{G_1\Box G_2}((u_1,u_2))}\\ &= \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\Big(\frac{\epsilon_{G_1}(u_1)+\epsilon_{G_2}(u_2)}{d_{G_1}(u_1)+d_{G_2}(u_2)}\Big),\end{aligned}$$

by Lemma 2.2. By Lemma 2.1, we have

$$\begin{aligned}\xi_{ce}^{-1}(G_1\Box G_2) &\leqslant \frac{1}{4}\Big(\sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\Big(\frac{\epsilon_G(u_1)+\epsilon_{G_2}(u_2)}{d_{G_1}(u_1)}\\ &+ \frac{\epsilon_{G_1}(u_1)+\epsilon_{G_2}(u_2)}{d_{G_2}(u_2)}\Big)\Big)\\ &\leqslant \frac{1}{4}\Big(n_2\xi_{ce}^{-1}(G_1)+\zeta(G_2)ID(G_1)+\zeta(G_1)ID(G_2)\\ &+ n_1\xi_{ce}^{-1}(G_2)\Big).\end{aligned}$$

□

Finally, we obtain the upper bounds for some Cartesian product of related indices of $G_1\Box G_2$.

THEOREM 2.5. *Let G_1 and G_2 be two connected graphs. Then*
(i) $\zeta^{-1}(G_1\Box G_2)\leqslant \frac{1}{4}\Big(n_2\zeta^{-1}(G_1)+n_1\zeta^{-1}(G_2)\Big)$.
(ii) $ECP(G_1\Box G_2)=ECP(G_1,x)TECP(G_2,x)+ECP(G_2,x)TECP(G_1,x)$.
(iii) $TECP(G_1\Box G_2)=TECP(G_1,x)TECP(G_2,x)$.

PROOF. By the definition of inverse total eccentricity index,

$$\begin{aligned}(i)\ \zeta^{-1}(G_1\Box G_2) &= \sum_{(u_1,u_2)\in V(G_1\Box G_2)}\frac{1}{\epsilon_{G_1\Box G_2}((u_1,u_2))}\\ &= \sum_{(u_1,u_2)\in V(G_1\Box G_2)}\Big(\frac{1}{\epsilon_{G_1}(u_1)+\epsilon_{G_2}(u_2)}\Big),\quad \text{by Lemma 2.2.}\end{aligned}$$

By Lemma 2.1, we have

$$\frac{1}{\epsilon_{G_1}(u_1)+\epsilon_{G_2}(u_2)}\leqslant\frac{1}{4}\Big(\frac{1}{\epsilon_{G_1}(u_1)}+\frac{1}{\epsilon_{G_2}(u_2)}\Big)$$

with equality holds if and only if $\epsilon_{G_1}(u_1)=\epsilon_{G_2}(u_2)$.

$$\begin{aligned}\zeta^{-1}(G_1\Box G_2) &\leqslant \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\frac{1}{4}\Big(\frac{1}{\epsilon_{G_1}(u_1)}+\frac{1}{\epsilon_{G_2}(u_2)}\Big)\\ &= \frac{1}{4}\Big(n_2\zeta^{-1}(G_1)+n_1\zeta^{-1}(G_2)\Big).\end{aligned}$$

(ii) By the definition of eccentric connectivity polynomial,

$$\begin{aligned}ECP((G_1\Box G_2),x) &= \sum_{(u_1,u_2)\in V(G_1\Box G_2)} d_{G_1\Box G_2}x^{\epsilon_{G_1\Box G_2}((u_1,u_2))}\\ &= \sum_{(u_1,u_2)\in V(G_1\Box G_2)}(d_{G_1}(u_1)+d_{G_2}(u_2))x^{\epsilon_{G_1}(u_1)+\epsilon_{G_2}(u_2)},\end{aligned}$$

by Lemma 2.2

$$\begin{aligned} ECP((G_1\Box G_2),x) &= \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)}\Big(d_{G_1}(u_1)x^{\epsilon_{G_1}(u_1)+\epsilon_{G_2}(u_2)} \\ &+ d_{G_2}(u_2)x^{\epsilon_{G_1}(u_1)+\epsilon_{G_2}(u_2)}\Big) \\ &= \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)} d_{G_1}(u_1)x^{\epsilon_{G_1}(u_1)}x^{\epsilon_{G_2}(u_2)} \\ &+ \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)} d_{G_2}(u_2)x^{\epsilon_{G_1}(u_1)}x^{\epsilon_{G_2}(u_2)} \end{aligned}$$

Finally

$$ECP((G_1\Box G_2),x) = ECP(G_1,x)TECP(G_2,x) + ECP(G_2,x)TECP(G_1,x).$$

(iii) By the definition of total eccentric polynomial,

$$\begin{aligned} TECP((G_1\Box G_2),x) &= \sum_{(u_1,u_2)\in V(G_1\Box G_2)} x^{\epsilon_{G_1\Box G_2}(u_1,u_2)} \\ &= \sum_{(u_1,u_2)\in V(G_1\Box G_2)} x^{\epsilon_{G_1}(u_1)+\epsilon_{G_2}(u_2)}, \end{aligned}$$

by Lemma 2.2 and

$$\begin{aligned} TECP((G_1\Box G_2),x) &= \sum_{u_1\in V(G_1)}\sum_{u_2\in V(G_2)} x^{\epsilon_{G_1}(u_1)}x^{\epsilon_{G_2}(u_2)} \\ &= \Big(\sum_{u_1\in V(G_1)} x^{\epsilon_{G_1}(u_1)}\Big)\Big(\sum_{u_2\in V(G_2)} x^{\epsilon_{G_2}(u_2)}\Big) \end{aligned}$$

Finally

$$TECP((G_1\Box G_2),x) = TECP(G_1,x)TECP(G_2,x).$$

□

3. Conclusion

In this paper, we study some eccentricity based indices such as adjacent eccentric distance, eccentric distance index, inverse degree, inverse total eccentricity and eccentric connective polynomial for Cartesian product of two given graphs. Further, we will study and develop these indices of some other graph operations.

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Department of Mathematics, Annamalai University, Annamalainagar 608 002, India
E-mail address: pramank@gmail.com and suganyatpr@gmail.com