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MINIMAL PRIME S-IDEALS IN 0-DISTRIBUTIVE ALMOST SEMILATTICES

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Abstract. Obtained necessary and sufficient conditions for a prime S-ideal to become minimal prime S-ideal in terms of filters in 0-distributive almost semilattices. Some properties of minimal prime S-ideals in 0-distributive almost semilattices are established. Also, proved that the set $\mathscr{B}(L)$, of all annihilator S-ideals of a 0-distributive ASL is a complete Boolean algebra. Finally, we characterized the minimal prime annihilator S-ideals in a 0-distributive ASL L.

1. Introduction

The concept of minimal prime ideal was put to advantage by Kist [**2**] by investigating commutative semigroups. Later, Thakare and Pawar [**7**] obtained some properties of minimal prime ideals in 0-distributive semilattices. They characterized minimal prime ideals in a 0-distributive semilattices in terms of maximal filters. Also, they provided useful tools for established properties of minimal prime ideals. The concept of S-ideals and prime S-ideals in almost semilattice (ASL) was introduced by Nanaji Rao, Swapna, Terefe [**3**] and established the set of all principal S-ideals in ASL form a semilattice. Also, they introduced the concept of annihilator and annihilator ideal and proved several results on annihilator ideals [**4**]. The concept of S-ideals and prime S-ideals in almost semilattice (ASL) was introduced by Nanaji Rao, Swapna, Terefe [**5**] and established the set of all principal S-ideals in ASL form a semilattice. Also, they proved an isomorphism of the semilattice $SI(L)$ of all S-ideals in an ASL L onto the semilattice of all ideals of a semilattice $PSI(L)$, moreover, this isomorphism gives one-to-one correspondence between the

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prime S-ideals of L and those of $PSI(L)$. Later, the concept of 0-distributive almost semilattice (0-distributive ASL) was introduced by Nanaji Rao and Swapna [**6**] and proved some basic properties of 0-distributive almost semilattices.

In this paper, we obtained necessary and sufficient conditions for a prime S-ideal to become minimal prime S-ideal. Certain basic properties of minimal prime S-ideals in 0-distributive almost semilattices are established. Derived a set of identities for a prime S-ideal to become minimal prime S-ideal. Next, we introduce the concept of annihilator S-ideal and proved that the set $\mathscr{B}(L)$, of all annihilator S-ideals of a 0-distributive almost semilattice is a complete Boolean algebra. Finally, we derived a set of identities for any nonempty subset A of a 0-distributive ASL L, the annihilator S-ideal A^* to become minimal prime annihilator S-ideal.

2. Preliminaries

In this section we collect a few important definitions and results which are already known and which will be used more frequently in the text.

Definition 2.1. An ASL with 0 is an algebra $(L, \circ, 0)$ of type $(2, 0)$ satisfies the following conditions:
1. $(x \circ y) \circ z = x \circ (y \circ z)$
2. $(x \circ y) \circ z = (y \circ x) \circ z$
3. $x \circ x = x$
4. $0 \circ x = 0,\ for\ all\ x, y, z \in L.$

Definition 2.2. Let L be an ASL. A nonempty subset I of L is said to be an S-ideal if it satisfies the following conditions:
1) If $x \in I\ and\ a \in L$, then $x \circ a \in I$.
2) If $x, y \in I$, then there exists $d \in I$ such that $d \circ x = x, d \circ y = y$.

Definition 2.3. Let L be an ASL and $a \in L$. Then $(a] = \{a \circ x : x \in L\}$ is an S-ideal of L and is called principal S-ideal generated by a.

Definition 2.4. A nonempty subset F of an ASL L is said to be a *filter* if F satisfies the following conditions:
(1) $x,\ y \in F$ implies $x \circ y \in F$
(2) If $x \in F\ and\ a \in L\ such\ that\ a \circ x = x, then\ a \in F$

Definition 2.5. A proper S-ideal P of an ASL L is said to be a prime S-ideal if for any $x, y \in L, x \circ y \in P$ imply $x \in P\ or\ y \in P$.

Definition 2.6. A proper filter F of L is said to be a prime filter if for any filters $F_1\ and\ F_2$ of L, $F_1 \cap F_2 \subseteq F$ imply $F_1 \subseteq F\ or\ F_2 \subseteq F$.

Definition 2.7. A proper filter F of L is said to be *maximal* if for any filter G of L such that $F \subseteq G \subseteq L$, then either $F = G$ or $G = L$.

Definition 2.8. Let L be an ASL with 0. Then L is said to be 0-distributive ASL if for any $x, y, z \in L,\ x \circ y = 0$ and $x \circ z = 0$ then there exists $d \in L$ such that $d \circ y = y,\ d \circ z = z$ and $d \circ x = 0$.

DEFINITION 2.9. Let L be an ASL with 0. Then for any nonempty subset A of L, $A^* = \{x \in L : x \circ a = 0 \text{ for all } a \in A\}$ is called the annihilator of A, and is denoted by A^*.

Note that if $A = \{a\}$, then we denote $A^* = \{a\}^*$ by $[a]^*$.

THEOREM 2.1. *Let L be an ASL with 0. Then for any nonempty subsets I, J of L, we have the following.*

(1) $I^* = \bigcap_{a \in I} [a]^*$
(2) $(I \cap J)^* = (J \cap I)^*$
(3) $I \subseteq J \implies J^* \subseteq I^*$
(4) $I^* \cap J^* \subseteq (I \cap J)^*$
(5) $I \subseteq I^{**}$
(6) $I^{***} = I^*$
(7) $I^* \subseteq J^* \Leftrightarrow J^{**} \subseteq I^{**}$
(8) $I \cap J = (0] \Leftrightarrow I \subseteq J^* \Leftrightarrow J \subseteq I^*$
(9) $(I \cup J)^* = I^* \cap J^*$

THEOREM 2.2. *Let L be an ASL with 0. Then for any $x, y \in L$, we have the following.*

(1) $x \leqslant y \Rightarrow [y]^* \subseteq [x]^*$
(2) $[x]^* \subseteq [y]^* \Rightarrow [y]^{**} \subseteq [x]^{**}$
(3) $x \in [x]^{**}$
(4) $(x]^* = [x]^*$
(5) $(x] \cap [x]^* = \{0\}$
(6) $[x \circ y]^* = [y \circ x]^*$
(7) $[x]^* \cap [y]^* \subseteq [x \circ y]^*$
(8) $[x \circ y]^{**} = [x]^{**} \cap [y]^{**}$
(9) $[x]^{***} = [x]^*$
(10) $[x]^* \subseteq [y]^* \Leftrightarrow [y]^{**} \subseteq [x]^{**}$

THEOREM 2.3. *Let L be an ASL with 0. A proper filter M of L is maximal if and only if for any $a \in L - M$, there exists $b \in M$ such that $a \circ b = 0$.*

THEOREM 2.4. *Let L be an ASL with 0, in which intersection of any family of S-ideals is again an S-ideal. Then the following are equivalent:*

(1) *L is 0-distributive ASL.*
(2) *A^* is an S-ideal, for all $A(\neq \emptyset) \subseteq L$.*
(3) *$SI(L)$ is pseudo-complemented semilattice.*
(4) *$SI(L)$ is 0-distributive semilattice.*
(5) *$PSI(L)$ is 0-distributive semilattice.*

THEOREM 2.5. *Let L be 0-distributive ASL. Then every maximal filter of L is a prime filter.*

DEFINITION 2.10. An element a in an ASL L with 0 is said to be dense element if $[a]^* = \{0\}$.

Note that the set of all dense elements in an ASL with 0 is denoted by D.

DEFINITION 2.11. Let $(P, \leqslant)$ be a poset. Then P is said to satisfy ascending chain condition (acc) if every ascending chain in P is terminate.

DEFINITION 2.12. An element x in a semilattice L is said to be meet-prime if $for\ any\ a, b \in L, a \wedge b \leqslant x$ implies $a \leqslant x\ or\ b \leqslant x$.

DEFINITION 2.13. Let L be a lattice with greatest element 1. An element $a \in L$ is said to be a dual atom if a is covered by 1.

THEOREM 2.6. *Let $(L, \vee, \wedge)$ be a lattice. Then for any $x, y, z \in L$, the following conditions are equivalent:*
(1) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$
(2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(3) $(x \vee y) \wedge z \leqslant x \vee (y \wedge z)$.

THEOREM 2.7. *If P is a partial ordered set bounded above each of whose non-viod subsets R has an infimum, then each non-void subset P will have a supremum, too, and by the definitions $\bigcap R = inf(R), \bigcup R = sup(R)$, then P becomes a complete lattice.*

DEFINITION 2.14. A complemented distributive lattices is called Boolean algebra.

3. Minimal Prime S-ideals

It is well-known that a minimal prime ideal in a semilattice S is a minimal element in the set of all prime ideals of S. Analogy, a minimal prime S-ideal in an ASL L is a minimal element in the set of all prime S-ideal of L. In this section, we derive a necessary and sufficient conditions for a prime S-ideal to become a minimal prime S-ideal in a 0-distributive ASL. Also, we prove some basic properties of minimal prime S-ideals in a 0-distributive ASL. Obtain a set of identities for a prime S-ideal to become a minimal prime S-ideal. First, we begin with the following.

LEMMA 3.1. *Let L be an ASL. Then a subset P of L is a prime S-ideal if and only if $L - P$ is a prime filter.*

PROOF. Suppose P is a prime S-ideal of L. Now, we shall prove that $L - P$ is a prime filter. Clearly $L - P$ is a nonempty proper subset of L. Now, let $x, y \in L - P$. Then $x, y \notin P$. Since P is prime, $x \circ y \notin P$. Thus $x \circ y \in L - P$. Let $x \in L - P\ and\ t \in L\ such\ that\ t \circ x = x$. Now, if $t \notin L - P$, then $t \in P$ and hence $x = t \circ x \in P$, a contradiction. Therefore $t \in L - P$. Thus $L - P$ is a filter. Now, suppose $F_1,\ F_2$ are filters of L such that $F_1 \nsubseteq L - P$ and $F_2 \nsubseteq L - P$. Then choose $a \in F_1\ such\ that\ a \notin L - P$ and $b \in F_2\ such\ that\ b \notin L - P$. Therefore $a, b \in P$. Since P is an S-ideal, there exists $d \in P\ such\ that\ d \circ a = a,\ d \circ b = b$. It follows that $d \in F_1\ and\ d \in F_2$. Hence $d \in F_1 \cap F_2$ and also $d \notin L - P$. Hence $F_1 \cap F_2 \nsubseteq L - P$. Thus $L - P$ is a prime filter.

Conversely, suppose $L - P$ is a prime filter. Now, we shall prove that P is a prime S-ideal of L. Since $L - P$ is nonempty proper subset of L, P is a nonempty proper subset of L. Now, let $x \in P$ *and* $t \in L$. Then $x \notin L-P$. Now, if $x \circ t \in L-P$, then $t \circ x \in L-P$ and hence $x \in L-P$, a contradiction. Therefore $x \circ t \notin L-P$. Thus $x \circ t \in P$. Now, let $x, y \in P$. Then $x, y \notin L-P$. It follows that $[x) \nsubseteq L-P, [y) \nsubseteq L-P$. Hence $[x) \cap [y) \nsubseteq L-P$. Thus there exists $z \in [x) \cap [y)$ *such that* $z \notin L-P$. Since $z \in [x)$, $z \circ x = x$. Similarly, we get $z \circ y = y$. Therefore $z \in P$ such that $z \circ x = x$ *and* $z \circ y = y$. Thus P is an S-ideal. Now, let $x, y \in L$ *such that* $x, y \notin P$. Then $x, y \in L-P$ *and hence* $x \circ y \in L-P$ *since* $L-P$ *is a filter*. Therefore $x \circ y \notin P$. Thus P is a prime S-ideal. □

It is well-known that every proper filter of an ASL L is contained in a maximal filter and hence every non-zero element is contained in a maximal filter. In the following we give necessary and sufficient condition for a subset of a 0-distributive ASL to become a minimal prime S-ideal.

Theorem 3.1. *Let L be a 0-distributive ASL. Then a subset M of L is a minimal prime S-ideal if and only if $L-M$ is a maximal filter.*

Proof. Suppose M is a minimal prime S-ideal of L. Now, we shall prove that $L-M$ is a maximal filter. Since M is a prime S-ideal, by Lemma 3.1, $L-M$ is a prime filter. Therefore there exists a maximal filter (say) F such that $L-M$ is contained in a maximal filter F. Since L is 0-distributive, F is a prime filter. It follows that $L-F$ is a prime S-ideal which is contained in M. Since M is minimal prime S-ideal, $L-F = M$. Hence $L-M$ is a maximal filter.

Conversely, suppose $L-M$ is a maximal filter in L. Since L is 0-distributive, $L-M$ is a prime filter and hence by Lemma 3.1, M is a prime S-ideal. Suppose J is a prime S-ideal of L such that $J \subsetneqq M$. Then $L-M$ is filter which properly contained in a proper filter $L-J$, a contradiction. Thus M is a minimal prime S-ideal. □

Corollary 3.1. *Let L be a 0-distributive ASL. Then every prime S-ideal contains a minimal prime S-ideal.*

Proof. Suppose P is a prime S-ideal of L. Then by Lemma 3.1, $L-P$ is a proper filter. Hence there exists a maximal filter H of L such that $L-P$ is contained in H. It follows that P contains a minimal prime S-ideal $L-H$. □

In the following theorem we derive necessary and sufficient conditions for a prime S-ideal to become minimal prime S-ideal.

Theorem 3.2. *Let L be a 0-distributive ASL. Then a prime S-ideal M of L is minimal if and only if $[x]^* - M \neq \emptyset$ for any $x \in M$.*

Proof. Suppose M is a minimal prime S-ideal of L and $x \in M$. Then by Theorem 3.1, $L-M$ is a maximal filter. Since $x \in L-(L-M)$, there exists $y \in L-M$ such that $x \circ y = 0$. Hence $y \in [x]^*$ *and* $y \notin M$. Thus $[x]^* - M \neq \emptyset$.

Conversely, assume the condition. Now, we shall prove that M is minimal prime S-ideal. Let $z \notin L-M$. Then $z \in M$ and hence $[z]^* - M \neq \emptyset$. Hence choose $y \in [z]^*$ such that $y \notin M$. Thus there exists $y \in L-M$ such that $y \circ z = 0$.

Therefore $L-M$ is a maximal filter. Hence by Theorem 3.1, M is a minimal prime S-ideal. □

Corollary 3.2. *Let L be a 0-distributive ASL. Then a prime S-ideal M of L is minimal if and only if it contains precisely one of $\{x\}, [x]^*$ for every $x \in L$.*

Proof. Suppose M is a minimal prime S-ideal in L and $x \in L$. Now, if $x \in M$, then by Theorem 3.2, $[x]^* - M \neq \emptyset$. Therefore there exists $t \in [x]^*$ such that $t \notin M$. Hence $[x]^* \nsubseteq M$. Suppose $[x]^* \subseteq M$ and suppose $x \in M$. Then $x \notin L-M$ and hence $x \in L-(L-M)$. Since M is a minimal prime S-ideal, $L-M$ is a maximal filter. Therefore there exists $y \in L-M$ such that $x \circ y = 0$. It follows that $y \in [x]^*$ and $y \notin M$. Hence $[x]^* \nsubseteq M$, a contradiction. Therefore $x \notin M$.

Conversely, assume the condition. Let $y \in L-(L-M)$. Then $y \in M$. Therefore by assumption, $[y]^* \nsubseteq M$. Hence there exists $z \in [y]^*$ *such that* $z \notin M$. Therefore $z \in L-M$ such that $y \circ z = 0$. Hence $L-M$ is a maximal filter. It follows by Theorem 3.1, M is a minimal prime S-ideal. □

Corollary 3.3. *Let L be a 0-distributive ASL, in which intersection of any family of S-ideals is again an S-ideal and let M be a minimal prime S-ideal. Then $x \in M$ if and only if $[x]^{**} \subseteq M$.*

Proof. Suppose M is a minimal prime S-ideal in L and $x \in M$. Then by Theorem 3.1, $L-M$ is a maximal filter. Since $x \notin L-M$, there exists $y \in L-M$ such that $x \circ y = 0$ and hence $y \in [x]^*$. Suppose $[x]^{**} \nsubseteq M$. Then there exists $z \in [x]^{**}$ such that $z \notin M$ and hence $z \in L-M$. Since $L-M$ is a filter, $y \circ z \in L-M$. On the other hand, since $y \in [x]^*$, $y \circ z \in [x]^*$. Similarly, $y \circ z \in [x]^{**}$. It follows that $y \circ z \in [x]^* \cap [x]^{**} = \{0\}$. Hence $y \circ z = 0$. Therefore $0 = y \circ z \in L-M$. Hence $L-M = L$, a contradiction to $L-M$ is a maximal filter. Thus $[x]^{**} \subseteq M$.

Converse is clear, since $x \in [x]^{**}$. □

Recall that I^* is the pseudo-complement of an S-ideal I in the semilattice $SI(L)$, of all S-ideals in a 0-distributive ASL. Also, note that the set of all minimal prime S-ideals in 0-distributive ASL is denoted by $\mathfrak{M}$. In the following we characterize the pseudo-complement I^* of an S-ideal I in 0-distributive ASL.

Theorem 3.3. *Let L be a 0-distributive ASL, in which intersection of any family of S-ideals is again an S-ideal. Then for any S-ideal I of L, I^* is the intersection of all minimal prime S-ideals not containing I.*

Proof. Suppose I is an S-ideal of L. Then we have, $I \cap I^* = (0]$. Now, let M be a minimal prime S-ideal of L. Since $I \cap I^* = (0] \subseteq M$, either $I \subseteq M$ or $I^* \subseteq M$. Therefore $I^* \subseteq \cap\{M \in \mathfrak{M} : I \subsetneq M\}$. Suppose $I^* \subset \cap\{M \in \mathfrak{M} : I \subsetneq M\}$. Then there exists $x \in \cap\{M \in \mathfrak{M} : I \subsetneq M\}$ such that $x \notin I^*$. Therefore there exists $y \in I$ such that $x \circ y \neq 0$. Hence $[x \circ y)$ is a proper filter. Therefore there exists a maximal filter (say) F of L such that $[x \circ y) \subseteq F$. This implies $x \circ y \in F$. Since $x \circ y \leqslant y$, $y \in F$. It follows that $y \notin L-F$. Hence $I \nsubseteq L-F$. Since $L-F$ is a minimal prime S-ideal, $\cap\{M \in \mathfrak{M} : I \subsetneq M\} \subseteq L-F$. On the other hand, we have $x \circ y \in F$ and hence $y \circ x \in F$ and $y \circ x \leqslant x$. Hence $x \in F$, a contradiction to $x \in L-F$. Therefore $\cap\{M \in \mathfrak{M} : I \subsetneq M\} = I^*$. □

Next, we introduce the concept of an annihilator S-ideal in an ASL with 0 and characterize annihilator S-ideals in terms of minimal prime S-ideals.

DEFINITION 3.1. Let L be an ASL with 0. Then an S-ideal I of L is said to be an annihilator S-ideal if $I = A^*$ for some nonempty subset A of L.

It can be easily seen that if I is an annihilator S-ideal in an ASL L with 0 then $I = I^{**}$.

EXAMPLE 3.1. Let $L = \{0, a, b, c\}$ and define a binary operation $\circ$ on L as follows:

∘	0	a	b	c
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
c	0	a	b	c

Then clearly $(L, \circ)$ is an ASL with 0. Now, put $I = \{0, a\}$. Then clearly $I = I^{**}$. Hence I is an annihilator S-ideal.

In the following we characterize annihilator S-ideals in terms of minimal prime S-ideals.

THEOREM 3.4. *Let L be a 0-distributive ASL, in which intersection of any family of S-ideals is again an S-ideal. Then any annihilator S-ideal I of L is the intersection of all minimal prime S-ideals containing it.*

PROOF. Suppose I is an annihilator S-ideal in a 0-distributive ASL L. Then $I = I^{**}$. Therefore by Theorem 3.3, we have $(I^*)^* = \cap\{M \in \mathfrak{M} : I^* \nsubseteq M\}$. Since $I \cap I^* = (0] \subseteq M$ and M is prime, $I \subseteq M$. It follows that

$$I = (I^*)^* = \cap\{M \in \mathfrak{M} : I \subseteq M\}.$$

□

COROLLARY 3.4. *Let L be a 0-distributive ASL, in which intersection of any family of S-ideals is again an S-ideal. Then a principal S-ideal of L is an annihilator S-ideal if and only if it is the intersection of all minimal prime S-ideals containing it.*

PROOF. Suppose $I = (a]$ is an annihilator S-ideal. Then $(a] = (a]^{**}$. Therefore by Theorem 3.3,

$$((a]^*]^* = \cap\{M \in \mathfrak{M} : (a]^* \nsubseteq M\} = \cap\{M \in \mathfrak{M} : (a] \subseteq M\}.$$

Conversely, assume the condition. Now, we shall prove that every principal S-ideal of L is an annihilator S-ideal. Let $I = (a]$ be a principal S-ideal of L. Then $(a] = \cap\{M \in \mathfrak{M} : (a] \subseteq M\}$. Consider

$$((a]^*]^* = \cap\{M \in \mathfrak{M} : (a]^* \nsubseteq M\} = \cap\{M \in \mathfrak{M} : (a] \subseteq M\} = (a].$$

Therefore $(a] = (a]^{**}$. Thus $(a]$ is an annihilator S-ideal. □

COROLLARY 3.5. *The intersection of all minimal prime S-ideals of a* 0*-distributive ASL is* $\{0\}$.

In the following we introduce the concept of dense S-ideal in 0-distributive ASL. An interesting property of non-dense S-ideal in a 0-distributive ASL is investigated in the following.

DEFINITION 3.2. An S-ideal I in a 0-distributive ASL L is called *dense* if $I^* = \{0\}$.

EXAMPLE 3.2. Let $L = \{0, a, b, c\}$ and define a binary operation $\circ$ on L as follows:

∘	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
c	0	a	b	c

Then clearly $(L, \circ)$ is an ASL with 0. Now, put $I = \{0, a\}$. Then clearly, I is an S-ideal and also $I^* = \{0\}$. Therefore I is dense.

THEOREM 3.5. *Any non-dense ideal of a* 0*-distributive ASL is contained in a minimal prime S-ideal and the converse is true for principal S-ideals.*

PROOF. Suppose I is a non-dense S-ideal of a 0-distributive ASL L. Then $I^* \neq \{0\}$. Therefore we can choose $t \in I^*$ *such that* $t \neq 0$. Hence there exists a maximal filter F such that $t \in F$. This implies $t \notin L - F$. Since $L - F$ is a minimal prime S-ideal and $I^* \nsubseteq L - F$, $I \subseteq L - F$.

Suppose $(a]$ is contained in a minimal prime S-ideal (say) M in L. This implies $a \in M$. Since $a \in M$, $[a]^* \nsubseteq M$. Hence $(a]^* = [a]^* \neq \{0\}$. Thus $(a]$ is non-dense. □

Recall that an element x in an ASL L with 0 is called dense element if $[x]^* = \{0\}$. Now, we prove the following.

THEOREM 3.6. *Let L be a* 0*-distributive ASL. Then an element in L belongs to some minimal prime S-ideal of L if and only if it is non-dense.*

PROOF. Suppose $x \in L$ such that x is in a minimal prime S-ideal (say) M of L. Then by Corollary 3.2, $[x]^* \nsubseteq M$. Hence $[x]^* \neq \{0\}$. Therefore x is non-dense element.

Converse is clear. □

In the following, we derive a set of identities for a prime S-ideal to become minimal prime S-ideal.

THEOREM 3.7. *Let L be a* 0*-distributive ASL. Then the following are equivalent:*

(1) *Every prime S-ideal is minimal prime.*
(2) *Every prime filter is minimal prime.*
(3) *Every prime filter is maximal.*

PROOF. (1) $\Rightarrow$ (2): Assume (1). Now, we shall prove that every prime filter is minimal prime. Suppose F is a prime filter and suppose F is not minimal. Then there exists a prime filter F_1 such that $F_1 \subset F$. This implies $L - F$ is contained in $L - F_1$ and both $L - F$, $L - F_1$ are prime S-ideals. But, by (1), $L - F_1$ is minimal prime and $L - F$ is contained in $L - F_1$, a contradiction. Therefore F is minimal prime filter.

(2) $\Rightarrow$ (3): Assume (2). Now, we shall prove that every prime filter is maximal. Suppose H is a prime filter and suppose H is not maximal. Then there exists a maximal filter M such that H is contained in M. Since L is 0-distributive, M is a prime filter. Hence by (2), M is a minimal prime filter. It follows that $H = M$. Thus H is maximal.

(3) $\Rightarrow$ (1): Assume (3). Now, we shall prove that every prime S-ideal is minimal prime. Suppose P is a prime S-ideal in L. Suppose Q is a prime S-ideal of L such that $Q \subseteq P$. Then $L - P \subseteq L - Q$ and we have both $L - P$ and $L - Q$ are prime filters. Hence by (3), we get $Q = P$. Thus P is minimal prime S-ideal. □

Finally, we give a necessary and sufficient condition for a prime S-ideal to become minimal prime S-ideal.

THEOREM 3.8. *A prime S-ideal P is a minimal prime S-ideal in a 0-distributive ASL L if and only if P consists precisely of all elements $x \in L$ such that $x \circ y = 0$ for some $y \notin P$.*

PROOF. Suppose P is a minimal prime S-ideal in L and $x \in P$. Then by Theorem 3.2, $[x]^* - P \neq \emptyset$. Hence we can choose $y \in [x]^*$ such that $y \notin P$. It follows that $x \circ y = 0$ *and* $y \notin P$. Suppose $z \in L$ such that $z \circ x = 0$ for some $x \notin P$. Then $z \circ x = 0 \in P$. Since P is prime, $z \in P$. Thus P consists precisely of all elements $x \in L$ such that $x \circ y = 0$ for some $y \notin P$.

Conversely, assume the condition. Suppose Q is a prime S-ideal of L such that $Q \subseteq P$. Suppose $Q \subset P$. Then there exists $x \in P$ such that $x \notin Q$. Therefore by assumption, there exists $y \notin P$ such that $x \circ y = 0$. Since $x \circ y = 0 \in Q$ *and* Q is prime, $x \in Q$ *or* $y \in Q$. It follows that $y \in Q \subset P$. Hence $y \in P$, a contradiction to $y \notin P$. Therefore $Q = P$. Thus P is minimal. □

4. Minimal prime annihilator S-ideal

Recall that if L is a 0-distributive ASL in which intersection of any family of S-ideals is again an S-ideal then for any nonempty subset A of L, A^* is an S-ideal and also, clearly A^* is an annihilator S-ideal (since $A^* = A^{***}$). Let L be a 0-distributive ASL in which intersection of any family of S-ideals is again an S-ideal. Denote by $\mathscr{B}(L)$ the set $\{A^* : \emptyset \neq A \subseteq L\}$. Thus $\mathscr{B}(L)$ is the set of all annihilator S-ideals in L. In this section, we prove that $\mathscr{B}(L)$ is a complete Boolean algebra. Also, derive a set of identities for any nonempty subset A of a 0-distributive ASL, in which intersection of any family of S-ideals is again an S-ideal, the annihilator S-ideal A^* to become a minimal prime annihilator S-ideal. First, we prove that $\mathscr{B}(L)$ is a complete Boolean algebra. For this we need the following.

LEMMA 4.1. *Let L be an ASL with 0. Then for any S-ideals I, J of L,*

$$(I \cap J)^{**} = I^{**} \cap J^{**}.$$

PROOF. Let $I, J \in SI(L)$. Then we have $I \cap J \subseteq I, J$. Hence by Theorem 2.1(3), we get $I^*, J^* \subseteq (I \cap J)^*$. It follows that $(I \cap J)^{**} \subseteq I^{**}, J^{**}$. Thus $(I \cap J)^{**} \subseteq I^{**} \cap J^{**}$.

Conversely, let $x \in I^{**} \cap J^{**}$ *and* $y \in (I \cap J)^*$. Then for any $i \in I$ *and* $j \in J$, we have $i \circ j \in I \cap J$. Hence $(y \circ i) \circ j = y \circ (i \circ j) = 0$. Therefore $y \circ i \in J^*$. Again, since $x \in J^{**}$ *and* $y \circ i \in J^*$, we get $(x \circ y) \circ i = x \circ (y \circ i) = 0$. Hence $x \circ y \in I^*$. Since $x \in I^{**}$, we get $x \circ y \in I^{**}$. Thus $x \circ y \in I^* \cap I^{**} = \{0\}$. Hence $x \circ y = 0$. Therefore $x \in (I \cap J)^{**}$. Thus $I^{**} \cap J^{**} \subseteq (I \cap J)^{**}$. Hence $(I \cap J)^{**} = I^{**} \cap J^{**}$. □

COROLLARY 4.1. *If $\{I_i \mid i \in \Delta\}$ is a family of S-ideals of L, then*

$$(\bigcap_{i \in \Delta} I_i)^{\star\star} = \bigcap_{i \in \Delta} (I_i)^{\star\star}.$$

THEOREM 4.1. *Let L be a 0-distributive ASL,in which intersection of any family of S-ideals is again an S-ideal. Then the set $\mathscr{B}(L)$, of all annihilator S-ideals of L is a complete Boolean algebra.*

PROOF. Clearly, the set $\mathscr{B}(L)$, of all annihilator S-ideals of a 0-distributive ASL L is nonempty, since $\{0\}$, $L \in \mathscr{B}(L)$. Also, clearly, $\mathscr{B}(L)$ is a poset with respect to set inclusion. Now, for any $A^*, B^* \in \mathscr{B}(L)$, define $A^* \wedge B^* = A^* \cap B^*$ and $A^* \vee B^* = (A^{**} \cap B^{**})^*$. Then clearly $\wedge, \vee$ are binary operations on $\mathscr{B}(L)$ and also, clearly $(\mathscr{B}(L), \vee, \wedge)$ is a bounded lattice with bounds $\{0\}$ *and* L. Let $A^* \in \mathscr{B}(L)$. Then we have $A^{**} \in \mathscr{B}(L)$. Now, $A^* \wedge A^{**} = A^* \cap A^{**} = \{0\}$ *and* $A^* \vee A^{**} = (A^{**} \cap A^{***})^* = \{0\}^* = L$. Thus $\mathscr{B}(L)$ is a complemented lattice. Finally, we shall prove that $\mathscr{B}(L)$ is a distributive lattice. That is, enough to prove that for any $A^*, B^*, C^* \in \mathscr{B}(L)$, $(A^* \vee B^*) \wedge C^* \subseteq A^* \vee (B^* \wedge C^*)$. We have $A^* \cap C^* \cap [A^{**} \cap (B^* \cap C^*)^*] = (0]$. It follows that $C^* \cap [A^{**} \cap (B^* \cap C^*)^*] \subseteq A^{**}$. Again, we have $B^* \cap C^* \cap [A^{**} \cap (B^* \cap C^*)^*] = (0]$. Therefore $C^* \cap [A^{**} \cap (B^* \cap C^*)^*] \subseteq B^{**}$. Hence $C^* \cap [A^{**} \cap (B^* \cap C^*)^*] \subseteq A^{**} \cap B^{**}$. It follows that $(C^* \cap (A^{**} \cap (B^* \cap C^*)^*)) \cap (A^{**} \cap B^{**})^* = (0]$. Hence we get $(A^{**} \cap B^{**})^* \cap C^* \subseteq (A^{**} \cap (B^* \cap C^*)^*)^*$. Therefore $(A^* \vee B^*) \wedge C^* \subseteq A^* \vee (B^* \wedge C^*)$. Therefore by Theorem 2.6, $\mathscr{B}(L)$ is a distributive lattice. Then $\mathscr{B}(L)$ is a Boolean Algebra. Also, by Theorem 2.7, and by Corollary 4.1, $\mathscr{B}(L)$ is a complete Boolean algebra. □

In the following we establish a set of identities which characterize minimal prime annihilator S-ideals in 0-distributive ASLs. For, this first we need the following.

LEMMA 4.2. *Let L be a 0-distributive ASL, in which intersection of any family of S-ideals is again an S-ideal. Then an annihilator S-ideal A^* is a prime S-ideal in L if and only if A^* is a dual atom in $\mathscr{B}(L)$.*

PROOF. Suppose A^* is a prime S-ideal in L. Now, we shall prove that A^* is a dual atom in $\mathscr{B}(L)$. Suppose $A^* \subseteq B^*$ and suppose $B^* \neq L$. Then there exists $s \in L$ such that $s \notin B^*$. Hence $s \circ b \neq 0$ *for some* $b(\neq 0) \in B$. Now, let $c \in B^*$. Then $c \circ b = 0 \in A^*$. Therefore either $c \in A^*$ *or* $b \in A^*$ since A^* is a prime S-ideal.

Suppose $b \in A^*$. Then $b \in B^*$ and also $b \in B$ and hence $b \circ b = 0$. Therefore $b = 0$, a contradiction. Hence $b \notin A^*$. It follows that $c \in A^*$. Hence $B^* \subseteq A^*$. Therefore $A^* = B^*$. Thus A^* is a dual atom.

Conversely, suppose A^* is a dual atom in $\mathscr{B}(L)$. Now, we shall prove that A^* is a prime S-ideal. Clearly A^* is an S-ideal. Again, since A^* is a dual atom, $A^* \neq L$. Therefore there exists $s \in L$ such that $s \notin A^*$. This implies $s \circ a \neq 0$ *for some* $a(\neq 0) \in A$. It follows that $[a]^* \neq L$. Since $a \in A$, $A^* \subseteq [a]^* \neq L$. Therefore $A^* = [a]^*$ since A^* is a dual atom. Suppose $x, y \in L$ such that $x \circ y \in [a]^*$ and suppose $x \notin [a]^*$. Since $x \circ a \leqslant a$, $[a]^* \subseteq [x \circ a]^*$. Again, since $[a]^* = A^*$, which is a dual atom, either $[a]^* = [x \circ a]^*$ *or* $[x \circ a]^* = L$. Suppose $[x \circ a]^* = L$. Then $x \in L = [x \circ a]^*$. It follows that $x \circ (x \circ a) = 0$. Hence $x \circ a = 0$. Therefore $x \in [a]^*$, a contradiction to $x \notin [a]^*$. Hence $[a]^* = [x \circ a]^*$. Now, since $x \circ y \in [a]^*$, $(x \circ y) \circ a = 0$. It follows that $y \in [x \circ a]^* = [a]^*$. Therefore $y \in [a]^*$. Thus $A^* = [a]^*$ is a prime S-ideal. □

It is well-known that, in a Boolean algebra B, an element a is meet-prime if and only if it is a dual atom. Hence we have the following.

Corollary 4.2. *Let L be a 0-distributive ASL, in which intersection of any family of S-ideals is again an S-ideal. Then an annihilator S-ideal A^* is prime in L if and only if A^* is a meet-prime element of $\mathscr{B}(L)$.*

Lemma 4.3. *Let L be a 0-distributive ASL, in which intersection of any family of S-ideals is again an S-ideal. Then every prime annihilator S-ideal is minimal prime in L.*

Proof. Suppose P is an annihilator prime S-ideal. Then $P = A^*$ for some $A(\neq \emptyset) \subseteq L$. Suppose $A = \{0\}$. Then $A^* = [0]^* = L$. Hence $P = L$, a contradiction to P is a prime S-ideal. Therefore $A \neq \{0\}$. Now, let $x \in A^*$. Then $x \circ a = 0$ *for all* $a \in A$. It follows that $a \in [x]^*$ for all $a \in A$. Therefore $A \subseteq [x]^*$. Now, we shall prove that $[x]^* - A^* \neq \emptyset$. Let $a(\neq 0) \in A$. Then $a \in [x]^*$. Suppose $a \in A^*$. Then $a \circ a = 0$. It follows that $a = 0$, a contradiction to $a \neq 0$. Hence $a \notin A^*$. Therefore $A \subseteq [x]^* - A^*$. Hence $[x]^* - A^* \neq \emptyset$. It follows that $[x]^* - P \neq \emptyset$. Thus by Theorem 3.2, P is a minimal prime S-ideal. □

Theorem 4.2. *Let L be a 0-distributive ASL, in which intersection of any family of S-ideals is again an S-ideal. Then for any nonempty subset A of L, the following are equivalent:*

(1) *A^* is a dual atom in $\mathscr{B}(L)$.*
(2) *A^* is a meet-prime element of $\mathscr{B}(L)$.*
(3) *A^* is a minimal prime annihilator S-ideal.*
(4) *A^* is a prime annihilator S-ideal.*

It is well-known that if a Boolean algebra B satisfies ascending chain condition(acc) then B is finite. Thus there will be only finite number of dual atoms in B when it satisfies acc. In accordance with this observation and by Lemma 4.3, we have the following.

LEMMA 4.4. *Let L be a 0-distributive ASL, in which intersection of any family of S-ideals is again an S-ideal. Then L contains a finite family of minimal prime S-ideals with intersection $\{0\}$ when $\mathscr{B}(L)$ satisfies ascending chain condition.*

PROOF. Suppose $\mathscr{B}(L)$ satisfies ascending chain condition and suppose A_1^*,....., A_n^* are dual atoms in $\mathscr{B}(L)$. Since A_i^* is a dual atom, we get $A_i^* = [a_i]^*$ for some $a_i(\neq 0) \in A_i$ for $i = 1, 2, 3, ..., n$. Let $x(\neq 0) \in \bigcap_{i=1}^{n} A_i^*$. Then by Theorem 3.2 in [**1**], $[x]^* \subseteq A_j^*$ for j, $1 \leqslant j \leqslant n$. Since $x \in A_j^* = [a_j]^*$, $x \circ a_j = 0$. This implies $a_j \in [x]^* \subseteq A_j^*$. hence we get $a_j = 0$, a contradiction to $a_i's$ are non-zero. Thus $\bigcap_{i=1}^{n} A_i^* = \{0\}$. □

THEOREM 4.3. *Let L be a 0-distributive ASL, in which intersection of any family of S-ideals is again an S-ideal and let $\mathscr{B}(L)$ satisfies ascending chain condition. Then the set complement of union of dual atoms in $\mathscr{B}(L)$ is the set of all dense elements of L.*

PROOF. Suppose $\mathscr{B}(L)$ satisfies ascending chain condition. Then we have $\mathscr{B}(L)$ is finite. Suppose $A_1^*, ..., A_n^*$ are distinct dual atoms in L and suppose $x \in L - \bigcup_{i=1}^{n} A_i^*$. Now, we shall prove that $[x]^* = \{0\}$. Suppose $[x]^* \neq \{0\}$. Then there exists $y(\neq 0) \in L$ such that $x \circ y = 0$. This implies $x \in [y]^*$. Since $[y]^* \neq L$, it follows that $[y]^* \subseteq A_j^*$ *for some* $j \leqslant n$. Hence $x \in \bigcup_{i=1}^{n} A_j^*$, a contradiction. Therefore $[x]^* = \{0\}$. Hence x is a dense element. Conversely, suppose $x \notin L - \bigcup_{i=1}^{n} A_i^*$. Then $x \in \bigcup_{i=1}^{n} A_i^*$. Therefore $x \in A_j^*$ *for some* $j \leqslant n$. Put $A_j^* \circ (\bigcap_{i \neq j} A_i^*) = \{x \circ y : x \in A_j^*, y \in \bigcap_{i \neq j} A_i^*\}$. Let $x \circ y \in A_j^* \circ (\bigcap_{i \neq j} A_i^*)$. Then $x \in A_j^*$ *and* $y \in \bigcap_{i \neq j} A_i^*$. This implies $x \circ y \in A_j^*$ *and* $x \circ y \in \bigcap_{i \neq j} A_i^*$. It follows that $x \circ y \in A_j^* \circ (\bigcap_{i \neq j} A_i^*)$. Hence $x \circ y \in \bigcap_{i=1}^{n} A_i^*$. Therefore $A_j^* \circ (\bigcap_{i \neq j} A_i^*) \subseteq \bigcap_{i=1}^{n} A_i^*$. Since by Lemma 4.4, $\bigcap_{i=1}^{n} A_i^* = \{0\}, A_j^* \circ \bigcap_{i \neq j} A_i^* = \{0\}$. Since A_i^*'s are distinct, there exists $y(\neq 0) \in \bigcap_{i \neq j} A_i^*$ such that $y \circ x = 0$. Therefore $y \in [x]^*$. Hence $[x]^* \neq \{0\}$. Thus x is non-dense. Therefore the set complement of union of dual atoms in $\mathscr{B}(L)$ is the set of all dense elements of L. □

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