BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., Vol. **10**(2)(2020), 213-224
DOI: 10.7251/BIMVI2002213K

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

SYMMETRIC POSITIVE SOLUTIONS OF FOURTH ORDER BOUNDARY VALUE PROBLEMS ON TIME SCALES

Ilkay Yaslan Karaca and Aycan Sinanoğlu

Abstract. The aim of this article is to investigate the existence of symmetric positive solutions for a class of fourth-order boundary value problem. Some sufficient conditions for the existence of multiple positive solutions are obtained by using fixed point index theorem. An example which supports our theoretical results is also indicated.

1. Introduction

Calculus on time scales was introduced by Hilger [**13**] as a theory which includes both differential and difference calculus as special cases. In the past few years, it has found a considerable amount of interest and attracted the attention of many researchers. Time scale calculus would allow exploration of a variety of situations in economic, biological, heat transfer, stock market and epidemic models; see the monographs of Aulbach and Hilger [**1**], Bohner and Peterson ([**2, 3**]), and Lakshmikantham et al. [**15**] and the references therein.

There are many authors studied the existence of positive solutions fourth-order boundary value problems ([**6, 7, 8, 12, 14, 17, 21, 23, 24, 25, 26**]) However, concerning the existence of the symmetric positive solutions of fourth-order boundary value problems only a small amount of work ([**4, 5, 10, 11, 18, 19, 20, 22**]) can be found in the literature.

2010 *Mathematics Subject Classification.* 34B18; 34B37; 34K10.
Key words and phrases. Symmetric positive solutions, Fixed point index theorem, Boundary value problems.

In this paper, we study existence of symmetric positive solutions of the following fourth-order boundary value problem (BVP):

$$\begin{cases} (\varphi(x^{\triangle\nabla}(t)))^{\triangle\nabla} = f(t,x(t)),\ t\in(0,1),\\ x(0)=x(1),\ x^{\triangle}(0)-x^{\triangle}(1)=\alpha x(\frac{1}{2}),\ x^{\triangle\nabla}(0)=x^{\triangle\nabla}(1),\\ (\varphi(x^{\triangle\nabla}))^{\triangle}(0)-(\varphi(x^{\triangle\nabla}))^{\triangle}(1)=\beta\varphi(x^{\triangle\nabla}(\frac{1}{2}))\end{cases} \tag{1.1}$$

where $0<\alpha,\ \beta<4,\ \frac{1}{2}\in\mathbb{T}$.

Throughout this paper we assume that following conditions hold:

(H1) $f\in\mathcal{C}([0,1]\times[0,\infty),[0,\infty))$ is symmetric on [0,1], (i.e., $f(t,x)=f(1-t,x)$ for $t\in[0,1]$);

(H2) $\varphi:\mathbb{R}\to\mathbb{R}$ is increasing homeomorphism and homomorphism. Also, $\varphi(0)=0$ and $\varphi(-x)=-\varphi(x)$.

A projection $\varphi:\mathbb{R}\to\mathbb{R}$ is called an increasing homeomorphism and homomorphism if the following conditions are satisfied:

(i) If $x\leqslant y$ then $\varphi(x)\leqslant\varphi(y)$ for all $x,y\in\mathbb{R}$;

(ii) φ is continuous, bijection and its inverse mapping is also continuous;

(iii) $\varphi(xy)=\varphi(x)\varphi(y)$ for all $x,y\in\mathbb{R}$.

In this paper, a symmetric positive solution x of (1.1) means a solution of (1.1) satisfying $x>0$ and $x(t)=x(1-t)$, $t\in[0,1]$.

This paper is organized as follows:
In section 2, we give some preliminaries lemmas. In section 3, we give the proof of necessary and sufficient conditions for existence of symmetric positive solutions of BVP (1.1). In section 4, an example is also presented to illustrate our main results. The results are even new for the difference equations and differential equations as well as for dynamic equations on time scales.

2. Preliminaries

We will need the following lemmas to state the main results of this paper.

Lemma 2.1. *Assume (H2) holds. Then for any $y\in\mathcal{C}[0,1]$ the BVP*

$$\begin{cases} \varphi(x^{\triangle\nabla}(t))=y(t),\ t\in(0,1),\\ x(0)=x(1), x^{\triangle}(0)-x^{\triangle}(1)=\alpha x(\frac{1}{2})\end{cases} \tag{2.1}$$

has unique solution x and x can be expressed in the form

$$x(t)=-\int_0^1 G(t,s)\varphi^{-1}(y(s))\nabla s,$$

where

$$G_1(t,s)=\begin{cases} t(1-s),\ 0\leqslant t\leqslant s\leqslant 1,\\ s(1-t),\ 0\leqslant s\leqslant t\leqslant 1,\end{cases} \tag{2.2}$$

$$G_2(s)=\begin{cases}\frac{1}{\alpha}-\frac{s}{2}, & 0\leqslant s\leqslant\frac{1}{2},\\ \frac{1}{\alpha}-\frac{1}{2}+\frac{s}{2}, & \frac{1}{2}\leqslant s\leqslant 1,\end{cases}\tag{2.3}$$

$$G(t,s)=G_1(t,s)+G_2(s).\tag{2.4}$$

PROOF. Let x be a solution of the problem (2.1). Then

$$x^{\triangle\nabla}(t)=\varphi^{-1}(y(t)),\ t\in(0,1).$$

An integration from 0 to t of both sides of the above equality yields

$$x^{\triangle}(t)=x^{\triangle}(0)+\int_0^t\varphi^{-1}(y(s))\nabla s.\tag{2.5}$$

We integrate both sides of the (2.5) to get

$$x(t)=x(0)+tx^{\triangle}(0)+\int_0^t(t-s)\varphi^{-1}(y(s))\nabla s.\tag{2.6}$$

If we let $t=1$ in (2.6), then we have

$$x^{\triangle}(0)=-\int_0^1(1-s)\varphi^{-1}(y(s))\nabla s\tag{2.7}$$

By using (2.5) – (2.7) and the equality $x^{\triangle}(0)-x^{\triangle}(1)=\alpha x(\frac{1}{2})$, we get

$$\begin{aligned}x(0)&=\int_0^{\frac{1}{2}}(\frac{s}{2}-\frac{1}{\alpha})\varphi^{-1}(y(s))\nabla s+\int_{\frac{1}{2}}^1(\frac{1}{2}-\frac{s}{2}-\frac{1}{\alpha})\varphi^{-1}(y(s))\nabla s\\ &=-\int_0^1 G_2(s)\varphi^{-1}(y(s))\nabla s\end{aligned}\tag{2.8}$$

where $G_2(s)$ is defined in (2.3).

Subsituting (2.7) and (2.8) to (2.6), we have

$$\begin{aligned}x(t)&=-\int_0^1 G_2(s)\varphi^{-1}(y(s))\nabla s-\int_0^1 G_1(t,s)\varphi^{-1}(y(s))\nabla s\\ &=-\int_0^1 G(t,s)\varphi^{-1}(y(s))\nabla s\end{aligned}$$

where $G_1(t,s)$ and $G(t,s)$ are given in (2.2) and (2.4) respectively. The proof is complete. □

LEMMA 2.2. *Assume that $(H1)-(H2)$ hold. Then*

$$\begin{cases}(\varphi(x^{\triangle\nabla}(t)))^{\triangle\nabla}=f(t,x(t)),\ t\in(0,1),\\ x(0)=x(1),\ x^{\triangle}(0)-x^{\triangle}(1)=\alpha x(\frac{1}{2}),\ x^{\triangle\nabla}(0)=x^{\triangle\nabla}(1),\\ (\varphi(x^{\triangle\nabla}))^{\triangle}(0)-(\varphi(x^{\triangle\nabla}))^{\triangle}(1)=\beta\varphi(x^{\triangle\nabla}(\frac{1}{2}))\end{cases}$$

has a unique solution x and x can be expressed in the form

$$x(t) = -\int_0^1 G(t,s)\varphi^{-1}(-\int_0^1 H(s,\tau)f(\tau,x(\tau))\nabla\tau)\nabla s \qquad (2.9)$$

where

$$H(t,s) = G_1(t,s) + G_3(s), \qquad (2.10)$$

$$G_3(s) = \begin{cases} \frac{1}{\beta} - \frac{s}{2}, & 0 \leqslant s \leqslant \frac{1}{2}, \\ \frac{1}{\beta} - \frac{1}{2} + \frac{s}{2}, & 0 \leqslant s \leqslant 1, \end{cases}$$

and $G_1(t,s)$ is as in (2.2).

Proof. Let us consider the following BVP:

$$\begin{cases} y^{\triangle\nabla}(t) = f(t,x(t)),\ t \in (0,1), \\ y(0) = y(1),\ y^{\Delta}(0) - y^{\Delta}(1) = \beta y(\frac{1}{2}). \end{cases} \qquad (2.11)$$

The BVP (2.11) has a unique solution

$$y(t) = -\int_0^1 H(t,s)f(s,x(s))\nabla s,$$

where $H(t,s)$ is as in (2.10). This completes the proof. □

Lemma 2.3. *Assume that $(H1)$ holds. Then we have for $s,t \in [0,1]$*

(i) $G(t,s) \geqslant 0,\ H(t,s) \geqslant 0,$

(ii) $G(t,s) = G(1-t,1-s),\ H(t,s) = H(1-t,1-s),$

(iii) $\Gamma G(s,s) \leqslant G(t,s) \leqslant G(s,s)$ *and* $\delta H(s,s) \leqslant H(t,s) \leqslant H(s,s),$

where

$$\Gamma = \min\{\frac{1}{\alpha} - \frac{1}{4}, \frac{3}{4}\},\ ,\ and\ \ \delta = \min\{\frac{1}{\beta} - \frac{1}{4}, \frac{3}{4}\}.$$

Proof. One can easily see the properties (i), (ii). We only give the proof of (iii). By the expression of $G_2(s)$, for $t \in [0,1]$ $s \in [0,\frac{1}{2}]$,

$$\begin{aligned} G(t,s) &= G_1(t,s) + G_2(s) \\ &\geqslant G_2(s) \\ &\geqslant s(1-s)G_2(s) + \frac{3}{4}G_2(s) \\ &\geqslant (\frac{1}{\alpha} - \frac{1}{4})s(1-s) + \frac{3}{4}G_2(s) \\ &\geqslant \min\{\frac{1}{\alpha} - \frac{1}{4}, \frac{3}{4}\}G(s,s) \\ &= \Gamma G(s,s). \end{aligned}$$

Similarly, for $t \in [0,1], s \in [\frac{1}{2},1]$, we get $G(t,s) \geqslant \Gamma G(s,s)$. Hence for $t \in [0,1], s \in [0,1]$ we obtain $G(t,s) \geqslant \Gamma G(s,s)$.

Similarly, it is easy to see that $\delta H(s,s) \leqslant H(t,s) \leqslant H(s,s)$. □

LEMMA 2.4. *$\max_{t,s\in[0,1]} G(t,s) \leqslant M$, where $M = 2\max\{\frac{1}{\alpha}, \frac{1}{4}\}$.*

PROOF. It is obvious that for $t, s \in [0,1]$, we get $\max_{t,s\in[0,1]} G_1(t,s) = \frac{1}{4}$ and $\max_{s\in[0,1]} G_2(s) = \frac{1}{\alpha}$. Hence,

$$\begin{aligned}\max_{t,s\in[0,1]} G(t,s) &= \max_{t,s\in[0,1]}[G_1(t,s) + G_2(s)]\\ &\leqslant \max_{t,s\in[0,1]} G_1(t,s) + \max_{t,s\in[0,1]} G_2(s)\\ &= 2\max\{\frac{1}{\alpha}, \frac{1}{4}\}\\ &= M.\end{aligned}$$

□

LEMMA 2.5. *$|H(t_1,s) - H(t_2,s)| \leqslant 2|t_1 - t_2|$ for $t_1, t_2 \in [0,1]$, $s \in [0,1]$.*

PROOF. If we show $|G_1(t_1,s) - G_1(t_2,s)| \leqslant 2|t_1 - t_2|$, the proof is complete. Let $t_1 > t_2$. We divide the proof into three steps.

Step 1: $0 \leqslant t_2 \leqslant t_1 \leqslant s \leqslant 1$.

Since $G_1(t_1,s) = t_1(1-s)$ and $G_1(t_2,s) = t_2(1-s)$, we have

$$\begin{aligned}|G_1(t_1,s) - G_1(t_2,s)| &= |t_1 - t_1 s - t_2 + t_2 s|\\ &= |s(t_2 - t_1) - (t_2 - t_1)|\\ &= (1-s)|t_2 - t_1|\\ &\leqslant 2|t_2 - t_1|.\end{aligned}$$

Step 2: $0 \leqslant s \leqslant t_2 \leqslant t_1 \leqslant 1$.

By $G_1(t_1,s) = s(1-t_1)$ and $G_1(t_2,s) = s(1-t_2)$, we get

$$\begin{aligned}|G_1(t_1,s) - G_1(t_2,s)| &= |s - t_1 s - s + t_2 s|\\ &= |s(t_2 - t_1)|\\ &= s|t_2 - t_1|\\ &\leqslant 2|t_2 - t_1|.\end{aligned}$$

Step 3 : $0 \leqslant t_2 \leqslant s \leqslant t_1 \leqslant 1$.

By $G_1(t_1,s) = s(1-t_1)$ and $G_1(t_2,s) = t_2(1-s)$, we obtain

$$\begin{aligned}|G_1(t_1,s) - G_1(t_2,s)| &= |s - t_1 s - t_2 + t_2 s|\\ &= |s(t_2 - t_1) - (s - t_2)|\\ &\leqslant |s(t_2 - t_1)| + |s - t_2|\\ &\leqslant 2|t_2 - t_1|.\end{aligned}$$

Similarly, it can easily see that $|G_1(t_1,s) - G_1(t_2,s)| \leqslant 2|t_1 - t_2|$ for $t_2 > t_1$. The proof is complete. □

Let $\mathbb{B}$ denote the Banach space $C_{ld}[0,1]$ with the norm $\|x\| = \max_{t\in[0,1]} |x(t)|$.

Define the cone $K \subset \mathbb{B}$ by

$K = \{x \in \mathbb{B} : x(t)$ is symmetric, concave, positive on $[0,1]$ and $\min_{t\in[0,1]} x(t) \geqslant \Gamma\|x\|\}$,

where $\Gamma = \min\{\frac{1}{\alpha} - \frac{1}{4}, \frac{3}{4}\}$. We can define the operator

$$Tx(t) = -\int_0^1 G(t,s)\varphi^{-1}(-\int_0^1 H(s,\tau)f(\tau,x(\tau))\nabla\tau)\nabla s,$$

where $x \in K$. Therefore solving (1.1) in K is equivalent to finding fixed points of the operator T.

Lemma 2.6. $\min_{t\in[0,1]} x(t) \geqslant \Gamma\|x\|$, *where* $\Gamma = \min\{\frac{1}{\alpha} - \frac{1}{4}, \frac{3}{4}\}$.

Proof. If we use Lemma 2.3 (iii) and (2.9), we have

$$\|x\| \leqslant -\int_0^1 G(s,s)\varphi^{-1}(-\int_0^1 H(s,\tau)f(\tau,x(\tau))\nabla\tau)\nabla s. \quad (2.12)$$

By using Lemma 2.3 (iii), (2.9) and (2.12), we obtain

$$\begin{aligned} x(t) &= -\int_0^1 G(t,s)\varphi^{-1}(-\int_0^1 H(s,\tau)f(\tau,x(\tau))\nabla\tau)\nabla s \\ &\geqslant \Gamma[-\int_0^1 G(s,s)\varphi^{-1}(-\int_0^1 H(s,\tau)f(\tau,x(\tau))\nabla\tau)\nabla s] \\ &\geqslant \Gamma\|x\|. \end{aligned}$$

□

Lemma 2.7. $\|x\| \leqslant M\|x^{\triangle\nabla}\|$, *where* $M = 2\max\{\frac{1}{\alpha}, \frac{1}{4}\}$.

Proof. From (2.9) and Lemma 2.4, we have

$$\begin{aligned} |x(t)| &= |-\int_0^1 G(t,s)\varphi^{-1}(-\int_0^1 H(s,\tau)f(\tau,x(\tau))\nabla\tau\nabla s)| \\ &= |-\int_0^1 G(t,s)x^{\triangle\nabla}(s)\nabla s| \\ &\leqslant M \max_{s\in[0,1]} |x^{\triangle\nabla}(s)| \\ &= M\|x^{\triangle\nabla}\|. \end{aligned}$$

So, we get $\|x\| \leqslant M\|x^{\triangle\nabla}\|$. The proof is complete. □

Lemma 2.8. *Suppose that* $(H1)-(H2)$ *hold. Then* $T : K \to K$ *is completely continuous.*

Proof. For all $x \in K$, we have

$$(Tx)^{\triangle\nabla}(t) = \varphi^{-1}(\int_0^1 H(t,s))f(s,x(s))\nabla s) \leqslant 0$$

which implies Tx is concave on $[0,1]$. From the definition of T, we get

$$\begin{aligned}
Tx(1-t) &= -\int_0^1 G(1-t,s)\varphi^{-1}(-\int_0^1 H(s,\tau)f(\tau,x(\tau))\nabla\tau)\nabla s \\
&= -\int_0^1 G(1-t,1-s)\varphi^{-1}(-\int_0^1 H(1-s,\tau)f(\tau,x(\tau))\nabla\tau)\nabla(1-s) \\
&= -\int_0^1 G(t,s)\varphi^{-1}(-\int_0^1 H(1-s,\tau)f(\tau,x(\tau))\nabla\tau)\nabla s \\
&= -\int_0^1 G(t,s)\varphi^{-1}(-\int_1^0 H(1-s,1-\tau)f(1-\tau,x(1-\tau))\nabla(1-\tau))\nabla s \\
&= -\int_0^1 G(t,s)\varphi^{-1}(-\int_0^1 H(s,\tau)f(\tau,x(\tau))\nabla\tau)\nabla s \\
&= Tx(t).
\end{aligned}$$

$Tx(t) \geqslant \Gamma\|Tx\|$ is obvious from Lemma 2.6. Thus $T(K) \subset K$.

On the other hand, by using Lemma 2.7 and the conditions $(H1)-(H2)$, from the definition of T, it is clear that $T: K \to K$ is continuous.

Next, our purpose is to prove that T maps bounded sets into precompact sets on $t \in [0,1]$. Let Ω be a bounded subset on K. Then we have r positive real number for $x \in \Omega$ such that $\|x\| \leqslant r$. We have

$$\|(Tx)^{\triangle\nabla}\| = \max_{t\in[0,1]} |(Tx)^{\triangle\nabla}(t)| = \max_{t\in[0,1]} |\varphi^{-1}(-\int_0^1 H(t,s)f(s,x(s))\nabla s)|.$$

Set

$$S_r := \sup\{|f(t,y)| : (t,y) \in [0,1]\times[0,r]\}.$$

By using the condition $(H2)$ and Lemma 2.3(iii), we get

$$\|(Tx)^{\triangle\nabla}\| \leqslant |\varphi^{-1}(S_r)\varphi^{-1}(\int_0^1 H(s,s)\nabla s))|.$$

We know that $\varphi^{-1}(S_r)$ and $\varphi^{-1}(\int_0^1 H(s,s)\nabla s))$ are bounded. Hence, there exist a positive number C such that $\|(Tx)^{\triangle\nabla}\| \leqslant C$. From Lemma 2.7, we get $\|(Tx)\| \leqslant CM$, so $T\Omega$ is uniformly bounded.

Finally, we show that $T\Omega$ is equicontinuous. For $s, t_1, t_2 \in [0,1]$, we have

$$\begin{aligned}
&|\varphi(Tx)^{\triangle\nabla}(t_1) - \varphi(Tx)^{\triangle\nabla}(t_2)| \\
&= |-\int_0^1 H(t_1,s)f(s,x(s))\nabla s + \int_0^1 H(t_2,s)f(s,x(s))\nabla s \\
&\leqslant \int_0^1 |H(t_1,s) - H(t_2,s)|f(s,x(s))\nabla s.
\end{aligned}$$

From definition of S_r and Lemma 2.5, we get

$$|\varphi(Tx)^{\triangle\nabla}(t_1) - \varphi(Tx)^{\triangle\nabla}(t_2)| \leqslant 2S_r|t_1 - t_2| \ \to 0,\ t_1 \to t_2.$$

Therefore, $T\Omega$ is equicontinuous. By Arzela-Ascoli Theorem, $T : K \to K$ is compact operator. Hence, the proof is complete. □

3. Main Results

In this section, we show that BVP (1.1) has multiple symmetric positive solutions by using fixed point index theorem which is given below.

LEMMA 3.1. [**9, 16**] *Let K be a cone in a Banach space $\mathbb{B}$. Let D be an open bounded subset of $\mathbb{B}$ with $D_k = D \cap K \neq \emptyset$ and $\overline{D}_k \neq K$. Suppose that $T : \overline{D}_k \to K$ is completely continuous map such that $x \neq Tx$ for all $x \in \partial D_k$. Then the following results hold:*

(i) *If $\|Tx\| \leqslant \|x\|,\ x \in \partial D_k$ then $i_k(T, D_k) = 1$;*
(ii) *If there exists $x_0 \in K \setminus \{0\}$ such that $x \neq Tx + \lambda x_0$, for all $x \in \partial D_k$ and all $\lambda > 0$, then $i_k(T, D_k) = 0$;*
(iii) *Let U be open in K such that $\overline{U} \subset D_k$. If $i_k(T, D_k) = 1$ and $i_k(T, U_k) = 0$, then T has a fixed point in $D_k \setminus \overline{U}_k$. The same result holds if $i_k(T, D_k) = 0$ and $i_k(T, U_k) = 1$.*

We define $K_\rho = \{x \in K : \|x\| < \rho\}$, and

$$\Omega_\rho = \{x \in K : \min_{t\in[0,1]} x(t) \leqslant \Gamma\rho\} = \{x \in K : \Gamma\|x\| \leqslant \min_{t\in[0,1]} x(t) < \Gamma\rho\}.$$

LEMMA 3.2. *Ω_ρ has the following properties:*
(a) *Ω_ρ is open relative to K.*
(b) *$K_{\Gamma\rho} \subset \Omega_\rho \subset K_\rho$.*
(c) *$u \in \partial\Omega_\rho$ if and only if $\min_{t\in[0,1]} u(t) = \Gamma\rho$.*
(d) *If $u \in \partial\Omega_\rho$, then $\Gamma\rho \leqslant u(t) \leqslant \rho$ for $t \in [0, 1]$.*

Now for convenience we introduce the following notations:

$$f^{\rho}_{\Gamma\rho} = \min\{\min_{t\in[0,1]} \frac{f(t,x)}{\varphi(\rho)} : x \in [\Gamma\rho, \rho]\}, \qquad f^{\rho}_{0} = \max\{\max_{t\in[0,1]} \frac{f(t,x)}{\varphi(\rho)} : x \in [0, \rho]\},$$

$$A = [\int_0^1 G(s,s)\varphi^{-1}(\int_0^1 H(\tau,\tau)\nabla\tau)\triangle s]^{-1},$$

$$B = [\Gamma\int_0^1 G(s,s)\varphi^{-1}(\int_0^1 \delta H(\tau,\tau)\nabla\tau)\triangle s]^{-1}.$$

Now we give our results on existence of multiple positive solutions of BVP(1.1).

THEOREM 3.1. *Suppose $(H1)-(H2)$ and one of the following conditions holds;*

(H3) *There exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ with $\rho_1 < \Gamma\rho_2$ and $\rho_2 < \rho_3$ such that $f^{\rho_1}_0 < \varphi(A)$, $f^{\rho_2}_{\Gamma\rho_2} > \varphi(B\Gamma)$, $f^{\rho_3}_0 < \varphi(A)$.*
(H4) *There exist $\rho_1, \rho_2, \rho_3 \in (0, \infty)$ with $\rho_1 < \rho_2 < \rho_3$ such that $f^{\rho_1}_{\Gamma\rho_1} > \varphi(B\Gamma)$, $f^{\rho_2}_0 < \varphi(A)$, $f^{\rho_3}_{\Gamma\rho_1} > \varphi(B\Gamma)$.*

Then problem (1.1) has at least two positive solutions x_1, x_2 *with* $x_1 \in \Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$, $x_2 \in K_{\rho_3} \setminus \overline{\Omega}_{\rho_2}$.

PROOF. We only consider the condition $(H3)$. If $(H4)$ holds, then the proof is similar to that of the case when $(H3)$ holds.

Firstly, we show that $i_k(T, K_{\rho_1}) = 1$. By using the condition $(H3)$ and Lemma 2.3 (iii), we obtain

$$\int_0^1 H(s,\tau)f(\tau,x(\tau))\nabla\tau \leqslant \varphi(A)\varphi(\rho_1)\int_0^1 H(\tau,\tau)\nabla\tau.$$

By performing above both side φ^{-1}, we have

$$-\varphi^{-1}(-\int_0^1 H(s,\tau)f(\tau,x(\tau))\nabla\tau) \leqslant A\rho_1\varphi^{-1}(-\int_0^1 H(\tau,\tau)\nabla\tau).$$

If we use the condition $(H3)$ and Lemma 2.3 (iii), we get

$$\begin{aligned}
Tx(t) &= -\int_0^1 G(t,s)\varphi^{-1}(-\int_0^1 H(s,\tau)f(\tau,x(\tau))\nabla\tau)\nabla s \\
&\leqslant A\rho_1\int_0^1 G(t,s)\varphi^{-1}(\int_0^1 H(\tau,\tau)\nabla\tau)\nabla s \\
&\leqslant A\rho_1\int_0^1 G(s,s)\varphi^{-1}(\int_0^1 H(\tau,\tau)\nabla\tau)\nabla s \\
&= \rho_1 = \|x\|.
\end{aligned}$$

Thus we have $Tx(t) \leqslant \|x\|$, which means $\|Tx(t)\| \leqslant \|x\|$ for $x \in \partial K_{\rho_1}$. By Lemma 3.1 (i), we have $i_k(T, K_{\rho_1}) = 1$.

Secondly, we show that $i_k(T, \Omega_{\rho_2}) = 0$. Let $e(t) = 1$ for $\forall t \in [0,1]$, then $e \in \partial K_1$. We claim that $x \neq Tx + \lambda e$ for $\lambda > 0$, $x \in \partial\Omega_{\rho_2}$. In fact, if not, there exist $x_0 \in \partial\Omega_{\rho_2}$ and $\lambda_0 > 0$ such that $x_0 = Tx_0 + \lambda_0 e$.
If we use the condition $(H3)$ and Lemma 2.3 (iii), we obtain

$$\int_0^1 H(s,\tau)f(\tau,x_0(\tau))\nabla\tau > \varphi(\rho_2)\varphi(B\Gamma)\int_0^1 \delta H(\tau,\tau)\nabla\tau.$$

By performing above both side φ^{-1}, we have

$$-\varphi^{-1}(-\int_0^1 H(s,\tau)f(\tau,x_0(\tau))\nabla\tau) > \rho_2 B\Gamma\varphi^{-1}(\int_0^1 \delta H(\tau,\tau)\nabla\tau).$$

Hence,

$$\begin{aligned}
x_0(t) &= Tx_0(t) + \lambda_0 e(t) \\
&= -\int_0^1 G(t,s)\varphi^{-1}(-\int_0^1 H(s,\tau)f(\tau,x_0(\tau))\nabla\tau)\nabla s + \lambda_0 e(t)
\end{aligned}$$

and

$$\begin{aligned} x_0(t) &= Tx_0(t) + \lambda_0 e(t) \\ &\geqslant \rho_2 B\Gamma \int_0^1 G(t,s)\varphi^{-1}(\int_0^1 \delta H(\tau,\tau)\nabla\tau)\nabla s + \lambda_0 1 \\ &\geqslant \rho_2 B\Gamma^2 \int_0^1 G(s,s)\varphi^{-1}(\int_0^1 \delta H(\tau,\tau)\nabla\tau)\nabla s + \lambda_0 \\ &= \rho_2\Gamma + \lambda_0. \end{aligned}$$

So, we get $\rho_2\Gamma \geqslant x_0(t) \geqslant \rho_2\Gamma + \lambda_0$ which is a contradiction. Hence by Lemma 3.1 (*ii*), it follows that $i_k(T, \Omega_{\rho_2}) = 0$. By Lemma 3.2 (*b*) and $\rho_1 < \Gamma\rho_2$, we have $\overline{K}_{\rho_1} \subset K_{\Gamma\rho_2} \subset \Omega_{\rho_2}$. It follows from Lemma 3.1 (*iii*) that T has a fixed point x_1 in $\Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$.

Finally, similar to proof of $i_k(T, \Omega_{\rho_1}) = 1$, we can prove that $i_k(T, \Omega_{\rho_3}) = 1$. By Lemma 3.2 (*b*) and $\rho_2 < \rho_3$, we have $\overline{\Omega}_{\rho_2} \subset \Omega_{\rho_3} \subset K_{\rho_3}$. It follows from Lemma 3.1 (iii) that T has a fixed point x_2 in $K_{\rho_3} \setminus \overline{\Omega}_{\rho_2}$.

Thus we can get the problem (1.1) has at least two positive solutions x_1, x_2 with $x_1 \in \Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$, $x_2 \in K_{\rho_3} \setminus \overline{\Omega}_{\rho_2}$ of BVP (1.1). □

Next we establish the existence of symmetric positive many solutions.

THEOREM 3.2. *Assume* $(H1) - (H2)$ *hold. Then we have the following assertions.*

(i) *There exist* $\{\rho_i\}_{i=1}^{2m_0+1} \subset (0,\infty)$ *with* $\rho_1 < \Gamma\rho_2 < \rho_2 < \rho_3 < \Gamma\rho_4 < ... < \rho_{2m_0+1}$ *such that*

$$f_0^{\rho_{2m-1}} < \varphi(A), \ \ m = 1, 2..., m_0 + 1, \ \ f_{\Gamma\rho_{2m}}^{\rho_{2m}} > \varphi(\Gamma B), \ \ m = 1, 2..., m_0.$$

Then problem (1.1) *has at least* $2m_0$ *solutions in* K.

(ii) *There exist* $\{\rho_i\}_{i=1}^{2m_0} \subset (0,\infty)$ *with* $\rho_1 < \Gamma\rho_2 < \rho_2 < \rho_3 < \Gamma\rho_4 < ... < \rho_{2m_0}$ *such that*

$$f_0^{\rho_{2m-1}} < \varphi(A), \ \ f_{\Gamma\rho_{2m}}^{\rho_{2m}} > \varphi(\Gamma B), \ \ m = 1, 2, 3, ..., m_0.$$

Then problem (1.1) *has at least* $2m_0 - 1$ *solutions in* K.

THEOREM 3.3. *Assume that* $(H1) - (H2)$ *hold. Then we have the following assertions.*

(i) *There exist* $\{\rho_i\}_{i=1}^{2m_0+1} \subset (0,\infty)$ *with* $\rho_1 < \rho_2 < \Gamma\rho_3 < \rho_3 < \rho_4 < ... < \rho_{2m_0+1}$ *such that*

$$f_0^{\rho_{2m}} < \varphi(A), \ \ m = 1, 2, 3, ..., m_0, \ \ f_{\Gamma\rho_{2m-1}}^{\rho_{2m-1}} > \varphi(\Gamma B), \ \ m = 1, 2, 3, ..., m_0 + 1.$$

Then problem(1.1) has at least $2m_0$ *solutions in* K.

(ii) *There exist* $\{\rho_i\}_{i=1}^{2m_0} \subset (0,\infty)$ *with* $\rho_1 < \rho_2 < \Gamma\rho_3 < \rho_3 < \rho_4 < ... < \rho_{2m_0}$ *such that*

$$f_0^{\rho_{2m}} < \varphi(A), \ \ f_{\Gamma\rho_{2m-1}}^{\rho_{2m-1}} > \varphi(\Gamma B), \ \ m = 1, 2, 3, ..., m_0.$$

Then problem (1.1) has at least $2m_0 - 1$ *solutions in* K.

4. Example

To illustrate how our main results can be used in practice we present an example.

EXAMPLE 4.1. Let $\mathbb{T} = [0, \frac{1}{3}] \cup \{\frac{1}{2}\} \cup [\frac{2}{3}, 1]$. We consider the following BVP:

$$\begin{cases} (x^{\triangle\nabla}(t))^{\triangle\nabla} = f(t, x(t)), \ t \in (0,1) \\ x(0) = x(1), \ x^{\triangle}(0) - x^{\triangle}(1) = 2x(\frac{1}{2}), \ x^{\triangle\nabla}(0) = x^{\triangle\nabla}(1), \\ (x^{\triangle\nabla})^{\triangle}(0) - (x^{\triangle\nabla})^{\triangle}(1) = 3x^{\triangle\nabla}(\frac{1}{2}), \end{cases}$$

Here

$$f(t, x(t)) = \begin{cases} 4.5125x(t), \ x \in [0, \ 0.02], \\ 1201.2x(t) - 23.93375, \ x \in [0.02, \ 0.025], \\ 51.33x(t) + 4.813, \ x \in [0.025, \ 0.1], \\ 9.946, \ x \in [0.1, \ 2]. \end{cases}$$

Since $\alpha = 2$, $\beta = 3$, we obtain $\Gamma = \frac{1}{4}$, $\delta = \frac{1}{12}$. Also, we choose $\varphi(x) = x$. It is obvious that φ satisfies assumption $(H2)$.

Also, we take $\rho_1 = 0.02, \ \rho_2 = 0.1, \ \rho_3 = 2$. After a simple calculation, we get $A \cong 4.987, B \cong 234.902$. Hence we have

$$\begin{aligned} f(t, x(t)) &< 0.09025, && for \ x \in [0, 0.02], \\ f(t, x(t)) &< 9.974, && for \ x \in [0.02, 0.025], \\ 6.096 < f(t, x(t)) &< 9.974, && for \ x \in [0.025, 0.1], \\ f(t, x(t)) &< 9.974, && for \ x \in [0.1, 2]. \end{aligned}$$

Thus the condition $(H3)$ of Theorem 3.1 is satisfied. Then by Theorem 3.1 the problem (1.1) has at least two symmetric positive solutions x_1, x_2 with $x_1 \in \Omega_{\rho_2} \setminus \overline{K}_{\rho_1}$, $x_2 \in K_{\rho_3} \setminus \overline{\Omega}_{\rho_2}$.

References

[1] B. Aulbach and S. Hilger. Linear dynamic processes with inhomogeneous time scale. in: G. A. Leonov (Ed.). *Nonlinear dynamics and quantum dynamical systems*(Mathematical research ; 59)(pp. 9–20). Berlin : Akademie Verlag, 1990.

[2] M. Bohner and A. Peterson. *Dynamic Equations on Time Scales. An Introduction with Applications.* Birkhäuser, Boston, 2001. ISBN: 0817642250

[3] M. Bohner and A. Peterson. *Advances in Dynamic Equations on Time Scales.* Birkhäuser, Boston, 2003. ISBN 0-8176-8230-9

[4] E. Çetin and F. S. Topal. Symmetric positive solutions of fourth-order boundary value problems for an increasing homeomorphism and homomorpishm on time scales. *Comput. Math. Appl.*, **63**(3)(2012) 669–678.

[5] E. Çetin and F. S. Topal. Existence of three symmetric positive solutions of fourth-order boundary value problems on symmetric time-scales. *Bull. Int. Math. Virtual Inst.*, **9**(1)(2019), 11-23.

[6] F. T. Fen and I. Y. Karaca. Existence of positive solutions for fourth-order impulsive integral boundary value problems on time scales. *Math. Methods Appl. Sci.*, **40**(16)(2017), 5727-5741.

[7] J. R. Graef, L. Kong and B. Yang. Existence, nonexistence and uniqueness of positive solutions to a three point fourth-order boundary value problem. *Nonlinear Stud.*, **18**(4)(2011), 565–575.

[8] J. R. Graef, L. Kong, Q. Kong and B. Yang. Positive solutions to a fourth-order boundary value problem. *Results Math.*, **59**(1-2)(2011), 141–155.

[9] D. Guo and V. Lakshmikantham. *Nonlinear Problems in Abstract Cones.* Academic Press, San Diego, 1988.

[10] M. Feng, P. Li and S. Sun. Symmetric positive solutions for fourth-order n-dimensional m-Laplace systems. *Bound. Value Probl.*, **2018** Article number: 63 (2018) .

[11] N. A. Hamal and F. Yörük. Symmetric positive solutions of fourth- order integral BVP for an increasing homeomorphism amd homomorphism with sing-changing nonlinearity on time scales. *Comput. Math. Appl.*, **59**(11)(2010), 3603–3611.

[12] J. Harjani and K. Sadarangani. Existence and uniqueness of positive solutions for a nonlinear fourth-order boundary value problem. *Positivity*, **14**(4)(2010), 849–858.

[13] S. Hilger. Analysis on measure chains-a unified approach to continuous and discrete calculus. *Results Math.*, **18**(1-2) (1990), 18–56.

[14] I. Y. Karaca. Fourth-order four-point boundary value problem on time scale. *Appl. Math. Lett.*, **21**(10) (2008), 1057–1063.

[15] V. Lakshmikantham, S. Sivasundaram and B. Kaymakçalan. *Dynamic Systems on Measure Chains*, Kluwer Academic Publishers Group, Dordrecht, Netherlands, 1996.

[16] K. Q. Lan. Multiple positive solutions of semilinear differential equations with singularities. *J. London Math. Soc.*, **63**(3)(2001), 690–704.

[17] Y. Li. Positive solutions of fourth-order boundary value problems with two parameters. *J. Math. Anal. Appl.*, **281**(2)(2003), 477–484.

[18] S. X. Luan and Y. L. Zhao. Symmetric positive solutions of fourth-order four-point boundary value problems with p-Laplacian operator. *Acta Math. Appl. Sin.*, **34**(2011), 801–812 (in Chinese)

[19] D.-X. Ma and W.-G. Ge. Multiple symmetric positive solutions of fourth-order two point boundary value problem. *J. Appl. Math. Comput.*, **2006**, Article 22: 295–306.

[20] H. Ma. Symmetric positive solutions for nonlocal boundary value problems of fourth-order. *Nonlinear Anal.*, **68**(3)(2008), 645–651.

[21] R. Ma and H. Wang. On the existence of positive solutions of fourth-order differential equations. *Appl. Anal.*, **59**(1-4)(1995), 225–231.

[22] T. Senlik and N. A. Hamal. Existence and multiplicity of symmetric positive solutions for nonlinear boundary-value problems with p-Laplacian operator. *Bound. Value Probl.*, **2014** Article number 44(2014), 12 pp.

[23] Y. J. Wang and J. Zhu. Existence and multiplicity of positive solutions for a class of fourth-order boundary value problems. *Acta Sci. Natur. Univ. Sunyatseni*, **2012**(1), 25–29.

[24] J. Xu and Z. Yang. Positive solutions for a fourth-order p-Laplacian boundary value problem. *Nonlinear Anal.*, **74**(7)(2011), 2612–2623.

[25] Z. Yang and J. Sun. Positive solutions of a fourth-order boundary value problem involving derivatives of all orders. *Commun. Pure Appl. Anal.*, **11**(5)(2012), 1615–1628.

[26] Q. Yao. Positive solutions and eigenvalue intervals of a nonlinear singular fourth-order boundary value problem. *Appl. Math.*, **58**(1)(2013), 93–110.

Received by editors 08.08.2019; Revised version 25.09.2019; Available online 30.09.2019.

Department of Mathematics Ege University,
35100 Bornova, Izmir, Turkey
E-mail address: ilkay.karaca@ege.edu.tr, aycansinanoglu@gmail.com