BULLETIN OF THE INTERNATIONAL MATHEMATICAL VIRTUAL INSTITUTE
ISSN (p) 2303-4874, ISSN (o) 2303-4955
www.imvibl.org /JOURNALS / BULLETIN
Bull. Int. Math. Virtual Inst., Vol. **10**(2)(2020), 201-212
DOI: 10.7251/BIMVI2002201Ö

Former
BULLETIN OF THE SOCIETY OF MATHEMATICIANS BANJA LUKA
ISSN 0354-5792 (o), ISSN 1986-521X (p)

A NOTE ON DERIVATIONS ON PRIME GAMMA RINGS WITH CHARACTERISTIC 2

Mehmet Ali Öztürk, Hasret Durna, and Tuba Acet

Abstract. In this paper we will study the relationship between the quotient Γ-ring and the existence of certain specific types of derivation of Γ-ring with characteristic 2.

1. Introduction

In 1957, Posner introduced derivation in ring in [**15**]. The different definitions of derivation such as semi-derivation, orthogonal derivation, θ-derivation, (σ, τ)-derivation, symmetric bi-derivation, objects more general than derivation, were introduced by many researchers (see, for example [**1, 2, 4, 5, 8, 17**]).

The notion of the Γ-ring was introduced by Nobusawa in [**9**]. The Γ-ring is a generalization of ring. In [**3**], the conditions in the Γ-ring defined by Nobusawa, were weakened by Barnes. In [**11**], [**12**] and [**14**], Öztürk and Jun studied extended centroid of prime Γ-ring and generalized centroid of semi-prime Γ-ring. In [**6**], Jing defined derivation in prime Γ-rings. Let M be a Γ-ring. A map $d : M \to M$ is called a derivation if $d(x+y) = d(x) + d(y)$ and $d(x\gamma y) = d(x)\gamma y + x\gamma d(y)$ for all $x, y \in M$ and $\gamma \in \Gamma$. In [**13**], Öztürk, Jun and Kim investigated the relationship between the quotient Γ-ring and derivation of Γ-ring M with $charM = 2$. In this paper we study the relationship between the quotient Γ-ring and the existence of certain specific types of derivation of Γ-ring M with $charM = 2$.

Throught in this paper, M in a Γ-ring in the sense of Barnes.

2010 *Mathematics Subject Classification.* 16N60, 16W25, 16Y99.
Key words and phrases. Gamma ring, extended centroid, θ-derivation, (θ, φ)-derivation.

2. Preliminaries

Let M and Γ be (additive) abelian groups. If the following conditions are hold in M, then we say that M is a Γ-*ring* (in the sense of Barnes),

(1) $a\alpha b \in M$,

(2) $(a+b)\,\alpha c = a\alpha b + a\alpha c.$
$a\,(\alpha+\beta)\,b = a\alpha b + a\beta b.$
$a\alpha\,(b+c) = a\alpha b + a\alpha c.$

(3) $(a\alpha b)\,\beta c = a\alpha\,(b\beta c)\,.$

for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. The center of Γ-ring M is defined by

$$Z = \{x \in M : x\gamma m = m\gamma x \text{ for all } m \in M \text{ and } \gamma \in \Gamma\}\,.$$

If U is an additive subgroup of M and $U\Gamma M \subseteq U$ (resp. $M\Gamma U \subseteq U$), then U is called a *right* (resp. *left*) *ideal of a* Γ-ring M. If U is both a right and a left ideal, then U is called an *ideal* of M. Let $a \in M$. The *principal right ideal* generated by a is the smallest right ideal of Γ-ring M containning a. This ideal is denoted by $\langle a\rangle_r$. Similarly one define $\langle a\rangle_l$ the *principal left ideal* generated by a and $\langle a\rangle$ the *principal two sided ideal* generated by a. Let $P \neq M$ is an ideal of a Γ-ring M. For any ideals U and V of M, if $U\Gamma V \subseteq P$ implies $U \subseteq P$ or $V \subseteq P$, then P is said to be *prime ideal of* M. A Γ-ring M is said to be *prime* if the zero ideal is prime.

THEOREM 2.1 ([**7**, Theorem 4]). *If M is a Γ-ring, the following conditions are equivalent*:

i) *M is prime Γ-ring.*

ii) *If $a, b \in M$ and $a\Gamma M\Gamma b = \langle 0\rangle$, then $a = 0$ or $b = 0$.*

iii) *If $\langle a\rangle$ and $\langle b\rangle$ are principal ideals of M such that $\langle a\rangle\,\Gamma\,\langle b\rangle = \langle 0\rangle$, then $a = 0$ or $b = 0$.*

iv) *If U and V are right ideals of M such that $U\Gamma V = \langle 0\rangle$, then $U = \langle 0\rangle$ or $V = \langle 0\rangle$.*

v) *If U and V are left ideals of M such that $U\Gamma V = \langle 0\rangle$, then $U = \langle 0\rangle$ or $V = \langle 0\rangle$.*

Let M and M' be two Γ-rings. A mapping $f : M \to M'$ of Γ-rings is called a Γ-ring homomorphism if $f\,(x+y) = f\,(x) + f\,(y)$ and $f\,(x\gamma y) = f\,(x)\,\gamma f\,(y)$ for all $x, y \in M$ and $\gamma \in \Gamma$.

Let M be a Γ-ring and $(A, +)$ be an abelian group. A is called a left M-module over M Γ-ring with respect to a mapping $\cdot : M \times \Gamma \times A \to A$ if for all $m, m' \in M$, $x, y \in A$ and $\alpha, \beta \in \Gamma$,

(i) $m\alpha\,(x+y) = m\alpha x + m\alpha y$,

(ii) $(m+m')\,\alpha x = m\alpha x + m\alpha y$,

(iii) $m\alpha\,(m'\beta x) = (m\alpha m')\,\beta x$.

Similarly, one can define a right M-module.

Let M be a Γ-ring and A, B be two left M-modules. A additive mapping f: $A \to B$ is called a left M-module homomorphism if $f\,(m\alpha x) = m\alpha f\,(x)$ for all $m \in M$, $x \in A$ and $\alpha \in \Gamma$.

Let M be a prime Γ-ring such that $M\Gamma M \neq M$. Denote

$$\mathcal{M} := \{(U,f) : \ U \ (\neq 0) \text{ is an ideal of } M \text{ and } f : U \to M \text{ is a right } M\text{-module homomorphism}\}.$$

Define a relation $\sim$ on $\mathcal{M}$ by

$$(U,f) \sim (V,g) \Leftrightarrow \exists W\,(\neq 0) \subset U \cap V \text{ such that } f = g \text{ on } W.$$

Since M is prime Γ-ring, it is possible to find a non-zero W and so "$\sim$" is an equivalent relation. This gives a chance for us to get a partition of $\mathcal{M}$. We denote the equivalence class by $Cl\,(U,f) = \widehat{f}$, where

$$\widehat{f} := \{\, g : V \to M | \ \ (U,f) \sim (V,g)\},$$

and denote by Q the set of all equivalence classes. Then Q is a Γ-ring, which is called the quotient Γ-ring of M (see [**11**], [**12**], and [**14**]).

Let M be a Γ-ring with unity. An element u in M is called a *unity* of M if it has a multiplicative inverse in M. If every nonzero element of M is a unity, we say that M is a Γ-*division ring.* A Γ-ring M is a Γ-*field* if it is a commutative Γ-division ring.

Lemma 2.1 ([**12**, Lemma 3.3]). *Let M be a prime Γ-ring such that $M\Gamma M \neq M$. Then the extended centroid C_Γ of M is a Γ-field.*

Let M be a Γ-ring and A be a M-module. A subset $B = \{b_i : \ i \in I\}$ of A is called linearly independent, if for every distinct $b_1, b_2, ..., b_n \in B$, $m_1, m_2, ..., m_n \in M$ and $\beta_1, \beta_2, ..., \beta_n \in \Gamma$ such that $\sum\limits_{i=0}^{n} m_i\beta_i b_i = 0$ implies that $m_1 = m_2 = ... = m_n = 0$.

Lemma 2.2 ([**11**, p. 476]). *Let M be a prime Γ-ring such that $M\Gamma M \neq M$ and the extended centroid C_Γ of M. If a_i and b_i are non-zero elements of M such that $\sum a_i\gamma_i x\beta_i b_i = 0$ for all $x \in M$ and $\gamma_i, \beta_i \in \Gamma$, then the a_i's (also b_i's) are linearly dependent over C_Γ. Moreover, if $a\gamma x\beta b = b\gamma x\beta a$ for all $x \in M$ and $\gamma, \beta \in \Gamma$ where $a\,(\neq 0)$, $b \in M$ are fixed, then there exists $\lambda \in C_\Gamma$ such that $b = \lambda\alpha a$ for all $\alpha \in \Gamma$.*

Lemma 2.3 ([**16**, Lemma 2]). *Let M be a prime Γ-ring, U a non-zero right (resp. left) ideal of M and $a \in M$. If $U\Gamma a = \langle 0\rangle$ (resp. $a\Gamma U = \langle 0\rangle$), then $a = 0$.*

Lemma 2.4 ([**10**, Lemma 1]). *Let M be a semi-prime Γ-ring and U a non-zero ideal of M. Then $Ann_l U = Ann_r U$.*

Let M be a semi-prime Γ-ring and U a non-zero ideal of M. In this case, we will write $Ann_l U = Ann_r U = Ann U$ by Lemma 2.4. Let us denote by F a set of all ideals of M which have zero annihilator in M. In this case, the set F is closed under multiplication by Lemma 2.4.

Theorem 2.2 ([**14**, Theorem 3.5]). *Let M be a semi-prime Γ-ring and Q the quotient Γ-ring of M. Then the Γ-ring Q satisfies the following properties*:

i) *For any element $q \in Q$, there exists an ideal $U_q \in F$ such that $q\,(U_q) \subseteq M$ (or $q\Gamma U_q \subseteq M$ for all $\gamma \in \Gamma$).*

ii) *If* $q \in Q$ *and* $q(U) = \langle 0 \rangle$ *for some* $U \in F$ *(or* $q\gamma U_q = \langle 0 \rangle$ *for some* $U \in F$ *and for all* $\gamma \in \Gamma$ *), then* $q = 0$*.*

iii) *If* $U \in F$ *and* $\Psi : U \to M$ *is a right* M*-module homomorphism, then there exists an element* $q \in Q$ *such that* $\Psi(u) = q(u)$ *for all* $u \in U$ *(or* $\Psi(u) = q\gamma u$ *for all* $u \in U$ *and* $\gamma \in \Gamma$ *).*

iv) *Let* W *be a submodule (an* (M, M)*-subbimodule) in* Q *and* $\Psi : W \to Q$ *a right* M*-module homomorphism. If* W *contains the ideal* U *ideal of the* Γ*-ring* M *such that* $\Psi(U) \subseteq M$ *and* $AnnU = Ann_r W$*, then there is an element* $q \in Q$ *such that* $\Psi(b) = q(b)$ *for any* $b \in W$ *(or* $\Psi(b) = q\gamma b$ *for any* $b \in W$ *and* $\gamma \in \Gamma$ *) and* $q(a) = 0$ *for any* $a \in Ann_r W$ *(or* $q\gamma a = 0$ *for any* $a \in Ann_r W$ *and* $\gamma \in \Gamma$ *).*

Lemma 2.5 ([**16**, Lemma 1]). *Let* M *be a prime* Γ*-ring and* Z *the center of* M*. If* $a, b, c \in M$ *and* $\beta, \gamma \in \Gamma$*, then*

$$[a\gamma b, c]_\beta = a\gamma [b, c]_\beta + [a, c]_\beta \gamma b + a\gamma (c\beta b) - a\beta (c\gamma b)$$

where $[a, b]_\gamma$ *is* $a\gamma b - b\gamma a$ *for all* $a, b \in M$ *and* $\gamma \in \Gamma$*.*

Definition 2.1. Let M be a Γ-ring and $\theta : M \to M$ be a function. An additive mapping $d : M \to M$ is called θ-derivation if $d(x\alpha y) = d(x)\alpha\theta(y) + x\alpha d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

Definition 2.2. Let M be a Γ-ring, $\theta : M \to M$ and $\varphi : M \to M$ be functions. An additive mapping $d : M \to M$ is called (θ, φ)-derivation if $d(x\alpha y) = d(x)\alpha\theta(y) + \varphi(x)\alpha d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

3. θ-derivation on prime Γ-ring

In this section, M is a prime Γ-ring such that $M\Gamma M \neq M$, Z is the center of M, C_Γ is the extended centroid of M and $[a, b]_\gamma = a\gamma b - b\gamma a$ for all $a, b \in M$ and $\gamma \in \Gamma$.

Lemma 3.1. *Let* M *be a prime* Γ*-ring,* U *be a non-zero ideal of* M*,* $\theta : M \to M$ *be a* Γ*-ring epimorphism and* d *be a* θ*-derivation of* M*. If* $a\Gamma d(U) = \langle 0 \rangle$ $(d(U)\Gamma a = \langle 0 \rangle)$ *for all* $a \in M$*, then* $a = 0$ *or* $d = 0$*.*

Proof. Let $a\Gamma d(U) = \langle 0 \rangle$. For all $u \in U$, $x \in M$ and $\alpha, \beta \in \Gamma$, we get

$$\begin{aligned} 0 = a\alpha d(u\beta x) &= a\alpha d(u)\beta\theta(x) + a\alpha u\beta d(x) \\ &= a\alpha u\beta d(x). \end{aligned}$$

From Lemma 2.3, we obtain that $a = 0$ or $d = 0$, since M is a prime Γ-ring. □

Lemma 3.2. *Let* M *be a prime* Γ*-ring,* U *be a non-zero ideal of* M*,* $\theta, \varphi : M \to M$ *be* Γ*-ring epimorphisms and* d *be a* (θ, φ)*-derivation of* M*. If* $a\Gamma d(U) = \langle 0 \rangle$ $(d(U)\Gamma a = \langle 0 \rangle)$ *for all* $a \in M$*, then* $a = 0$ *or* $d = 0$*.*

Proof. Let $a\Gamma d(U) = \langle 0 \rangle$. For all $u \in U$, $x \in M$ and $\alpha, \beta \in \Gamma$, we get

$$\begin{aligned} 0 = a\alpha d(u\beta x) &= a\alpha d(u)\beta\theta(x) + a\alpha\varphi(u)\beta d(x) \\ &= a\alpha\varphi(u)\beta d(x). \end{aligned}$$

From Lemma 2.3, we obtain that $a = 0$ or $d = 0$, since M is a prime Γ-ring and φ is a Γ-ring epimorphism. □

THEOREM 3.1. *Let M be a prime Γ-ring with $charM = 2$, $\theta : M \to M$ be a Γ-ring epimorphism and $0 \neq d_1, d_2$ be θ-derivaion on M such that $d_1\theta = \theta d_1$ and $d_2\theta = \theta d_2$. If for all $x \in M$,*

$$d_1 d_2 (x) = 0 \tag{3.1}$$

then there exists $\lambda \in C_\Gamma$ such that $d_2 (x) = \lambda\alpha d_1 (x)$ for all $x \in M$ and $\alpha \in \Gamma$.

PROOF. Let $\alpha \in \Gamma$ and $x, y \in M$. Replacing x by $x\alpha y$ in (3.1) and using (3.1), we get

$$0 = d_2 (x) \alpha d_1 (\theta (y)) + d_1 (x) \alpha d_2 (\theta (y)), \tag{3.2}$$

since $charM = 2$ and $d_2\theta = \theta d_2$.

Replacing x by $x\beta z$ in (3.2) and using (3.2), we get

$$d_2 (x) \beta\theta (z) \alpha d_1 (\theta (y)) = d_1 (x) \beta\theta (z) \alpha d_2 (\theta (y)), \tag{3.3}$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Since θ is Γ-ring epimorphism, for all $x, w, m \in M$ and $\alpha, \beta \in \Gamma$, we get

$$d_2 (x) \beta m\alpha d_1 (w) = d_1 (x) \beta m\alpha d_2 (w). \tag{3.4}$$

Now if we replace w by x in (3.4), then we obtain

$$d_2 (x) \beta m\alpha d_1 (x) = d_1 (x) \beta m\alpha d_2 (x), \tag{3.5}$$

for all $x, m \in M$ and $\beta, \alpha \in \Gamma$. If $d_1 (x) \neq 0$, then there exists $\lambda (x) \in C_\Gamma$ such that $d_2 (x) = \lambda (x) \gamma d_1 (x)$ for all $x \in M$ and $\gamma \in \Gamma$. Thus, if $d_1 (x) \neq 0 \neq d_1 (w)$, then (3.4) implies that

$$(\lambda (y) - \lambda (x)) \gamma d_1 (x) \beta m\alpha d_1 (w) = 0. \tag{3.6}$$

Since M is a prime Γ-ring, we conclude by using Lemma 3.1 that $\lambda (y) = \lambda (x)$ for all $x, y \in M$. Hence we prove that there exists $\lambda \in C_\Gamma$ such that $d_2 (x) = \lambda\gamma d_1 (x)$ for all $x \in M$ and $\gamma \in \Gamma$ with $d_1 (x) \neq 0$. On the other hand, if $d_1 (x) = 0$, then $d_2 (x) = 0$ as well. Therefore, $d_2 (x) = \lambda\gamma d_1 (x)$ for all $x \in M$ and $\gamma \in \Gamma$. This completes the proof. □

PROPOSITION 3.1. *Let M be a prime Γ-ring with $charM = 2$, $\theta : M \to M$ be a Γ-ring epimorphism and $0 \neq d$ be a θ-derivation of M such that $d\theta = \theta d$. If for all $x \in M$,*

$$d (x) \in Z, \tag{3.7}$$

then there exists $\lambda (m) \in C_\Gamma$ such that $d (m) = \lambda (m) \alpha d (z)$ for all $m, z \in M$ and $\alpha \in \Gamma$ or M is commutative.

PROOF. From (3.7), we have

$$[d (x), y]_\beta = 0, \tag{3.8}$$

for all $x, y \in M$ and $\beta \in \Gamma$.

Replacing x by $x\gamma z$ in (3.8), we get

$$[d(x\gamma z), y]_\beta = d(x)\gamma\theta(z)\beta y + x\gamma d(z)\beta y - y\beta d(x)\gamma\theta(z) - y\beta x\gamma d(z). \tag{3.9}$$

Since θ is Γ-ring epimorphism, we get

$$\begin{aligned} 0 &= d(x)\gamma(m\beta y - y\beta m) + d(z)\gamma(x\beta y - y\beta x) \\ &= d(x)\gamma[m, y]_\beta + d(z)\gamma[x, y]_\beta, \end{aligned} \tag{3.10}$$

for all $x, y, z, m \in M$ and $\gamma, \beta \in \Gamma$. Replacing x by $d(x)$ in (3.10), we get

$$0 = d^2(x)\gamma[m, y]_\beta + d(z)\gamma[d(x), y]_\beta, \tag{3.11}$$

for all $x, y, z, m \in M$. Using (3.8) in (3.11), we get

$$d^2(x)\gamma[m, y]_\beta = 0, \tag{3.12}$$

for all $x, y, m \in M$ and $\gamma, \beta \in \Gamma$.

Now, substituting $x\alpha z$ for x in (3.12), it follows from $charM = 2$ that

$$0 = d^2(x)\alpha\theta^2(z)\gamma[m, y]_\beta, \tag{3.13}$$

for all $x, y, z, m \in M$ and $\gamma, \beta \in \Gamma$. Since M is prime Γ-ring and θ is Γ-ring epimorphism, we obtain

$$d^2(x) = 0 \text{ for all } x \in M \text{ or } [m, y]_\beta = 0 \text{ for all } y, m \in M \text{ and } \beta \in \Gamma. \tag{3.14}$$

From (3.14), if $d^2(x) = 0$ for all $x \in M$, then replacing x by $x\gamma y$ in this last relation, it follows from $d(x) \in Z$ that

$$d(x)\gamma d(m) = d(m)\gamma d(x) \text{ for all } x, m \in M \text{ and } \gamma \in \Gamma. \tag{3.15}$$

Replacing x by $x\alpha z$ in (3.15), it follows from (3.15) that for all $x, z, m \in M$ and $\alpha \in \Gamma$,

$$d(x)\alpha y\gamma d(m) = d(m)\alpha y\gamma d(x)$$

since θ is Γ-ring epimorphism.

If $d(m) \neq 0$, then there exists $\lambda(m) \in C_\Gamma$ such that $d(x) = \lambda(x)\alpha d(m)$ for all $x, m \in M$ and $\alpha \in \Gamma$ by Lemma 2.2. On the other hand, it follows from (3.14) that if $[m, y]_\beta = 0$ for all $y, m \in M$ and $\beta \in \Gamma$, then M is commutative. This completes the proof. □

THEOREM 3.2. *Let M be a prime Γ-ring with $charM = 2$, U be a non-zero ideal of M, $\theta : M \to M$ be a Γ-ring epimorphism and $0 \neq d_1, d_2$ be θ-derivation on M such that $d_1\theta = \theta d_1$ and $d_2\theta = \theta d_2$. If $d_2(U) \subseteq U$ and for all $u \in U$,*

$$d_1 d_2(u) = 0 \tag{3.16}$$

then there exists $\lambda \in C_\Gamma$ such that $d_2(x) = \lambda\alpha d_1(x)$ for all $x \in M$ and $\alpha \in \Gamma$.

PROOF. Let $\gamma \in \Gamma$ and $u, v \in U$. Replacing u by $d_2(u)\gamma v$ in (3.16) and using hypothesis, we have

$$d_2^2(u)\gamma d_1(w) = 0, \tag{3.17}$$

for all $u \in U$, $w \in M$ and $\gamma \in \Gamma$. Since $d_1 \neq 0$, for all $u \in U$, $d_2^2(u) = 0$ from Lemma 3.1. Replacing u by $u\gamma x$ in (3.16) and using hypothesis, we get

$$0 = d_2(u)\gamma d_1(\theta(x)) + d_1(u)\gamma\theta(d_2(x)) + u\gamma d_1(d_2(x)), \tag{3.18}$$

for all $u \in U$, $x \in M$ and $\gamma \in \Gamma$. Replacing u by $d_2(u)$ in (3.18) and using (3.17), we get

$$d_2(u)\gamma d_1(d_2(x)) = 0.$$

Since $d_2 \neq 0$, $d_1(d_2(x)) = 0$ for all $x \in M$ from Lemma 3.1. From here, there exists $\lambda \in C_\Gamma$ such that $d_2(x) = \lambda\alpha d_1(x)$ for all $x \in M$ and $\alpha \in \Gamma$. □

THEOREM 3.3. *Let M be a prime Γ-ring, U be a non-zero right ideal of M, $\theta : M \to M$ be Γ-ring epimorphism and $0 \neq d$ be a θ-derivation of M such that $d\theta = \theta d$. If*

$$d(u)\gamma a = 0 \text{ for all } u \in U \text{ and } \gamma \in \Gamma, \tag{3.19}$$

where a is a fixed element of M, then there exists an element of Q such that $q\gamma a = 0$ implies $q\gamma u = 0$ for all $u \in U$ and $\gamma \in \Gamma$.

PROOF. Let $u \in U$, $x \in M$ and $\beta \in \Gamma$. Since U is a right ideal of M, we have $u\beta x \in U$. Replacing u by $u\beta x$ in (3.19), we get

$$d(u)\beta\theta(x)\gamma a + u\beta d(x)\gamma a = 0 \tag{3.20}$$

for all $u \in U$, $x \in M$ and $\gamma, \beta \in \Gamma$. And so,

$$d(u)\beta\left(\sum \theta(x)\gamma a\alpha m\right) = -\left(u\beta\left(\sum d(x)\gamma a\alpha m\right)\right)$$

for all $m \in M$, $\alpha \in \Gamma$. Therefore, for any $v \in V = M\Gamma a\Gamma M$ which is a non-zero ideal of M, we have

$$d(u)\beta v = u\beta f(v) \tag{3.21}$$

for all $u \in U$. $f(v)$ is independent of u but it is dependent on v. Since M is a prime Γ-ring, $f(v)$ is well-defined. Note that $v\alpha y \in V$ for any $y \in M$, $v \in V$ and $\alpha \in \Gamma$. Replacing v by $v\alpha y$ in (3.21), we get

$$d(u)\beta v\alpha y = u\beta f(v\alpha y),$$
$$u\beta f(v)\alpha y = u\beta f(v\alpha y)$$
$$u\beta(f(v)\alpha y - f(v\alpha y)) = 0$$

which implies from Lemma 2.3 that

$$f(v)\alpha y = f(v\alpha y) \tag{3.22}$$

for all $y \in M$, $v \in V$ and $\alpha \in \Gamma$. It follows from (3.22) that $f : V \to M$ is a right M-module homomorphism. In this case, $q = Cl(V, f) \in Q$. Moreover, $f(v) = q\beta v$ for all $v \in V$ and $\alpha \in \Gamma$ by Theorem 2.2. Let $x \in M$, $v \in V$, $u \in U$ and $\gamma, \beta \in \Gamma$. Replacing v by $\theta(x)\gamma v$ in (3.21), we get

$$d(u)\beta\theta(x)\gamma v = u\beta f(\theta(x)\gamma v) = u\beta q\beta\theta(x)\gamma v. \tag{3.23}$$

Also, replacing u by $u\gamma x$ in (3.21), we get

$$d(u\gamma x)\beta v = u\gamma x\beta f(v)$$

$$d(u)\gamma\theta(x)\beta v + u\gamma d(x)\beta v = u\gamma x\beta f(v)$$

$$d(u)\gamma\theta(x)\beta v = u\gamma x\beta q\beta v - u\gamma d(x)\beta v. \quad (3.24)$$

Now, replacing β by γ and replacing γ by β in (3.24), we get

$$d(u)\beta\theta(x)\gamma v = u\beta x\gamma q\gamma v - u\beta d(x)\gamma v. \quad (3.25)$$

Thus, from (3.23) and (3.25), we obtain

$$u\beta q\beta\theta(x)\gamma v = u\beta x\gamma q\gamma v - u\beta d(x)\gamma v$$

$$u\beta(q\beta\theta(x) - x\gamma q + d(x))\gamma v = 0$$

for all $x \in M$, $v \in V$, $u \in U$ and $\gamma, \beta \in \Gamma$. Hence $d(x) = x\gamma q - q\beta\theta(x)$ for all $x \in M$ and $\gamma, \beta \in \Gamma$ by Lemma 2.3. Let $u \in U$ and $x \in M$, $d(u) = q\alpha u - \theta(u)\beta q$ and $d(x) = q\beta x - \theta(x)\alpha q$. Then we have

$$0 = d(u\beta x)\gamma a = (q\alpha(u\beta x) - \theta(u\beta x)\beta q)\gamma a.$$

Thus, $q\alpha u\beta x\gamma a = \theta(u\beta x)\beta q\gamma a$. If $q\gamma a = 0$, then $q\alpha u\beta x\gamma a = 0$, and so since M is prime Γ-ring, we get $q\Gamma U = \langle 0\rangle$. □

THEOREM 3.4. *Let M be a prime Γ-ring with $char M \neq 2$, U a non-zero right ideal of M and $0 \neq d$ be a θ-derivation of M such that $d\theta = \theta d$ and $\theta(U) \subseteq U$. Then the subring of M generated by $d(U)$ contains no nonzero right ideals of M if and only if $d(U)\Gamma\theta(U) = \langle 0\rangle$.*

PROOF. Let A be the subring generated by $d(U)$. Let $S = A \cap U$, $u \in U$, $s \in S$ and $\gamma \in \Gamma$. Then $d(s\gamma u) = d(s)\gamma\theta(u) + s\gamma d(u)$, and so we have $d(s)\gamma\theta(u) \in S$. Thus $d(S)\Gamma\theta(U)$ is a right ideal of M. In this case, $d(S)\Gamma\theta(U) = \langle 0\rangle$ by hypothesis. $d(u\gamma a) = d(u)\gamma\theta(a) + u\gamma d(a) \in S$ and $d(u)\gamma\theta(a) \in S$ where $u \in U$, $a \in A$. Thus, we have $u\gamma d(a) \in S$. Therefore, $0 = d(u\gamma d(a))\beta\theta(u) = \left(d(u)\gamma\theta(d(a)) + u\gamma d^2(a)\right)\beta\theta(u)$. Since M is prime Γ-ring, it follows from Lemma 2.3 that

$$d(u)\gamma\theta(d(a)) + u\gamma d^2(a) = 0 \quad (3.26)$$

for all $u \in U$, $\gamma \in \Gamma$ and $a \in A$. Replacing u by $u\beta v$ where $v \in U$, $\beta \in \Gamma$ in (3.26), we get, for all $u, v \in U$, $\beta, \gamma \in \Gamma$ and $a \in A$

$$d(u)\beta v\gamma\theta(d(a)) = 0.$$

Since M is prime Γ-ring , we get $d(U)\Gamma U = \langle 0\rangle$ or $d(A)\Gamma U = \langle 0\rangle$. If $d(A)\Gamma U = \langle 0\rangle$, then $d^2(U)\Gamma U = \langle 0\rangle$. Let u, $v \in U$ and $\beta \in \Gamma$. Then

$$0 = d(d(u\beta v)) = d^2(u)\beta\theta^2(v) + 2d(u)\beta d(\theta(v)) + u\beta d^2(v),$$

and so we have $d(u)\beta d(\theta(v)) = 0$ for all $u, v \in U$ and $\beta \in \Gamma$ by $char M \neq 2$. Replacing u by $u\gamma w$ where $w \in U$, $\gamma \in \Gamma$ in last relation, we have

$$d(u)\gamma\theta(w)\beta d(\theta(v)) = 0$$

which yields $d(u)\gamma\theta(w) = 0$ for all $u, w \in U$ and $\gamma \in \Gamma$.

Conversely, assume that $d(U)\Gamma\theta(U) = \langle 0\rangle$ .Then $A\Gamma d(U) = \langle 0\rangle$. Since M is a prime Γ-ring, A contains no non-zero right ideals. □

4. (θ, φ)-derivation on prime Γ-ring

THEOREM 4.1. *Let M be a prime Γ-ring with $charM = 2$, $\theta : M \to M$ and $\varphi : M \to M$ be Γ-ring epimorphisms and $0 \neq d_1, d_2$ be (θ, φ)-derivaions on M such that $d_i\theta = \theta d_i$ and $d_i\varphi = \varphi d_i$, $i = 1, 2$. If for all $x \in M$,*

$$d_1 d_2 (x) = 0 \tag{4.1}$$

then there exists $\lambda \in C_\Gamma$ such that $d_2 (x) = \lambda\alpha d_1 (x)$ for all $x \in M$ and $\alpha \in \Gamma$.

PROOF. Let $\alpha \in \Gamma$ and $x, y \in M$. Replacing x by $x\alpha y$ in (4.1) and using (4.1), we get

$$d_2 (\varphi (x)) \alpha d_1 (\theta (y)) = d_1 (\varphi (x)) \alpha d_2 (\theta (y)), \tag{4.2}$$

since $charM = 2$ and $d_i\theta = \theta d_i$, $d_i\varphi = \varphi d_i$ for $i = 1, 2$.

Since φ is Γ-ring epimorphism, for all $m, y \in M$ and $\alpha \in \Gamma$, we get

$$d_2 (m) \alpha d_1 (\theta (y)) = d_1 (m) \alpha d_2 (\theta (y)). \tag{4.3}$$

Replacing m by $m\beta z$ in (4.2) and using (4.2), we get

$$d_2 (m) \beta\theta (z) \alpha d_1 (\theta (y)) = d_1 (m) \beta\theta (z) \alpha d_2 (\theta (y)), \tag{4.4}$$

for all $m, y, z \in M$ and $\alpha, \beta \in \Gamma$. Since θ is Γ-ring epimorphism, for all $m, n, k \in M$ and $\alpha, \beta \in \Gamma$, we get

$$d_2 (m) \beta n\alpha d_1 (k) = d_1 (m) \beta n\alpha d_2 (k). \tag{4.5}$$

Now if we replace k by m in (4.5), then we obtain

$$d_2 (m) \beta n\alpha d_1 (m) = d_1 (m) \beta n\alpha d_2 (m), \tag{4.6}$$

for all $n, m \in M$ and $\beta, \alpha \in \Gamma$. If $d_1 (m) \neq 0$, then there exists $\lambda (m) \in C_\Gamma$ such that $d_2 (m) = \lambda (m) \gamma d_1 (m)$ for all $x \in M$ and $\gamma \in \Gamma$. Thus, if $d_1 (m) \neq 0 \neq d_1 (k)$, then (4.5) implies that

$$(\lambda (k) - \lambda (m)) \gamma d_1 (m) \beta z\alpha d_1 (k) = 0. \tag{4.7}$$

Since M is a prime Γ-ring, we conclude by using Lemma 2.2 that $\lambda (k) = \lambda (m)$ for all $k, m \in M$. Hence we prove that there exists $\lambda \in C_\Gamma$ such that $d_2 (m) = \lambda\gamma d_1 (m)$ for all $m \in M$ and $\gamma \in \Gamma$ with $d_1 (m) \neq 0$. On the other hand, if $d_1 (m) = 0$, then $d_2 (m) = 0$ as well. Therefore, $d_2 (m) = \lambda\gamma d_1 (m)$ for all $m \in M$ and $\gamma \in \Gamma$. This completes the proof. □

PROPOSITION 4.1. *Let M be a prime Γ-ring with $charM = 2$, $\theta : M \to M$ and $\varphi : M \to M$ be Γ-ring epimorphisms and $0 \neq d$ be a (θ, φ)-derivation of M such that $d\theta = \theta d$, $d\varphi = \varphi d$. If for all $x \in M$,*

$$d (x) \in Z, \tag{4.8}$$

then there exists $\lambda (m) \in C_\Gamma$ such that $d (m) = \lambda (m) \alpha d (z)$ for all $m, z \in M$ and $\alpha \in \Gamma$ or M is commutative.

Proof. From (4.8), we have

$$[d(x), y]_\beta = 0, \tag{4.9}$$

for all $x, y \in M$ and $\beta \in \Gamma$.

Replacing x by $x\gamma z$ in (4.9), we get

$$[d(x\gamma z), y]_\beta = d(x)\gamma\theta(z)\beta y + \varphi(x)\gamma d(z)\beta y - y\beta d(x)\gamma\theta(z) - y\beta\varphi(x)\gamma d(z). \tag{4.10}$$

Since θ is Γ-ring epimorphism, we get

$$\begin{aligned} 0 &= d(x)\gamma(m\beta y - y\beta m) + d(z)\gamma(\varphi(x)\beta y - y\beta\varphi(x)) \\ &= d(x)\gamma[m, y]_\beta + d(z)\gamma[\varphi(x), y]_\beta, \end{aligned} \tag{4.11}$$

for all $x, y, z, m \in M$ and $\gamma, \beta \in \Gamma$. Replacing x by $d(x)$ in (4.11), we get

$$0 = d^2(x)\gamma[m, y]_\beta + d(z)\gamma[\varphi(d(x)), y]_\beta, \tag{4.12}$$

for all $x, y, z, m \in M$. Since φ is Γ-ring epimorphism, we get

$$0 = d^2(x)\gamma[m, y]_\beta + d(z)\gamma[d(n), y]_\beta, \tag{4.13}$$

for all $x, y, z, m, n \in M$ and $\gamma, \beta \in \Gamma$.

Using (4.9) in (4.13), we get

$$d^2(x)\gamma[m, y]_\beta = 0, \tag{4.14}$$

for all $x, y, m \in M$ and $\gamma, \beta \in \Gamma$.

Now, substituting $x\alpha z$ for x in (4.14), we get

$$\begin{aligned} 0 \;=\; & (d^2(x)\alpha\theta^2(z) + \varphi(d(x))\alpha d(\theta(z)) \\ & + d(\varphi(x))\alpha\theta(d(z)) + \varphi^2(x)\alpha d^2(z))\gamma[m, y]_\beta, \end{aligned}$$

for all $x, y, z, m \in M$ and $\alpha, \gamma, \beta \in \Gamma$.

Using (4.13) in last relation, we have

$$0 = d^2(x)\alpha\theta^2(z)\gamma[m, y]_\beta, \tag{4.15}$$

for all $x, y, z, m \in M$ and $\alpha, \gamma, \beta \in \Gamma$. Since M is prime Γ-ring and θ is Γ-ring epimorphism, we obtain

$$d^2(x) = 0 \text{ for all } x \in M \text{ or } [m, y]_\beta = 0 \text{ for all } y, m \in M \text{ and } \beta \in \Gamma. \tag{4.16}$$

From (4.16), if $d^2(x) = 0$ for all $x \in M$, then replacing x by $x\gamma y$ in this last relation, it follows from $d(x) \in Z$ that

$$d(x)\gamma d(m) = d(m)\gamma d(x) \text{ for all } x, m \in M \text{ and } \gamma \in \Gamma. \tag{4.17}$$

Replacing x by $x\alpha n$ in (4.17), it follows from (4.17) that for all $x, n, m \in M$ and $\alpha, \gamma \in \Gamma$,

$$d(x)\alpha\theta(n)\gamma d(m) = d(m)\gamma d(x)\alpha\theta(n) \tag{4.18}$$

Since θ is Γ-ring epimorphism, we have

$$d(x)\alpha k\gamma d(m) = d(m)\gamma d(x)\alpha k,$$

for all $x, m, k \in M$ and $\alpha, \gamma \in \Gamma$

If $d(x) \neq 0$, then there exists $\lambda(m) \in C_\Gamma$ such that $d(x) = \lambda(x)\alpha d(m)$ for all $x, m \in M$ and $\alpha \in \Gamma$ by Lemma 2.2. On the other hand, it follows from (4.16) that if $[m, y]_\beta = 0$ for all $y, m \in M$ and $\beta \in \Gamma$, then M is commutative. This completes the proof. □

THEOREM 4.2. *Let M be a prime Γ-ring with $charM = 2$, U be a non-zero ideal of M, $\theta : M \to M$ and $\varphi : M \to M$ be Γ-ring epimorphisms and $0 \neq d_1, d_2$ be (θ, φ)-derivations on M such that $d_i\theta = \theta d_i$ and $d_i\varphi = \varphi d_i$, $i = 1, 2$. If $d_2(U) \subseteq U$ and for all $u \in U$,*

$$d_1 d_2(u) = 0 \tag{4.19}$$

then there exists $\lambda \in C_\Gamma$ such that $d_2(x) = \lambda\alpha d_1(x)$ for all $x \in M$ and $\alpha \in \Gamma$.

PROOF. Let $\gamma \in \Gamma$ and $u, v \in U$. Replacing u by $d_2(u)\gamma v$ in (4.19) and using hypothesis, we have

$$\varphi\left(d_2^2(u)\right)\gamma d_1(\theta(v)) = 0, \tag{4.20}$$

for all $u, v \in U$ and $\gamma \in \Gamma$. Since θ and φ are Γ-ring epimorphisms, we get

$$d_2^2(y)\gamma d_1(z) = 0, \tag{4.21}$$

for all $y, z \in M$ and $\gamma \in \Gamma$. Since $d_1 \neq 0$, for all $y \in M$, $d_2^2(y) = 0$ from Lemma 3.2. Replacing u by $u\gamma x$ in (4.19) and using hypothesis, we get

$$0 = d_2(u)\gamma d_1(\theta(x)) + d_1(u)\gamma\theta(d_2(x)) + u\gamma d_1(d_2(x)), \tag{4.22}$$

for all $u \in U$, $x \in M$ and $\gamma \in \Gamma$. Replacing u by $d_2(u)$ in (4.22) and using (4.21), we get

$$d_2(u)\gamma d_1(d_2(x)) = 0.$$

Since $d_2 \neq 0$, $d_1(d_2(x)) = 0$ for all $x \in M$ from Lemma 3.2. From here, there exists $\lambda \in C_\Gamma$ such that $d_2(x) = \lambda\alpha d_1(x)$ for all $x \in M$ and $\alpha \in \Gamma$. □

References

[1] M. Ashraf and N. Rehman. On (σ, τ)-derivations in prime rings. *Arch. Math., Brno* , **38**(4)(2002), 259–264.
[2] N. Aydın and K. Kaya. Some generalizations in prime rings with (σ, τ)-derivation. *Turk. J. Math.*, **16**(1992), 169–176.
[3] W. E. Barnes. On the Γ-rings of Nobusawa. *Pasific J. Math.*, **18**(3)(1966), 411–422.
[4] J. C. Chang. On semi-derivation on prime rings. *Chin. J. Math.*, **12**(4)(1984), 225–262.
[5] J. C. Chang. α-derivations with invertible values. *Bull. Inst. Math. Acad. Sinica*, **13**(4)(1985), 323–333.
[6] F. J. Jing. On derivations of Γ-rings. *Qufu Shifin Daxue Xuebeo Ziran Kexue Ban.*, **13**(4)(1987), 159–161.
[7] S. Kyuno. On prime gamma rings. *Pasific J. Math.*, **75**(1)(1978), 185–190.
[8] G. Maksa. On the trace of symmetric bi-derivations. *C. R. Math. Rep. Acad. Sci. Canada*, **9**(1987), 303–307.
[9] N. Nobusawa. On a generalization of the ring theory. *Osaka J. Math.*, **1**(1)(1964), 81–89.
[10] M. A. Öztürk, M. Sapanci, M. Soytürk and K. H. Kim. Symmetric bi-derivation on prime gamma rings. *Sci. Math.*, **3**(2)(2000), 273–281.
[11] M. A. Öztürk and Y. B. Jun. On the centroid of the prime gamma rings. *Comm. Korean Math. Soc.*, **15**(3)(2000), 469–479.

[12] M. A. Öztürk and Y. B. Jun. On the centroid of the prime gamma rings II. *Turk J. Math.*, **25**(3)(2001), 367–377.
[13] M. A. Öztürk, Y. B. Jun and K. H. Kim. On derivations of prime gamma rings. *Turk J. Math.*, **26**(3)(2002), 317–327.
[14] M. A. Öztürk and Y. B. Jun. Regularity of the generalized centroid of semiprime gamma rings. *Comm. Korean Math. Soc.*, **19**(2)(2004), 233–242.
[15] E. C. Posner. Derivations in prime rings. *Proc. Amer. Math. Soc.*, **8**(6)(1957), 1093–1100.
[16] M. Soytürk. The commutativity in prime gamma rings with derivation. *Turk J. Math.*, **18**(2)(1994), 149–155.
[17] M. S. Yenigul and N. Argaç. On ideals and orthogonal derivations. *J. of Southwest China Normal Univ.*, **20**(2)(1995), 137–140.

Received by editors 05.03.2019; Revised version 14.09.2019; Available online 23.09.2019.

Adiyaman University, Faculty of Arts and Sciences, Department of Mathematics, 02040 Adiyaman, TURKEY
E-mail address: mehaliozturk@gmail.com

Sivas Cumhuriyet University, Faculty of Sciences, Department of Mathematics, 58140 Sivas, TURKEY
E-mail address: hyazarli@cumhuriyet.edu.tr

Adiyaman University, Faculty of Arts and Sciences, Department of Mathematics, 02040 Adiyaman, TURKEY
E-mail address: acettuba02@gmail.com