MINIMAL PRIME S-IDEALS IN
0-DISTRIBUTIVE ALMOST SEMILATTICES

G. Nanaji Rao and Ch. Swapna

Abstract. Obtained necessary and sufficient conditions for a prime S-ideal to become minimal prime S-ideal in terms of filters in 0-distributive almost semilattices. Some properties of minimal prime S-ideals in 0-distributive almost semilattices are established. Also, proved that the set $\mathcal{F}(L)$, of all annihilator S-ideals of a 0-distributive ASL is a complete Boolean algebra. Finally, we characterized the minimal prime annihilator S-ideals in a 0-distributive ASL $L$.

1. Introduction

The concept of minimal prime ideal was put to advantage by Kist [2] by investigating commutative semigroups. Later, Thakare and Pawar [7] obtained some properties of minimal prime ideals in 0-distributive semilattices. They characterized minimal prime ideals in a 0-distributive semilattices in terms of maximal filters. Also, they provided useful tools for established properties of minimal prime ideals. The concept of $S$-ideals and prime $S$-ideals in almost semilattice (ASL) was introduced by Nanaji Rao, Swapna, Terefe [3] and established the set of all principal $S$-ideals in ASL form a semilattice. Also, they introduced the concept of annihilator and annihilator ideal and proved several results on annihilator ideals [4]. The concept of $S$-ideals and prime $S$-ideals in almost semilattice (ASL) was introduced by Nanaji Rao, Swapna, Terefe [5] and established the set of all principal $S$-ideals in ASL form a semilattice. Also, they proved an isomorphism of the semilattice $SI(L)$ of all $S$-ideals in an ASL $L$ onto the semilattice of all ideals of a semilattice $PSI(L)$, moreover, this isomorphism gives one-to-one correspondence between the

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prime $S$-ideals of $L$ and those of $PSI(L)$. Later, the concept of 0-distributive almost semilattice (0-distributive ASL) was introduced by Nanaji Rao and Swapna [6] and proved some basic properties of 0-distributive almost semilattices.

In this paper, we obtained necessary and sufficient conditions for a prime $S$-ideal to become minimal prime $S$-ideal. Certain basic properties of minimal prime $S$-ideals in 0-distributive almost semilattices are established. Derived a set of identities for a prime $S$-ideal to become minimal prime $S$-ideal. Next, we introduce the concept of annihilator $S$-ideal and proved that the set $B(L)$, of all annihilator $S$-ideals of a 0-distributive almost semilattice is a complete Boolean algebra. Finally, we derived a set of identities for any nonempty subset $A$ of a 0-distributive ASL $L$, the annihilator $S$-ideal $A^*$ to become minimal prime annihilator $S$-ideal.

2. Preliminaries

In this section we collect a few important definitions and results which are already known and which will be used more frequently in the text.

**Definition 2.1.** An ASL with 0 is an algebra $(L, \circ, 0)$ of type $(2,0)$ satisfies the following conditions:

1. $(x \circ y) \circ z = x \circ (y \circ z)$
2. $(x \circ y) \circ z = (y \circ x) \circ z$
3. $x \circ x = x$
4. $0 \circ x = 0$, for all $x, y, z \in L$.

**Definition 2.2.** Let $L$ be an ASL. A nonempty subset $I$ of $L$ is said to be an $S$-ideal if it satisfies the following conditions:

1) If $x \in I$ and $a \in L$, then $x \circ a \in I$.
2) If $x, y \in I$, then there exists $d \in I$ such that $d \circ x = x, d \circ y = y$.

**Definition 2.3.** Let $L$ be an ASL and $a \in L$. Then $(a) = \{a \circ x : x \in L\}$ is an $S$-ideal of $L$ and is called principal $S$-ideal generated by $a$.

**Definition 2.4.** A nonempty subset $F$ of an ASL $L$ is said to be a filter if $F$ satisfies the following conditions:

1) $x, y \in F$ implies $x \circ y \in F$
2) If $x \in F$ and $a \in L$ such that $a \circ x = x$, then $a \in F$.

**Definition 2.5.** A proper $S$-ideal $P$ of an ASL $L$ is said to be a prime $S$-ideal if for any $x, y \in L, x \circ y \in P$ imply $x \in P$ or $y \in P$.

**Definition 2.6.** A proper filter $F$ of $L$ is said to be a prime filter if for any filters $F_1$ and $F_2$ of $L$, $F_1 \cap F_2 \subseteq F$ imply $F_1 \subseteq F$ or $F_2 \subseteq F$.

**Definition 2.7.** A proper filter $F$ of $L$ is said to be maximal if for any filter $G$ of $L$ such that $F \subseteq G \subseteq L$, then either $F = G$ or $G = L$.

**Definition 2.8.** Let $L$ be an ASL with 0. Then $L$ is said to be 0-distributive ASL if for any $x, y, z \in L, x \circ y = 0$ and $x \circ z = 0$ then there exists $d \in L$ such that $d \circ y = y, d \circ z = z$ and $d \circ x = 0$. 
Definition 2.9. Let \( L \) be an ASL with 0. Then for any nonempty subset \( A \) of \( L \), \( A^* = \{ x \in L : x \circ a = 0 \text{ for all } a \in A \} \) is called the annihilator of \( A \), and is denoted by \( A^* \).

Note that if \( A = \{ a \} \), then we denote \( A^* = \{ a \}^* \) by \( [a]^* \).

Theorem 2.1. Let \( L \) be an ASL with 0. Then for any nonempty subsets \( I, J \) of \( L \), we have the following.

1. \( I^* = \bigcap_{a \in I} [a]^* \)
2. \( (I \cap J)^* = (J \cap I)^* \)
3. \( I \subseteq J \implies J^* \subseteq I^* \)
4. \( I^* \cap J^* \subseteq (I \cap J)^* \)
5. \( I \subseteq J^* \)
6. \( I^** = I^* \)
7. \( I^* \subseteq J^* \iff J** \subseteq I^* \)
8. \( I \cap J = \emptyset \iff I \subseteq J^* \iff J \subseteq I^* \)
9. \( (I \cup J)^* = I^* \cap J^* \)

Theorem 2.2. Let \( L \) be an ASL with 0. Then for any \( x, y \in L \), we have the following.

1. \( x \leq y \implies [y]^* \subseteq [x]^* \)
2. \( [x]^* \subseteq [y]^* \implies [y]^{**} \subseteq [x]^{**} \)
3. \( x \in [x]^{**} \)
4. \( (x)^* = [x]^* \)
5. \( (x) \cap [x]^* = \emptyset \)
6. \( [x \circ y]^* = [y \circ x]^* \)
7. \( [x]^* \cap [y]^* \subseteq [x \circ y]^* \)
8. \( [x \circ y]^{**} = [x]^{**} \cap [y]^{**} \)
9. \( [x]^{***} = [x]^* \)
10. \( [x]^* \subseteq [y]^* \iff [y]^{**} \subseteq [x]^{**} \)

Theorem 2.3. Let \( L \) be an ASL with 0. A proper filter \( M \) of \( L \) is maximal if and only if for any \( a \in L - M \), there exists \( b \in M \) such that \( a \circ b = 0 \).

Theorem 2.4. Let \( L \) be an ASL with 0, in which intersection of any family of \( S \)-ideals is again an \( S \)-ideal. Then the following are equivalent:

1. \( L \) is 0-distributive ASL.
2. \( A^* \) is an \( S \)-ideal, for all \( A(\neq \emptyset) \subseteq L \).
3. \( SI(L) \) is pseudo-complemented semilattice.
4. \( SI(L) \) is 0-distributive semilattice.
5. \( PSI(L) \) is 0-distributive semilattice.

Theorem 2.5. Let \( L \) be 0-distributive ASL. Then every maximal filter of \( L \) is a prime filter.

Definition 2.10. An element \( a \) in an ASL \( L \) with 0 is said to be dense element if \( [a]^* = \{0\} \).

Note that the set of all dense elements in an ASL with 0 is denoted by \( D \).
Definition 2.11. Let \((P, \leq)\) be a poset. Then \(P\) is said to satisfy ascending chain condition (acc) if every ascending chain in \(P\) is terminate.

Definition 2.12. An element \(x\) in a semilattice \(L\) is said to be meet-prime if for any \(a, b \in L\), \(a \land b \leq x\) implies \(a \leq x\) or \(b \leq x\).

Definition 2.13. Let \(L\) be a lattice with greatest element 1. An element \(a \in L\) is said to be a dual atom if \(a\) is covered by 1.

Theorem 2.6. Let \((L, \lor, \land)\) be a lattice. Then for any \(x, y, z \in L\), the following conditions are equivalent:

1. \(x \lor (y \land z) = (x \lor y) \land (x \lor z)\)
2. \(x \land (y \lor z) = (x \land y) \lor (x \land z)\)
3. \((x \lor y) \land z \leq x \lor (y \land z)\).

Theorem 2.7. If \(P\) is a partial ordered set bounded above each of whose non-void subsets \(R\) has an infimum, then each non-void subset \(P\) will have a supremum, too, and by the definitions \(\bigcap R = \inf(R), \bigcup R = \sup(R)\), then \(P\) becomes a complete lattice.

Definition 2.14. A complemented distributive lattices is called Boolean algebra.

3. Minimal Prime \(S\)-ideals

It is well-known that a minimal prime ideal in a semilattice \(S\) is a minimal element in the set of all prime ideals of \(S\). Analogy, a minimal prime \(S\)-ideal in an ASL \(L\) is a minimal element in the set of all prime \(S\)-ideal of \(L\). In this section, we derive a necessary and sufficient conditions for a prime \(S\)-ideal to become a minimal prime \(S\)-ideal in a 0-distributive ASL. Also, we prove some basic properties of minimal prime \(S\)-ideals in a 0-distributive ASL. Obtain a set of identities for a prime \(S\)-ideal to become a minimal prime \(S\)-ideal. First, we begin with the following.

Lemma 3.1. Let \(L\) be an ASL. Then a subset \(P\) of \(L\) is a prime \(S\)-ideal if and only if \(L - P\) is a prime filter.

Proof. Suppose \(P\) is a prime \(S\)-ideal of \(L\). Now, we shall prove that \(L - P\) is a prime filter. Clearly \(L - P\) is a nonempty proper subset of \(L\). Now, let \(x, y \in L - P\). Then \(x, y \notin P\). Since \(P\) is prime, \(x \lor y \notin P\). Thus \(x \lor y \in L - P\). Let \(x \in L - P\) and \(t \in L\) such that \(t \circ x = x\). Now, if \(t \notin L - P\), then \(t \in P\) and hence \(x = t \circ x \in P\), a contradiction. Therefore \(t \in L - P\). Thus \(L - P\) is a filter. Now, suppose \(F_1, F_2\) are filters of \(L\) such that \(F_1 \nsubseteq L - P\) and \(F_2 \nsubseteq L - P\). Then choose \(a \in F_1\) such that \(a \notin L - P\) and \(b \in F_2\) such that \(b \notin L - P\). Therefore \(a, b \in P\). Since \(P\) is an \(S\)-ideal, there exists \(d \in P\) such that \(d \circ a = a\), \(d \circ b = b\). It follows that \(d \in F_1\) and \(d \in F_2\). Hence \(d \in F_1 \cap F_2\) and also \(d \notin L - P\). Hence \(F_1 \cap F_2 \nsubseteq L - P\). Thus \(L - P\) is a prime filter.

Conversely, suppose \(L - P\) is a prime filter. Now, we shall prove that \(P\) is a prime \(S\)-ideal of \(L\). Since \(L - P\) is nonempty proper subset of \(L\), \(P\) is a nonempty
proper subset of $L$. Now, let $x \in P$ and $t \in L$. Then $x \notin L - P$. Now, if $x \circ t \in L - P$, then $t \circ x \in L - P$ and hence $x \in L - P$, a contradiction. Therefore $x \circ t \notin L - P$. Thus $x \circ t \in P$. Now, let $x, y \in P$. Then $x, y \notin L - P$. It follows that $[x] \notin L - P, [y] \notin L - P$. Hence $[x] \cap [y] \notin L - P$. Thus there exists $z \in [x] \cap [y]$ such that $z \notin L - P$. Since $z \in [x], z \circ x = x$. Similarly, we get $z \circ y = y$. Therefore $z \in P$ such that $z \circ x = x$ and $z \circ y = y$. Thus $P$ is an $S$-ideal. Now, let $x, y \in L$ such that $x, y \notin P$. Then $x, y \in L - P$ and hence $x \circ y \in L - P$ since $L - P$ is a filter. Therefore $x \circ y \notin P$. Thus $P$ is a prime $S$-ideal. □

It is well-known that every proper filter of an ASL $L$ is contained in a maximal filter and hence every non-zero element is contained in a maximal filter. In the following we give necessary and sufficient condition for a subset of a 0-distributive ASL to become a minimal prime $S$-ideal.

**Theorem 3.1.** Let $L$ be a 0-distributive ASL. Then a subset $M$ of $L$ is a minimal prime $S$-ideal if and only if $L - M$ is a maximal filter.

**Proof.** Suppose $M$ is a minimal prime $S$-ideal of $L$. Now, we shall prove that $L - M$ is a maximal filter. Since $M$ is a prime $S$-ideal, by Lemma 3.1, $L - M$ is a prime filter. Therefore there exists a maximal filter (say) $F$ such that $L - M$ is contained in a maximal filter $F$. Since $L$ is 0-distributive, $F$ is a prime filter. It follows that $L - F$ is a prime $S$-ideal which is contained in $M$. Since $M$ is minimal prime $S$-ideal, $L - F = M$. Hence $L - M$ is a maximal filter.

Conversely, suppose $L - M$ is a maximal filter in $L$. Since $L$ is 0-distributive, $L - M$ is a prime filter and hence by Lemma 3.1, $M$ is a prime $S$-ideal. Suppose $J$ is a prime $S$-ideal of $L$ such that $J \subseteq M$. Then $L - M$ is filter which properly contained in a proper filter $L - J$, a contradiction. Thus $M$ is a minimal prime $S$-ideal. □

**Corollary 3.1.** Let $L$ be a 0-distributive ASL. Then every prime $S$-ideal contains a minimal prime $S$-ideal.

**Proof.** Suppose $P$ is a prime $S$-ideal of $L$. Then by Lemma 3.1, $L - P$ is a proper filter. Hence there exists a maximal filter $H$ of $L$ such that $L - P$ is contained in $H$. It follows that $P$ contains a minimal prime $S$-ideal $L - H$. □

In the following theorem we derive necessary and sufficient conditions for a prime $S$-ideal to become minimal prime $S$-ideal.

**Theorem 3.2.** Let $L$ be a 0-distributive ASL. Then a prime $S$-ideal $M$ of $L$ is minimal if and only if $[x]^* - M \neq \emptyset$ for any $x \in M$.

**Proof.** Suppose $M$ is a minimal prime $S$-ideal of $L$ and $x \in M$. Then by Theorem 3.1, $L - M$ is a maximal filter. Since $x \in L - (L - M)$, there exists $y \in L - M$ such that $x \circ y = 0$. Hence $y \in [x]^*$ and $y \notin M$. Thus $[x]^* - M \neq \emptyset$.

Conversely, assume the condition. Now, we shall prove that $M$ is minimal prime $S$-ideal. Let $z \notin L - M$. Then $z \in M$ and hence $[z]^* - M \neq \emptyset$. Hence choose $y \in [z]^*$ such that $y \notin M$. Thus there exists $y \in L - M$ such that $y \circ z = 0$. □
Therefore $L - M$ is a maximal filter. Hence by Theorem 3.1, $M$ is a minimal prime $S$-ideal.

**Corollary 3.2.** Let $L$ be a 0-distributive ASL. Then a prime $S$-ideal $M$ of $L$ is minimal if and only if it contains precisely one of $\{x\}, [x]^*$ for every $x \in L$.

**Proof.** Suppose $M$ is a minimal prime $S$-ideal in $L$ and $x \in L$. Now, if $x \in M$, then by Theorem 3.2, $[x]^* - M \neq \emptyset$. Therefore there exists $t \in [x]^*$ such that $t \notin M$. Hence $[x]^* \not\subseteq M$ and suppose $x \in M$. Then $x \notin L - M$ and hence $x \in L - (L - M)$. Since $M$ is a minimal prime $S$-ideal, $L - M$ is a maximal filter. Therefore there exists $y \in L - M$ such that $x \circ y = 0$. It follows that $y \in [x]^*$ and $y \notin M$. Hence $[x]^* \not\subseteq M$, a contradiction. Therefore $x \notin M$.

Conversely, assume the condition. Let $y \in L - (L - M)$. Then $y \in M$. Therefore by assumption, $[y]^* \not\subseteq M$. Hence there exists $z \in [y]^*$ such that $z \notin M$. Therefore $z \in L - M$ such that $y \circ z = 0$. Hence $L - M$ is a maximal filter. It follows by Theorem 3.1, $M$ is a minimal prime $S$-ideal.

**Corollary 3.3.** Let $L$ be a 0-distributive ASL, in which intersection of any family of $S$-ideals is again an $S$-ideal and let $M$ be a minimal prime $S$-ideal. Then $x \in M$ if and only if $[x]^{**} \subseteq M$.

**Proof.** Suppose $M$ is a minimal prime $S$-ideal in $L$ and $x \in M$. Then by Theorem 3.1, $L - M$ is a maximal filter. Since $x \notin L - M$, there exists $y \in L - M$ such that $x \circ y = 0$ and hence $y \in [x]^*$. Suppose $[x]^{**} \not\subseteq M$. Then there exists $z \in [x]^{**}$ such that $z \notin M$ and hence $z \in L - M$. Since $L - M$ is a filter, $y \circ z \in L - M$. On the other hand, since $y \in [x]^*$, $y \circ z \in [x]^*$. Similarly, $y \circ z \in [x]^{**}$. It follows that $y \circ z \in [x]^* \cap [x]^{**} = \{0\}$. Hence $y \circ z = 0$. Therefore $0 = y \circ z \in L - M$. Hence $L - M = L$, a contradiction to $L - M$ is a maximal filter. Thus $[x]^{**} \not\subseteq M$.

Converse is clear, since $x \in [x]^{**}$.

Recall that $I^*$ is the pseudo-complement of an $S$-ideal $I$ in the semilattice $SI(L)$, of all $S$-ideals in a 0-distributive ASL. Also, note that the set of all minimal prime $S$-ideals in 0-distributive ASL is denoted by $\mathfrak{M}$. In the following we characterize the pseudo-complement $I^*$ of an $S$-ideal $I$ in 0-distributive ASL.

**Theorem 3.3.** Let $L$ be a 0-distributive ASL, in which intersection of any family of $S$-ideals is again an $S$-ideal. Then for any $S$-ideal $I$ of $L$, $I^*$ is the intersection of all minimal prime $S$-ideals not containing $I$.

**Proof.** Suppose $I$ is an $S$-ideal of $L$. Then we have, $I \cap I^* = (0)$. Now, let $M$ be a minimal prime $S$-ideal of $L$. Since $I \cap I^* = (0) \subseteq M$, either $I \subseteq M$ or $I^* \subseteq M$. Therefore $I^* \subseteq \cap\{M \in \mathfrak{M} : I \not\subseteq M\}$. Suppose $I^* \subseteq \cap\{M \in \mathfrak{M} : I \not\subseteq M\}$. Then there exists $x \in \cap\{M \in \mathfrak{M} : I \not\subseteq M\}$ such that $x \notin I^*$. Therefore there exists $y \in I$ such that $x \circ y \neq 0$. Hence $[x \circ y]$ is a proper filter. Therefore there exists a maximal filter (say) $F$ of $L$ such that $[x \circ y] \subseteq F$. This implies $x \circ y \in F$. Since $x \circ y \leq y$, $y \in F$. It follows that $y \notin L - F$. Hence $I \not\subseteq L - F$. Since $L - F$ is a minimal prime $S$-ideal, $\cap\{M \in \mathfrak{M} : I \not\subseteq M\} \subseteq L - F$. On the other hand, we have $x \circ y \in F$ and hence $y \circ x \in F$ and $y \circ x \leq x$. Hence $x \in F$, a contradiction to $x \in L - F$. Therefore $\cap\{M \in \mathfrak{M} : I \not\subseteq M\} = I^*$.
Next, we introduce the concept of an annihilator $S$-ideal in an ASL with 0 and characterize annihilator $S$-ideals in terms of minimal prime $S$-ideals.

**Definition 3.1.** Let $L$ be an ASL with 0. Then an $S$-ideal $I$ of $L$ is said to be an annihilator $S$-ideal if $I = A^*$ for some nonempty subset $A$ of $L$.

It can be easily seen that if $I$ is an annihilator $S$-ideal in an ASL $L$ with 0 then $I = I^{**}$.

**Example 3.1.** Let $L = \{0, a, b, c\}$ and define a binary operation $\circ$ on $L$ as follows:

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<th>0</th>
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Then clearly $(L, \circ)$ is an ASL with 0. Now, put $I = \{0, a\}$. Then clearly $I = I^{**}$. Hence $I$ is an annihilator $S$-ideal.

In the following we characterize annihilator $S$-ideals in terms of minimal prime $S$-ideals.

**Theorem 3.4.** Let $L$ be a 0-distributive ASL, in which intersection of any family of $S$-ideals is again an $S$-ideal. Then any annihilator $S$-ideal $I$ of $L$ is the intersection of all minimal prime $S$-ideals containing it.

**Proof.** Suppose $I$ is an annihilator $S$-ideal in a 0-distributive ASL $L$. Then $I = I^{**}$. Therefore by Theorem 3.3, we have $(I^*)^* = \cap\{M \in \mathfrak{M} : I^* \not\subseteq M\}$. Since $I \cap I^* = \{0\} \subseteq M$ and $M$ is prime, $I \subseteq M$. It follows that

$$I = (I^*)^* = \cap\{M \in \mathfrak{M} : I \subseteq M\}.$$

**Corollary 3.4.** Let $L$ be a 0-distributive ASL, in which intersection of any family of $S$-ideals is again an $S$-ideal. Then a principal $S$-ideal of $L$ is an annihilator $S$-ideal if and only if it is the intersection of all minimal prime $S$-ideals containing it.

**Proof.** Suppose $I = \langle a \rangle$ is an annihilator $S$-ideal. Then $(a) = (a)^{**}$. Therefore by Theorem 3.3,

$$((a)^*)^* = \cap\{M \in \mathfrak{M} : (a)^* \not\subseteq M\} = \cap\{M \in \mathfrak{M} : (a) \subseteq M\}.$$

Conversely, assume the condition. Now, we shall prove that every principal $S$-ideal of $L$ is an annihilator $S$-ideal. Let $I = \langle a \rangle$ be a principal $S$-ideal of $L$. Then $(a) = \cap\{M \in \mathfrak{M} : \langle a \rangle \subseteq M\}$. Consider

$$((a)^*)^* = \cap\{M \in \mathfrak{M} : (a)^* \not\subseteq M\} = \cap\{M \in \mathfrak{M} : (a) \subseteq M\} = (a).$$

Therefore $(a) = (a)^{**}$. Thus $(a)$ is an annihilator $S$-ideal.  

Corollary 3.5. The intersection of all minimal prime $S$-ideals of a $0$-distributive ASL is $\{0\}$.

In the following we introduce the concept of dense $S$-ideal in $0$-distributive ASL. An interesting property of non-dense $S$-ideal in a $0$-distributive ASL is investigated in the following.

Definition 3.2. An $S$-ideal $I$ in a $0$-distributive ASL $L$ is called dense if $I^* = \{0\}$.

Example 3.2. Let $L = \{0, a, b, c\}$ and define a binary operation $*$ on $L$ as follows:

$\begin{array}{cccc}
0 & 0 & a & b \\
0 & 0 & 0 & 0 \\
a & 0 & a & a \\
b & 0 & a & b \\
c & 0 & a & b & c
\end{array}$

Then clearly $(L, *)$ is an ASL with $0$. Now, put $I = \{0, a\}$. Then clearly, $I$ is an $S$-ideal and also $I^* = \{0\}$. Therefore $I$ is dense.

Theorem 3.5. Any non-dense ideal of a $0$-distributive ASL is contained in a minimal prime $S$-ideal and the converse is true for principal $S$-ideals.

Proof. Suppose $I$ is a non-dense $S$-ideal of a $0$-distributive ASL $L$. Then $I^* \neq \{0\}$. Therefore we can choose $t \in I^*$ such that $t \neq 0$. Hence there exists a maximal filter $F$ such that $t \in F$. This implies $t \notin L - F$. Since $L - F$ is a minimal prime $S$-ideal and $I^* \subseteq L - F$, $I \subseteq L - F$

Suppose $[a]$ is contained in a minimal prime $S$-ideal (say) $M$ in $L$. This implies $a \in M$. Since $a \in M$, $[a]^* \not\subset M$. Hence $[a]^* = [a]^* \neq \{0\}$. Thus $[a]$ is non-dense.

Recall that an element $x$ in an ASL $L$ with $0$ is called dense element if $[x]^* = \{0\}$. Now, we prove the following.

Theorem 3.6. Let $L$ be a $0$-distributive ASL. Then an element in $L$ belongs to some minimal prime $S$-ideal of $L$ if and only if it is non-dense.

Proof. Suppose $x \in L$ such that $x$ is in a minimal prime $S$-ideal (say) $M$ of $L$. Then by Corollary 3.2, $[x]^* \not\subset M$. Hence $[x]^* \neq \{0\}$. Therefore $x$ is non-dense element.

Converse is clear.

In the following, we derive a set of identities for a prime $S$-ideal to become minimal prime $S$-ideal.

Theorem 3.7. Let $L$ be a $0$-distributive ASL. Then the following are equivalent:

1. Every prime $S$-ideal is minimal prime.
2. Every prime filter is minimal prime.
3. Every prime filter is maximal.
PROOF. (1) ⇒ (2): Assume (1). Now, we shall prove that every prime filter is minimal prime. Suppose $F$ is a prime filter and suppose $F$ is not minimal. Then there exists a prime filter $F_1$ such that $F_1 \subset F$. This implies $L - F$ is contained in $L - F_1$ and both $L - F, L - F_1$ are prime $S$-ideals. But, by (1), $L - F_1$ is minimal prime and $L - F$ is contained in $L - F_1$, a contradiction. Therefore $F$ is minimal prime filter.

(2) ⇒ (3): Assume (2). Now, we shall prove that every prime filter is maximal. Suppose $H$ is a prime filter and suppose $H$ is not maximal. Then there exists a maximal filter $M$ such that $H$ is contained in $M$. Since $L$ is $0$-distributive, $M$ is a prime filter. Hence by (2), $M$ is a minimal prime filter. It follows that $H = M$. Thus $H$ is maximal.

(3) ⇒ (1): Assume (3). Now, we shall prove that every prime $S$-ideal is minimal prime. Suppose $P$ is a prime $S$-ideal in $L$. Suppose $Q$ is a prime $S$-ideal of $L$ such that $Q \subseteq P$. Then $L - P \subseteq L - Q$ and we have both $L - P$ and $L - Q$ are prime filters. Hence by (3), we get $Q = P$. Thus $P$ is minimal prime $S$-ideal. □

Finally, we give a necessary and sufficient condition for a prime $S$-ideal to become minimal prime $S$-ideal.

THEOREM 3.8. A prime $S$-ideal $P$ is a minimal prime $S$-ideal in a $0$-distributive ASL $L$ if and only if $P$ consists precisely of all elements $x \in L$ such that $x \circ y = 0$ for some $y \notin P$.

PROOF. Suppose $P$ is a minimal prime $S$-ideal in $L$ and $x \in P$. Then by Theorem 3.2, $[x]^+ - P \neq \emptyset$. Hence we can choose $y \in [x]^+$ such that $y \notin P$. It follows that $x \circ y = 0$ and $y \notin P$. Suppose $z \in L$ such that $z \circ x = 0$ for some $x \notin P$. Then $z \circ x = 0 \in P$. Since $P$ is prime, $z \in P$. Thus $P$ consists precisely of all elements $x \in L$ such that $x \circ y = 0$ for some $y \notin P$.

Conversely, assume the condition. Suppose $Q$ is a prime $S$-ideal of $L$ such that $Q \subseteq P$. Suppose $Q \subset P$. Then there exists $x \in P$ such that $x \notin Q$. Therefore by assumption, there exists $y \notin P$ such that $x \circ y = 0$. Since $x \circ y = 0 \in Q$ and $Q$ is prime, $x \in Q$ or $y \notin Q$. It follows that $y \in Q \subset P$. Hence $y \in P$, a contradiction to $y \notin P$. Therefore $Q = P$. Thus $P$ is minimal. □

4. Minimal prime annihilator $S$-ideal

Recall that if $L$ is a $0$-distributive ASL in which intersection of any family of $S$-ideals is again an $S$-ideal then for any nonempty subset $A$ of $L$, $A^*$ is an $S$-ideal and also, clearly $A^*$ is an annihilator $S$-ideal (since $A^* = A^{**}$). Let $L$ be a $0$-distributive ASL in which intersection of any family of $S$-ideals is again an $S$-ideal. Denote by $\mathcal{B}(L)$ the set $\{A^* : \emptyset \neq A \subseteq L\}$. Thus $\mathcal{B}(L)$ is the set of all annihilator $S$-ideals in $L$. In this section, we prove that $\mathcal{B}(L)$ is a complete Boolean algebra. Also, derive a set of identities for any nonempty subset $A$ of a $0$-distributive ASL, in which intersection of any family of $S$-ideals is again an $S$-ideal, the annihilator $S$-ideal $A^*$ to become a minimal prime annihilator $S$-ideal. First, we prove that $\mathcal{B}(L)$ is a complete Boolean algebra. For this we need the following.
Lemma 4.1. Let \( L \) be an \( ASL \) with \( 0 \). Then for any \( S \)-ideals \( I, J \) of \( L \),
\[
(I \cap J)^{**} = I^{**} \cap J^{**}.
\]

Proof. Let \( I, J \in SI(L) \). Then we have \( I \cap J \subseteq I, J \). Hence by Theorem 2.1(3), we get \( I^{**}, J^{**} \subseteq (I \cap J)^{*} \). It follows that \( (I \cap J)^{**} \subseteq I^{**}, J^{**} \). Thus \( (I \cap J)^{**} \subseteq I^{**} \cap J^{**} \).

Conversely, let \( x \in I^{**} \cap J^{**} \) and \( y \in (I \cap J)^{*} \). Then for any \( i \in I \) and \( j \in J \), we have \( i \circ j \in I \cap J \). Hence \( (y \circ i) \circ j = y \circ (i \circ j) = 0 \). Therefore \( y \circ i \in J^{*} \). Again, since \( x \in J^{**} \) and \( y \circ i \in J^{*} \), we get \( (x \circ y) \circ i = x \circ (y \circ i) = 0 \). Hence \( x \circ y \in I^{*} \). Since \( x \in I^{**} \), we get \( x \circ y \in I^{**} \). Thus \( x \circ y \in I^{*} \cap I^{**} = \{0\} \). Hence \( x \circ y = 0 \). Therefore \( x \in (I \cap J)^{*} \). Thus \( I^{**} \cap J^{**} \subseteq (I \cap J)^{*} \). Hence \( (I \cap J)^{**} = I^{**} \cap J^{**} \). \( \square \)

Corollary 4.1. If \( \{I_{i} \mid i \in \Delta \} \) is a family of \( S \)-ideals of \( L \), then
\[
(\bigcap_{i \in \Delta} I_{i})^{**} = \bigcap_{i \in \Delta} (I_{i})^{**}.
\]

Theorem 4.1. Let \( L \) be a \( 0 \)-distributive \( ASL \), in which intersection of any family of \( S \)-ideals is again an \( S \)-ideal. Then the set \( \mathcal{B}(L) \), of all annihilator \( S \)-ideals of \( L \) is a complete Boolean algebra.

Proof. Clearly, the set \( \mathcal{B}(L) \), of all annihilator \( S \)-ideals of a \( 0 \)-distributive \( ASL \) \( L \) is nonempty, since \( \{0\}, L \in \mathcal{B}(L) \). Also, clearly, \( \mathcal{B}(L) \) is a poset with respect to set inclusion. Now, for any \( A^{*}, B^{*} \in \mathcal{B}(L) \), define \( A^{*} \cap B^{*} = A^{*} \cap B^{*} \) and \( A^{*} \lor B^{*} = (A^{*} \cap B^{*})^{**} \). Then clearly \( \land, \lor \) are binary operations on \( \mathcal{B}(L) \) and also, clearly \( (\mathcal{B}(L), \lor, \land) \) is a bounded lattice with bounds \( \{0\} \) and \( L \). Let \( A^{*} \in \mathcal{B}(L) \). Then we have \( A^{**} \in \mathcal{B}(L) \). Now, \( A^{*} \land A^{**} = A^{*} \cap A^{**} = \{0\} \) and \( A^{*} \lor A^{**} = (A^{*} \cap A^{**})^{**} = \{0\}^{**} = L \). Thus \( \mathcal{B}(L) \) is a complemented lattice. Finally, we shall prove that \( \mathcal{B}(L) \) is a distributive lattice. That is, to prove that for any \( A^{*}, B^{*}, C^{*} \in \mathcal{B}(L) \), \( (A^{*} \lor B^{*}) \land C^{*} \subseteq A^{*} \lor (B^{*} \land C^{*}) \). We have \( A^{*} \lor C^{*} \lor [A^{**} \lor (B^{*} \lor C^{*})] = \{0\} \). It follows that \( C^{*} \lor [A^{**} \lor (B^{*} \lor C^{*})] \subseteq (A^{*} \lor C^{*})^{**} \). Again, we have \( B^{*} \lor C^{*} \lor [A^{**} \lor (B^{*} \lor C^{*})] = \{0\} \). Therefore \( C^{*} \lor [A^{**} \lor (B^{*} \lor C^{*})] \subseteq (A^{*} \lor C^{*})^{**} \). Hence \( C^{*} \lor (A^{*} \lor (B^{*} \lor C^{*})^{**}) \subseteq (A^{*} \lor (B^{*} \lor C^{*})^{**})^{**} \). Therefore \( (A^{*} \lor B^{*}) \lor C^{*} \subseteq A^{*} \lor (B^{*} \lor C^{*}) \). Therefore by Theorem 2.6, \( \mathcal{B}(L) \) is a distributive lattice. Then \( \mathcal{B}(L) \) is a Boolean Algebra. Also, by Theorem 2.7, and by Corollary 4.1, \( \mathcal{B}(L) \) is a complete Boolean algebra. \( \square \)

In the following we establish a set of identities which characterize minimal prime annihilator \( S \)-ideals in \( 0 \)-distributive \( ASLs \). For, this first we need the following.

Lemma 4.2. Let \( L \) be a \( 0 \)-distributive \( ASL \), in which intersection of any family of \( S \)-ideals is again an \( S \)-ideal. Then an annihilator \( S \)-ideal \( A^{*} \) is a prime \( S \)-ideal in \( L \) if and only if \( A^{*} \) is a dual atom in \( \mathcal{B}(L) \).

Proof. Suppose \( A^{*} \) is a prime \( S \)-ideal in \( L \). Now, we shall prove that \( A^{*} \) is a dual atom in \( \mathcal{B}(L) \). Suppose \( A^{*} \subseteq B^{*} \) and suppose \( B^{*} \neq L \). Then there exists \( s \in L \) such that \( s \not\in B^{*} \). Hence \( s \circ b \neq 0 \) for some \( b(\neq 0) \in B \). Now, let \( c \in B^{*} \).

Then \( c \circ b = 0 \in A^{*} \). Therefore either \( c \in A^{*} \) or \( b \in A^{*} \) since \( A^{*} \) is a prime \( S \)-ideal.
Let $A$. Suppose $A$ is a prime and only if it is a dual atom. Hence we have the following.

Then $x \in L$ such that $x \circ y \in [a]^*$ and suppose $x \notin [a]^*$. Since $x \circ a \leq a$, $[a]^* \subseteq [x \circ a]^*$. Again, since $[a]^* = A^*$, which is a dual atom, either $[a]^* = [x \circ a]^*$ or $[x \circ a]^* = L$. Suppose $[x \circ a]^* = L$. Then $x \in L = [x \circ a]^*$. It follows that $x \circ (x \circ a) = 0$. Hence $x \circ a = 0$. Therefore $x \in [a]^*$, a contradiction to $x \notin [a]^*$. Hence $[a]^* = [x \circ a]^*$. Now, since $x \circ y \in [a]^*$, $(x \circ y) \circ a = 0$. It follows that $y \in [x \circ a]^* = [a]^*$. Therefore $y \in [a]^*$. Thus $A^* = [a]^*$ is a prime $S$-ideal.

It is well-known that, in a Boolean algebra $B$, an element $a$ is meet-prime if and only if it is a dual atom. Hence we have the following.

**Corollary 4.2.** Let $L$ be a 0-distributive ASL, in which intersection of any family of $S$-ideals is again an $S$-ideal. Then an annihilator $S$-ideal $A^*$ is prime in $L$ if and only if $A^*$ is a meet-prime element of $\mathcal{B}(L)$.

**Lemma 4.3.** Let $L$ be a 0-distributive ASL, in which intersection of any family of $S$-ideals is again an $S$-ideal. Then every prime annihilator $S$-ideal is minimal prime in $L$.

**Proof.** Suppose $P$ is an annihilator prime $S$-ideal. Then $P = A^*$ for some $A(\neq \emptyset) \subseteq L$. Suppose $A = \{0\}$. Then $A^* = [0]^* = L$. Hence $P = L$, a contradiction to $P$ is a prime $S$-ideal. Therefore $A \neq \{0\}$. Now, let $x \in A^*$. Then $x \circ a = 0$ for all $a \in A$. It follows that $a \in [x]^*$ for all $a \in A$. Therefore $A \subseteq [x]^*$. Now, we shall prove that $[x]^* = A^* \neq \emptyset$. Let $a(\neq 0) \in A$. Then $a \in [x]^*$. Suppose $a \in A^*$. Then $a \circ a = 0$. It follows that $a = 0$, a contradiction to $a \neq 0$. Hence $a \notin A^*$. Therefore $A \subseteq [x]^* = A^*$. Hence $[x]^* = A^* \neq \emptyset$. It follows that $[x]^* = P \neq \emptyset$. Thus by Theorem 3.2, $P$ is a minimal prime $S$-ideal.

**Theorem 4.2.** Let $L$ be a 0-distributive ASL, in which intersection of any family of $S$-ideals is again an $S$-ideal. Then for any nonempty subset $A$ of $L$, the following are equivalent:

1. $A^*$ is a dual atom in $\mathcal{B}(L)$.
2. $A^*$ is a meet-prime element of $\mathcal{B}(L)$.
3. $A^*$ is a minimal prime annihilator $S$-ideal.
4. $A^*$ is a prime annihilator $S$-ideal.

It is well-known that if a Boolean algebra $B$ satisfies ascending chain condition (acc) then $B$ is finite. Thus there will be only finite number of dual atoms in $B$ when it satisfies acc. In accordance with this observation and by Lemma 4.3, we have the following.
LEMMA 4.4. Let $L$ be a 0-distributive ASL, in which intersection of any family of $S$-ideals is again an $S$-ideal. Then $L$ contains a finite family of minimal prime $S$-ideals with intersection $\{0\}$ when $\mathcal{B}(L)$ satisfies ascending chain condition.

Proof. Suppose $\mathcal{B}(L)$ satisfies ascending chain condition and suppose $A_1^\ast, \ldots, A_n^\ast$ are dual atoms in $\mathcal{B}(L)$. Since $A_i^\ast$ is a dual atom, we get $A_i^\ast = [a_i]^\ast$ for some $a_i(\neq 0) \in A_i$ for $i = 1, 2, 3, \ldots, n$. Let $x(\neq 0) \in \bigcap_{i=1}^n A_i^\ast$. Then by Theorem 3.2 in [1], $[x]^\ast \subseteq A_j^\ast$ for $j, 1 \leq j \leq n$. Since $x \in A_j^\ast = [a_j]^\ast$, $x \circ a_j = 0$. This implies $a_j \in [x]^\ast \subseteq A_j^\ast$. Hence we get $a_j = 0$, a contradiction to $a_j$'s are non-zero. Thus $\bigcap_{i=1}^n A_i^\ast = \{0\}$. □

THEOREM 4.3. Let $L$ be a 0-distributive ASL, in which intersection of any family of $S$-ideals is again an $S$-ideal and let $\mathcal{B}(L)$ satisfies ascending chain condition. Then the set complement of union of dual atoms in $\mathcal{B}(L)$ is the set of all dense elements of $L$.

Proof. Suppose $\mathcal{B}(L)$ satisfies ascending chain condition. Then we have $\mathcal{B}(L)$ is finite. Suppose $A_1^\ast, \ldots, A_n^\ast$ are distinct dual atoms in $L$ and suppose $x \in L - \bigcup_{i=1}^n A_i^\ast$. Now, we shall prove that $[x]^\ast = \{0\}$. Suppose $[x]^\ast \neq \{0\}$. Then there exists $y(\neq 0) \in L$ such that $x \circ y = 0$. This implies $x \in [y]^\ast$. Since $[y]^\ast \neq L$, it follows that $[y]^\ast \subseteq A_j^\ast$ for some $j \leq n$. Hence $x \in \bigcap_{i=1}^n A_i^\ast$, a contradiction. Therefore $[x]^\ast = \{0\}$. Hence $x$ is a dense element. Conversely, suppose $x \notin L - \bigcup_{i=1}^n A_i^\ast$. Then $x \in \bigcap_{i=1}^n A_i^\ast$. Therefore $x \in A_j^\ast$ for some $j \leq n$. Put $A_j^\ast \circ (\bigcap_{i \neq j} A_i^\ast) = \{x \circ y : x \in A_j^\ast, y \in \bigcap_{i \neq j} A_i^\ast\}$. Let $x \circ y \in A_j^\ast \circ (\bigcap_{i \neq j} A_i^\ast)$. Then $x \in A_j^\ast$ and $y \in \bigcap_{i \neq j} A_i^\ast$. This implies $x \circ y \in A_j^\ast$ and $x \circ y \in \bigcap_{i \neq j} A_i^\ast$. It follows that $x \circ y \in A_j^\ast \circ (\bigcap_{i \neq j} A_i^\ast)$. Hence $x \circ y \in \bigcap_{i=1}^n A_i^\ast$. Therefore $A_j^\ast \circ (\bigcap_{i \neq j} A_i^\ast) \subseteq \bigcap_{i=1}^n A_i^\ast$. Since by Lemma 4.4, $\bigcap_{i=1}^n A_i^\ast = \{0\}, A_j^\ast \circ (\bigcap_{i \neq j} A_i^\ast) = \{0\}$. Since $A_i^\ast$'s are distinct, there exists $y(\neq 0) \in \bigcap_{i \neq j} A_i^\ast$ such that $y \circ x = 0$. Therefore $y \in [x]^\ast$. Hence $[x]^\ast \neq \{0\}$. Thus $x$ is non-dense. Therefore the set complement of union of dual atoms in $\mathcal{B}(L)$ is the set of all dense elements of $L$. □

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G. Nanaji Rao: Department of Mathematics, Andhra University, Visakhapatnam-530003, India.
E-mail address: nani6us@yahoo.com, drganajirao.math@auvsp.edu.in,

Ch. Swapna: Department of Mathematics, Andhra University, Visakhapatnam-530003, India,
E-mail address: swapna_chettupilli@yahoo.com