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G. Nanaji Rao and R. Venkata Aravinda Raju

Abstract. The concept of annihilator of a nonempty subset of an AL with 0 is introduced and proved some basic properties of annihilators in an AL with 0. We introduced the concept of 0-distributive AL and obtained necessary and sufficient conditions for an AL with 0 to become 0-distributive AL in terms of annihilators, ideals and pseudo-complementations.

1. Introduction

J. C. Varlet [**1**], introduced the concept of 0-distributive lattices to generalize the notion of pseudo-complemented lattices and observed that every distributive lattice with 0 is 0-distributive and also, observed every pseudo-complemented lattice is 0-distributive. The concept of almost lattice (AL) was introduced by Nanaji Rao and Habtamu Tiruneh Alemu [**2**] as a common abstraction of almost all lattice theoretic generalizations of Boolean algebra like distributive lattices, almost distributive lattices and lattices and established necessary and sufficient conditions for an AL L to become lattice. Also, they introduced the concepts of initial segments, ideals and filters in an AL and proved that the set $\mathcal{I}(L)$ of all ideals in an AL L form a complete lattice and proved the $\mathcal{PI}(L)$ of all principal ideals of L is a sub lattice of the lattice $\mathcal{I}(L)$. The concept of pseudo-complemented almost lattices was introduced by G. Nanaji Rao and R. Venkata Aravinda Raju [**4**] and proved some basic properties of pseudo-complementation on an AL L. Also, they proved that pseudo-complementation on an AL L is equationally definable and proved that a one-to-one correspondence between set of pseudo-complementations on an AL L and the set of all maximal elements in L.

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In this paper, the concept of annihilator of a nonempty subset of an AL L is introduced and proved some basic properties of annihilators in L. Next, the concept of 0-distributive AL is introduced and gave certain examples of 0-distributive ALs. Obtained necessary and sufficient conditions for an AL with 0 to become 0-distributive AL in terms of annihilators, ideals and pseudo-complementations.

2. Preliminaries

In this section we collect few important definitions and results which are already known and which will be used more frequently in the paper.

Definition 2.1. An algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an AL with 0 if, for any $a, b, c \in L$, it satisfies the following conditions:

(1) $(a \wedge b) \wedge c = (b \wedge a) \wedge c$
(2) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
(4) $(a \vee b) \vee c = a \vee (b \vee c)$
(5) $a \wedge (a \vee b) = a$
(6) $a \vee (a \wedge b) = a$
(7) $(a \wedge b) \vee b = b$
(8) $0 \wedge a = a$

It can be easily seen that $a \wedge b = a$ if and only if, $a \vee b = b$ in an AL.

Definition 2.2. Let L be an AL and $a, b \in L$. Then we define a is less than or equal to b and write $a \leqslant b$ if and only if $a \wedge b = a$ or equivalently $a \vee b = b$.

Theorem 2.1. *The relation $\leqslant$ is a partial ordering on an AL L and hence $(L, \leqslant)$ is a poset.*

Definition 2.3. Let L be any nonempty set. Define, for any $x, y \in L$, $x \vee y = x = y \wedge x$. Then, clearly L is an AL and is called descrete AL.

Definition 2.4. Let $(P, \leqslant)$ be a poset. Then P is said to satisfy ascending condition (acc) if every ascending chain in P is terminate.

Definition 2.5. Let L be an AL with 0. Then a unary operation $a \mapsto a^*$ on L is called a pseudo-complementation on L if, for any $a, b \in L$, it satisfies the following conditions:
(P_1) $a \wedge b = 0 \Rightarrow a^* \wedge b = b$
(P_2) $a \wedge a^* = 0$
(P_3) $(a \vee b)^* = a^* \wedge b^*$

Definition 2.6. Let L be an AL. Then a nonempty subset I of L is said to be an ideal of L if it satisfies the following:

(1) If $x, y \in I$ then there exists $d \in I$ such that $d \wedge x = x$ and $d \wedge y = y$.
(2) If $x \in I$ and $a \in L$ then $x \wedge a \in I$.

LEMMA 2.1. *Let L be an AL and I be an ideal of L. Then the following are equivalent:*
(1) $x, y \in I$ *implies* $x \vee y \in I$. *and*
(2) $x, y \in I$ *implies there exists* $d \in I$ *such that* $d \wedge x = x$ *and* $d \wedge y = y$.

COROLLARY 2.1. *Let L be an AL and $a \in L$. Then $(a] = \{a \wedge x | \ x \in L\}$ is an ideal of L and is called principal ideal generated by a.*

COROLLARY 2.2. *Let L be an AL and $a,\ b \in L$. Then $a \in (b]$ if and only if $a = b \wedge a$.*

COROLLARY 2.3. *Let L be an AL and $a,\ b \in L$. Then $(a \wedge b] = (b \wedge a]$.*

THEOREM 2.2. *Let L be an AL. Then the set $\mathcal{I}(L)$ of all ideals of L form a lattice under set inclusion in which the glb and lub for any $I, J \in \mathcal{I}(L)$ are respectively $I \wedge J = I \cap J$ and $I \vee J = \{x \in L : (a \vee b) \wedge x = x$ for some $a \in I$ and $b \in J\}$.*

LEMMA 2.2. *Let L be an AL. Then for any $a, b \in L$, we have the following:*
(1) $(a] \vee (b] = (a \vee b] = (b \vee a]$ *and*
(2) $(a] \cap (b] = (a \wedge b] = (b \wedge a]$.

THEOREM 2.3. *Let L be an AL. Then the set $P\mathcal{I}(L)$ of all principal ideals of L is a sublattice of the lattice $\mathcal{I}(L)$ of all ideals of L.*

3. Annihilators

In this section, we introduce the concept of annihilator of a nonempty subset of an AL L with 0 and prove some basic properties of annihilators in L. First, we begin this section with the following definition.

DEFINITION 3.1. Let L be an AL with 0. Then for any nonempty subset A of L, define $A^* = \{x \in L : x \wedge a = 0$ for all $a \in A\}$. Here A^* is called the annihilator of A in L.

Note that if $A = \{a\}$ then we write $[a]^*$ instead of A^*. In the following we prove some basic properties of annihilators.

THEOREM 3.1. *Let L be an AL with 0. Then for any non-empty subsets A, B of L, we have the following.*

(1) $A^* = \bigcap_{a \in A} [a]^*$
(2) $(A \cap B)^* = (B \cap A)^*$
(3) If $A \subseteq B$, then $B^* \subseteq A^*$
(4) $A^* \cap B^* \subseteq (A \cap B)^*$
(5) $A \subseteq A^{**}$
(6) $A^{***} = A^*$
(7) $A^* \subseteq B^* \Leftrightarrow B^{**} \subseteq A^{**}$
(8) $(A \cup B)^* = A^* \cap B^* = (B \cup A)^*$

Proof. (1) Let $x \in A^*$. Then $x \wedge a = 0$ for all $a \in A$. Hence $x \in [a]^*$ for all $a \in A$. Therefore $x \in \bigcap_{a \in A} [a]^*$. Thus $A^* \subseteq \bigcap_{a \in A} [a]^*$. Conversely, suppose $t \in \bigcap_{a \in A} [a]^*$. Then $t \in [a]^*$ for all $a \in A$. This implies $t \wedge a = 0$ for all $a \in A$. Therefore $t \in A^*$. Thus $\bigcap_{a \in A} [a]^* \subseteq A^*$. Therefore $A^* = \bigcap_{a \in A} [a]^*$.

(2) Proof is clear, since $A \cap B = B \cap A$.

(3) Suppose $A \subseteq B$ and $x \in B^*$. Then $x \wedge a = 0$, for all $a \in B$. Hence $x \wedge a = 0$ for all $a \in A$. Thus $x \in A^*$. Therefore $B^* \subseteq A^*$.

(4) Since $A \cap B \subseteq A, B$, by (3), we get $A^*, B^* \subseteq (A \cap B)^*$. Therefore $A^* \cap B^* \subseteq (A \cap B)^*$.

(5) Suppose $x \in A$ and $y \in A^*$. Then $y \wedge a = 0$ for all $a \in A$. In particular, $y \wedge x = 0$. Hence $x \in A^{**}$. Thus $A \subseteq A^{**}$.

(6) From (5), we get $A \subseteq A^{**}$. Therefore from (3), we get $A^{***} \subseteq A^*$. Conversely, suppose $x \in A^*$ and $a \in A^{**}$. Then $x \wedge a = 0$. Therefore $x \in A^{***}$ and hence $A^* \subseteq A^{***}$. Thus $A^{***} = A^*$. Proof of (7) follows by conditions (3) and (6).

(8) we have $A, B \subseteq A \cup B$. Therefore by (3), we get $(A \cup B)^* \subseteq A^*, B^*$. Hence $(A \cup B)^* \subseteq A^* \cap B^*$. Conversely, suppose $x \in A^* \cap B^*$ and $a \in A \cup B$. Then we get $x \wedge a = 0$. Thus $x \in (A \cup B)^*$ and hence $A^* \cap B^* \subseteq (A \cup B)^*$. Therefore $(A \cup B)^* = A^* \cap B^*$. Since $A \cup B = B \cup A$, $A^* \cap B^* = (B \cup A)^*$. □

Corollary 3.1. *Let L be an AL with 0. Then for any ideals I, J of L. we have the following.*

(1) $I \cap I^* = (0]$
(2) $I^* = \bigcap_{a \in I} [a]^*$
(3) $(I \cap J)^* = (J \cap I)^*$
(4) $I \subseteq J \Rightarrow J^* \subseteq I^*$
(5) $I^* \cap J^* \subseteq (I \cap J)^*$
(6) $I \subseteq I^{**}$
(7) $I^{***} = I^*$
(8) $I^* \subseteq J^* \Leftrightarrow J^{**} \subseteq I^{**}$
(9) $(I \cup J)^* = I^* \cap J^* = (J \cup I)^*$

Corollary 3.2. *Let L be an AL with 0. Then for any ideals I, J of L, we have the following.*

(1) $(I \cap J)^{**} = I^{**} \cap J^{**}$
(2) $I \cap J = (0] \Leftrightarrow I \subseteq J^* \Leftrightarrow J \subseteq I^*$

Proof. (1) Clearly, $(I \cap J)^{**} \subseteq I^{**} \cap J^{**}$. Conversely, let $x \in I^{**} \cap J^{**}$ and $y \in (I \cap J)^*$. Then for any $i \in I$ and $j \in J$, we have $i \wedge j \in I \cap J$. It follows $y \wedge (i \wedge j) = 0$. Hence $(y \wedge i) \wedge j = 0$ Therefore $y \wedge i \in J^*$. Again, since $x \in J^{**}$ and $y \wedge i \in J^*$, we get $(x \wedge y) \wedge i = x \wedge (y \wedge i) = 0$. Therefore $x \wedge y \in I^*$. Since $x \in I^{**}$, $x \wedge (x \wedge y) = 0$. Hence $x \wedge y = 0$. Thus $x \in (I \cap J)^{**}$. Therefore $I^{**} \cap J^{**} \subseteq (I \cap J)^{**}$. Hence $(I \cap J)^{**} = I^{**} \cap J^{**}$.

(2) Suppose $I \cap J = (0]$. Let $x \in I$ and $y \in J$. Then $x \wedge y \in I \cap J = (0]$. Therefore $x \wedge y = 0$. Hence $x \in J^*$. Thus $I \subseteq J^*$. Now, suppose $I \subseteq J^*$. Then by condition (4) of Corollary 3.1., we get $J^{**} \subseteq I^*$. Hence by condition (6) of Corollary 3.1., we get $J \subseteq I^*$. Finally, suppose $J \subseteq I^*$. This implies $I \cap J \subseteq I \cap I^*$. It follows that $I \cap J = (0]$. □

COROLLARY 3.3. *Let L be an AL with 0. If $\{I_i : i \in \Delta\}$ is a family of ideals of L, then* $(\bigcap_{i\in\Delta} I_i)^{**} = \bigcap_{i\in\Delta} (I_i)^{**}$.

Finally, we prove the following.

THEOREM 3.2. *Let L be an AL with 0. Then for any $x, y \in L$, we have the following.*

(1) $(x] \cap [x]^* = (0]$
(2) $[x]^* \cap [x]^{**} = (0]$
(3) $(x]^* = [x]^*$
(4) $(x]^* \cap [x]^{**} = (0]$
(5) $x \leqslant y \Rightarrow [y]^* \subseteq [x]^*$
(6) $[x \wedge y]^* = [y \wedge x]^*$
(7) $[x \vee y]^* = [y \vee x]^*$
(8) $(x] \subseteq [x]^{**}$
(9) $[x]^{***} = [x]^*$
(10) $[x]^* \subseteq [y]^* \Leftrightarrow [y]^{**} \subseteq [x]^{**}$.
(11) $[x \wedge y]^{**} = [x]^{**} \cap [y]^{**}$

PROOF. (1) Let $t \in (x] \cap [x]^*$. Then $t \in (x]$ and $t \in [x]^*$. This implies $t = x \wedge t$ and $t \wedge x = 0$. Hence $t = 0$. Therefore $(x] \cap [x]^* = (0]$.

(2) Let $t \in [x]^* \cap [x]^{**}$. Then $t \in [x]^*$ and $t \in [x]^{**}$. This implies that $t \wedge t = 0$. Thus $t = 0$. Therefore $[x]^* \cap [x]^{**} = (0]$.

(3) Let $t \in (x]^*$. Then $t \wedge s = 0$ for all $s \in (x]$. In particular, $t \wedge x = 0$, since $x \in (x]$. Thus $t \in [x]^*$. Therefore $(x]^* \subseteq [x]^*$. Conversely, suppose $t \in [x]^*$. Then $t \wedge x = 0$. Let $s \in (x]$. Then $s = x \wedge s$. Now, $t \wedge s = t \wedge (x \wedge s) = (t \wedge x) \wedge s = 0 \wedge s = 0$. Therefore $t \in (x]^*$. Hence $[x]^* \subseteq (x]^*$. Thus $(x]^* = [x]^*$. Proof of (4) follows by condition (3).

(5) Suppose $x \leqslant y$. Let $t \in [y]^*$. Then $t \wedge y = 0$. Since $x \leqslant y$, $t \wedge x \leqslant t \wedge y$. It follows that $t \wedge x = 0$. Hence $t \in [x]^*$. Therefore $[y]^* \subseteq [x]^*$.

(6) Since $x \wedge y = 0$ if and only if $y \wedge x = 0$, $[x \wedge y]^* = [y \wedge x]^*$.

(7) Since $(x \vee y) \wedge t = 0$ if and only if $(y \vee x) \wedge t = 0$ for all $t \in L$, $[x \vee y]^* = [y \vee x]^*$.

(8) Let $t \in (x]$ and $s \in [x]^*$. Then $t = x \wedge t$ and $s \wedge x = 0$. Now, $s \wedge t = s \wedge (x \wedge t) = (s \wedge x) \wedge t = 0 \wedge t = 0$. Therefore $t \in [x]^{**}$. Thus $(x] \subseteq [x]^{**}$.

(9) From (8), we get $(x] \subseteq [x)^{**}$. Therefore, from condition (3) of theorem 3.1. we get $[x]^{***} \subseteq [x]^*$. Conversely, let $t \in [x]^*$ and $s \in [x]^{**}$. Then $t \wedge s = 0$. It follows that $t \in [x]^{***}$. Hence $[x]^* \subseteq [x]^{***}$. Thus $[x]^* = [x]^{***}$.

Proof of (10) follows by condition (3) of Theorem 3.1. and by condition (9).

(11) Clearly, $[x \wedge y]^{**} \subseteq [x]^{**} \cap [y]^{**}$. Conversely, suppose $t \in [x]^{**} \cap [y]^{**}$. Then $t \in [x]^{**}$ and $t \in [y]^{**}$. We shall prove that $t \in [x \wedge y]^{**}$. Now, let $s \in [x \wedge y]^*$. Then $s \wedge (x \wedge y) = 0$. Which implies $(s \wedge x) \wedge y = 0$. Therefore $s \wedge x \in [y]^*$ and we have $t \in [y]^{**}$. It follows that $t \wedge (s \wedge x) = 0$. Hence $(t \wedge s) \wedge x = 0$. Therefore $t \wedge s \in [x]^*$ and we have $t \in [x]^{**}$. We get $t \wedge (t \wedge s) = 0$ and hence $t \wedge s = 0$. Therefore $t \in [x \wedge y]^{**}$. Hence $[x]^{**} \cap [y]^{**} \subseteq [x \wedge y]^{**}$. Thus $[x \wedge y]^{**} = [x]^{**} \cap [y]^{**}$. □

4. 0-distributive Almost Lattices

In this section, we introduce the concept of 0-distributive AL and give certain examples of 0-distributive ALs. We establish necessary and sufficient conditions for an AL with 0 to become 0-distributive AL in terms of annihilators, ideals and pseudo-complementations. We begin this section with the following.

Definition 4.1. Let L be an AL with 0. Then L is said to be 0-distributive if for any $a, b, c \in L$, $a \wedge b = 0$ and $a \wedge c = 0$ imply $a \wedge (b \vee c) = 0$.

Example 4.1. Let $A = \{0, a\}$ and $B = \{0, b_1, b_2\}$ be two discrete ALs. Now, put $L = A \times B = \{(0,\ 0), (0,\ b_1), (0,\ b_2), (a,\ 0), (a,\ b_1), (a,\ b_2)\}$ and define operations $\vee$ and $\wedge$ on L as follows.

$\vee$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
$(0,0)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
$(0,b_1)$	$(0,b_1)$	$(0,b_1)$	$(0,b_1)$	(a,b_1)	(a,b_1)	(a,b_1)
$(0,b_2)$	$(0,b_2)$	$(0,b_2)$	$(0,b_2)$	(a,b_2)	(a,b_2)	(a,b_2)
$(a,0)$	$(a,0)$	(a,b_1)	(a,b_2)	$(a,0)$	(a,b_1)	(a,b_2)
(a,b_1)	(a,b_1)	(a,b_1)	(a,b_1)	(a,b_1)	(a,b_1)	(a,b_1)
(a,b_2)	(a,b_2)	(a,b_2)	(a,b_2)	(a,b_2)	(a,b_2)	(a,b_2)

and

$\wedge$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(0,b_1)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$
$(0,b_2)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$
$(a,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(a,0)$	$(a,0)$	$(a,0)$
(a,b_1)	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
(a,b_2)	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)

Then clearly, $(L, \vee, \wedge, (0, 0))$ is a $0-$distributive AL with $(0, 0)$ as its zero element.

Example 4.2. Let $L = \{0, a, b, c\}$ and define operations $\vee$ and $\wedge$ on L as follows:

$\vee$	0	a	b	c
0	0	a	b	c
a	a	a	b	c
b	b	b	b	c
c	c	c	c	c

and

$\wedge$	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
c	0	a	b	c

Then clearly $(L, \vee, \wedge, 0)$ is a $0-$distributive AL.

Now, we prove the following theorem which characterize $0-$distributive AL, in terms of annihilators.

THEOREM 4.1. *Let L be an AL with* 0*. Then L is $0-$distributive if and only if for any nonempty subset A of L, A^* is an ideal of L.*

PROOF. Clearly $0 \in A^*$ and hence A^* is a nonempty subset of L. Let $x, y \in A^*$. Then $x \wedge a = 0$, $y \wedge a = 0$ for all $a \in A$. Let $t \in A$. Then $t \wedge x = 0$ and $t \wedge y = 0$. Since L is $0-$distributive, $(x \vee y) \wedge t = 0$. Hence $x \vee y \in A^*$. Again, let $x \in A^*$ and $r \in L$. Then $x \wedge a = 0$ for all $a \in L$. For any $t \in A$, we have $t \wedge (x \wedge r) = (t \wedge x) \wedge r = 0 \wedge r = 0$. Hence $x \wedge r \in A^*$. Therefore A^* is an ideal of L. Conversely, assume the condition. Now, we shall prove that L is 0-distributive. Let $a, b, c \in L$ such that $a \wedge b = 0$ and $a \wedge c = 0$. Then $b, c \in [a]^*$. Since $\{a\}$ is nonempty, by assumption, $[a]^*$ is an ideal. Therefore $b \vee c \in [a]^*$. Hence $(b \vee c) \wedge a = 0$. Thus L is 0-distributive. $\square$

COROLLARY 4.1. *Let L be an AL with* 0 *and let $a \in L$. Then L is $0-$distributive if and only if $[a]^*$ is an ideal of L.*

Recall that the set $\mathcal{I}(L)$ of all ideals in an AL L is a lattice with respect to set inclusion where for any $I, J \in \mathcal{I}(L)$, $I \wedge J = I \cap J$ and $I \vee J = \{x \in L : (a \vee b) \wedge x = x$ for some $a \in I$ and $b \in J\}$. In the following we give necessary and sufficient condition for an AL with 0 to become 0-distributive AL in terms of ideal.

In many of the following statements, such as Theorem 4.2, Corollary 4.3, and Theorem 4.4, the evidence is based on the proven results in article [**3**], although this is not explicitly stated.

THEOREM 4.2. *Let L be an AL with* 0*. Then L is $0-$distributive if and only if the lattice $\mathcal{I}(L)$ is a $0-$distributive.*

PROOF. Suppose L is a $0-$distributive AL. Let $I, J, K \in \mathcal{I}(L)$ such that $I \cap J = (0]$ and $I \cap K = (0]$. We shall prove that $I \cap (J \vee K) = (0]$. Let $x \in I \cap (J \vee K)$. Then $x \in I$ and $x \in (J \vee K)$. This implies $x \in I$ and $(a \vee b) \wedge x = x$, for some $a \in J$ and $b \in K$. Therefore $x \wedge a \in I \cap J$ and $x \wedge b \in I \cap K$. Hence $x \wedge a = 0$ and $x \wedge b = 0$. Since L is $0-$distributive, $x \wedge (a \vee b) = 0$. It follows that $x = 0$. Therefore $I \cap (J \vee K) = (0]$. Thus $\mathcal{I}(L)$ is $0-$distributive. Conversely, suppose $\mathcal{I}(L)$ is a 0-distributive. Let $a, b, c \in L$ such that $a \wedge b = 0$ and $a \wedge c = 0$. Then $(a \wedge b] = (0]$ and $(a \wedge c] = (0]$. This implies $(a] \cap (b] = (0]$ and $(a] \cap (c] = (0]$. Since $\mathcal{I}(L)$ is a 0-distributive, $(a] \cap ((b] \vee (c]) = (0]$. Hence $(a] \cap (b \vee c] = (0]$. Then it follows that $(a \wedge (b \vee c)] = (0]$. Hence $a \wedge (b \vee c) = 0$. Therefore L is 0-distributive. $\square$

COROLLARY 4.2. *Let L be an AL with* 0*. Then L is $0-$distributive if and only if $\mathcal{PI}(L)$ is $0-$distributive.*

In the following we prove that every pseudo-complemented AL is a $0-$ distributive AL.

THEOREM 4.3. *Every pseudo-complemented AL is a $0-$distributive AL.*

PROOF. Suppose L is a pseudo-complemented AL. Now, we shall prove that L is a $0-$ distributive. Let $a, b, c \in L$ such that $a \wedge b = 0$ and $a \wedge c = 0$. Then $a^* \wedge b = b$ and $a^* \wedge c = c$. It follows that $a^* \vee b = a^*$ and $a^* \vee c = a^*$. Now, $a^* = a^* \vee a^* = (a^* \vee b) \vee (a^* \vee c)$. This implies $a^* = (a^* \vee b) \vee (a^* \vee c)$. Now, $a^* \wedge (b \vee c) = ((a^* \vee b) \vee (a^* \vee c)) \wedge (b \vee c) = ((a^* \vee a^*) \vee (b \vee c)) \wedge (b \vee c) = (a^* \vee (b \vee c)) \wedge (b \vee c) = b \vee c$. Therefore $a^* \wedge (b \vee c) = b \vee c$. Now, $a \wedge (b \vee c) = a \wedge (a^* \wedge (b \vee c)) = (a \wedge a^*) \wedge (b \vee c) = 0 \wedge (b \vee c) = 0$. Thus $a \wedge (b \vee c) = 0$. Therefore L is a $0-$distributive AL. □

But, the converse of the above theorem is not true. For, consider the following example.

EXAMPLE 4.3. Consider the following AL, whose Hasse diagram is as follows:

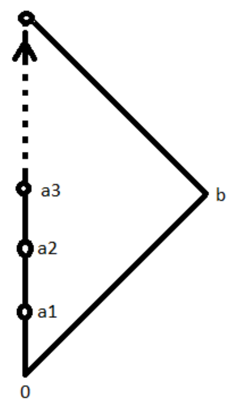

FIGURE 1

Then clearly this is a $0-$distributive AL. But, this AL is not pseudo-complemented since the element b has no pseudo-complement.

Next, we prove that if L is a 0-distributive AL then the lattice $\mathcal{I}(L)$ of all ideals of L is pseudo-complemented.

THEOREM 4.4. *Let L be a $0-$distributive AL. Then the lattice $\mathcal{I}(L)$ of all ideals of L is pseudo-complemented.*

PROOF. Clearly, $\mathcal{I}(L)$ is a lattice with respect to set inclusion, where for any $I, J \in \mathcal{I}(L)$, $I \wedge J = I \cap J$ and $I \vee J = \{x \in L : (a \vee b) \wedge x = x$ for some $a \in I$ and $b \in J\}$. Let $I \in \mathcal{I}(L)$. Then we have $I^* = \{x \in L : x \wedge a = 0$ for all $a \in I\}$ is an ideal of L. Clearly $I \cap I^* = (0]$. Now, let $J \in \mathcal{I}(L)$ such that $I \cap J = (0]$. Now, we shall prove that $J \subseteq I^*$. Let $x \in J$ and $a \in I$. Then clearly $x \wedge a \in I \cap J = (0]$. Hence $x \wedge a = 0$. Therefore $x \in I^*$. Thus $J \subseteq I^*$. Therefore $\mathcal{I}(L)$ is a Pseudo-complemented lattice. $\square$

Finally, we derive necessary and sufficient condition for an AL with 0 to become 0-distributive in terms of pseudo-complementation. For, this we need the following.

LEMMA 4.1. *Let L be an AL which satisfies ascending chain condition. Then every ideal in L is a principal ideal.*

PROOF. Suppose L satisfies ascending chain condition and suppose I is an ideal of L. Then clearly I is nonempty. Now, choose an element $a_1 \in I$. Then clearly $(a_1] \subseteq I$. Suppose $(a_1] \subsetneqq I$, choose $x_1 \in I$ such that $x_1 \notin (a_1]$. Now, put $a_2 = a_1 \vee x_1$. Then $a_2 \in I$. Suppose $a_1 = a_2$. Then $a_1 = a_1 \vee x_1$. Thus $a_1 \wedge x_1 = x_1$ and hence $x_1 \in (a_1]$, a contradiction to $x_1 \notin (a_1]$. Therefore $a_1 \neq a_2$. Hence $a_1 < a_2$. Continuing the above process, we get strictly ascending chain $a_1 < a_2 < a_3 <$ .... This chain must be terminate at some stage. Therefore there exist a positive integer k such that $a_k = a_{k+1} =$ ..... Now, we shall prove that $(a_k] = I$. Clearly, $(a_k] \subseteq I$, since $a_k \in I$. Conversely, let $x \in I$. Then $a_k \vee x \in I$ and $a_k \leqslant a_k \vee x$. It follows that $a_k = a_k \vee x$. This implies $a_k \wedge x = x$. Hence $x \in (a_k]$. Thus $I = (a_k]$. Therefore I is a principal ideal. $\square$

LEMMA 4.2. *Let L be a $0-$distributive AL and $I, J \in \mathcal{I}(L)$. Then $(I \vee J)^* = I^* \cap J^*$.*

PROOF. We have $I, J \subseteq I \vee J$. Therefore $(I \vee J)^* \subseteq I^*, J^*$ and hence $(I \vee J)^* \subseteq I^* \cap J^*$. Conversely suppose $t \in I^* \cap J^*$ and $x \in I \vee J$. Then $(a \vee b) \wedge x = x$, where $a \in I$ and $b \in J$. Now, since $t \in I^*$ and $a \in I$, $t \wedge a = 0$. Similarly, we get $t \wedge b = 0$. Hence $t \wedge (a \vee b) = 0$. Therefore $t \wedge x = t \wedge ((a \vee b) \wedge x) = (t \wedge (a \vee b)) \wedge x = 0 \wedge x = 0$. Hence $t \in (I \vee J)^*$. Thus $I^* \cap J^* \subseteq (I \vee J)^*$. Therefore $(I \vee J)^* = I^* \cap J^*$. $\square$

COROLLARY 4.3. *Let L be 0-distributive AL. Then for any $x, y \in L$, $[x \vee y]^* = [x]^* \cap [y]^*$.*

THEOREM 4.5. *Let L be an AL with 0 which satisfies ascending chain condition. Then L is $0-$ distributive if and only if L is pseudo-complemented.*

PROOF. Suppose L is $0-$distributive AL. Now, we shall prove that L is pseudo-complemented. Let $a \in L$. Then $[a]^*$ is an ideal. Since L satisfies ascending chain condition, $[a]^* = (b]$ for some $b \in L$. Define $* : L \to L$ by $*(a) = b \wedge m$ where $[a]^* = (b]$. Suppose $[a]^* = (b]$ and $[a]^* = (c]$. Then $(b] = (c]$. Since $b \in (b] = (c]$, $b = c \wedge b$. This implies $b \wedge m = (c \wedge b) \wedge m = (b \wedge c) \wedge m = c \wedge m$, since $c \in (c] = (b]$. Therefore $b \wedge m = c \wedge m$. Thus $*$ is well defined.

Let $x, y \in L$ such that $x \wedge y = 0$. Since $[x]^*$ is an ideal, $[x]^* = (t]$ for some $t \in L$. Now, Consider $x^* \wedge y = (t \wedge m) \wedge y = (m \wedge t) \wedge y = t \wedge y$. Again, since $x \wedge y = 0$, $y \in [x]^* = (t]$. Thus $y \in (t]$. Hence $y = t \wedge y$. Therefore $x^* \wedge y = y$. Also, $x \wedge x^* = x \wedge (t \wedge m) = (x \wedge t) \wedge m = 0 \wedge m$ (since $t \in [x]^*$)$= 0$. Therefore $x \wedge x^* = 0$.

Finally, we shall prove that for any $x, y \in L$, $(x \vee y)^* = x^* \wedge y^*$. Let $x, y \in L$. Then $[x]^*, [y]^*$ are ideals. Therefore $[x]^*, [y]^*$ are principal ideals. Thus $[x]^* = (s], [y]^* = (t]$ for some $s, t \in L$. Therefore by the definition of $*$, $x^* = s \wedge m$ and $y^* = t \wedge m$. Now, consider $[x \vee y]^* = [x]^* \cap [y]^* = (s] \cap (t] = (s \wedge t]$. Therefore $(x \vee y)^* = (s \wedge t) \wedge m = (s \wedge m) \wedge (t \wedge m) = x^* \wedge y^*$. Therefore $(x \vee y)^* = x^* \wedge y^*$. Therefore L is pseudo-complemented. Converse follows by Theorem 4.8. $\square$

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G. Nanaji Rao,
Department of Mathematics, Andhra University, Visakhapatnam-530003, India.
E-mail address: nani6us@yahoo.com, drgnanajirao.math@auvsp.edu.in

R. Venkata Aravinda Raju,
Department of Mathematics, Andhra University, Visakhapatnam-530003, India
E-mail address: aravindaraju.1@gmail.com