A NOTE ON DERIVATIONS ON PRIME GAMMA RINGS WITH CHARACTERISTIC 2

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Abstract. In this paper we will study the relationship between the quotient Γ-ring and the existence of certain specific types of derivation of Γ-ring with characteristic 2.

1. Introduction

In 1957, Posner introduced derivation in ring in [15]. The different definitions of derivation such as semi-derivation, orthogonal derivation, θ-derivation, (σ, τ)-derivation, symmetric bi-derivation, objects more general than derivation, were introduced by many researchers (see, for example [1, 2, 4, 5, 8, 17]).

The notion of the Γ-ring was introduced by Nobusawa in [9]. The Γ-ring is a generalization of ring. In [3], the conditions in the Γ-ring defined by Nobusawa, were weakened by Barnes. In [11], [12] and [14], Öztürk and Jun studied extended centroid of prime Γ-ring and generalized centroid of semi-prime Γ-ring. In [6], Jing defined derivation in prime Γ-rings. Let M be a Γ-ring. A map d : M → M is called a derivation if d(x + y) = d(x) + d(y) and d(xγy) = d(x)γy + xγd(y) for all x, y ∈ M and γ ∈ Γ. In [13], Öztürk, Jun and Kim investigated the relationship between the quotient Γ-ring and derivation of Γ-ring M with charM = 2. In this paper we study the relationship between the quotient Γ-ring and the existence of certain specific types of derivation of Γ-ring M with charM = 2.

Throught in this paper, M in a Γ-ring in the sense of Barnes.

2010 Mathematics Subject Classification. 16N60, 16W25, 16Y99.
Key words and phrases. Gamma ring, extended centroid, θ-derivation, (θ, ϕ)-derivation.
2. Preliminaries

Let $M$ and $\Gamma$ be (additive) abelian groups. If the following conditions are hold in $M$, then we say that $M$ is a $\Gamma$-ring (in the sense of Barnes),

1. $aab \in M$,
2. $(a + b) ac = aob + aac$.
3. $a(a + \beta) b = aob + a\beta b$.
4. $aa(b + c) = aob + aac$.

for all $a, b, c \in M$ and $\alpha, \beta \in \Gamma$. The center of $\Gamma$-ring $M$ is defined by

$$Z = \{x \in M : x\gamma m = m\gamma x \text{ for all } m \in M \text{ and } \gamma \in \Gamma\}.$$ 

If $U$ is an additive subgroup of $M$ and $U \Gamma M \subseteq U$ (resp. $MTU \subseteq U$), then $U$ is called a right (resp. left) ideal of a $\Gamma$-ring $M$. If $U$ is both a right and a left ideal, then $U$ is called an ideal of $M$. Let $a \in M$. The principal right ideal generated by $a$ is the smallest right ideal of $\Gamma$-ring $M$ containing $a$. This ideal is denoted by $\langle a \rangle_r$. Similarly one define $\langle a \rangle_l$ the principal left ideal generated by $a$ and $\langle a \rangle$ the principal two sided ideal generated by $a$. Let $P \neq M$ is an ideal of a $\Gamma$-ring $M$. For any ideals $U$ and $V$ of $M$, if $UTV \subseteq P$ implies $U \subseteq P$ or $V \subseteq P$, then $P$ is said to be prime ideal of $M$. A $\Gamma$-ring $M$ is said to be prime if the zero ideal is prime.

**Theorem 2.1** ([7, Theorem 4]). If $M$ is a $\Gamma$-ring, the following conditions are equivalent:

i) $M$ is prime $\Gamma$-ring.
ii) If $a, b \in M$ and $a\Gamma M b = \{0\}$, then $a = 0$ or $b = 0$.
iii) If $\langle a \rangle$ and $\langle b \rangle$ are principal ideals of $M$ such that $\langle a \rangle \Gamma \langle b \rangle = \{0\}$, then $a = 0$ or $b = 0$.
iv) If $U$ and $V$ are right ideals of $M$ such that $UTV = \{0\}$, then $U = \{0\}$ or $V = \{0\}$.
v) If $U$ and $V$ are left ideals of $M$ such that $UTV = \{0\}$, then $U = \{0\}$ or $V = \{0\}$.

Let $M$ and $M'$ be two $\Gamma$-rings. A mapping $f : M \to M'$ of $\Gamma$-rings is called a $\Gamma$-ring homomorphism if $f(x + y) = f(x) + f(y)$ and $f(xy\gamma) = f(x) \gamma f(y)$ for all $x, y \in M$ and $\gamma \in \Gamma$.

Let $M$ be a $\Gamma$-ring and $(A, +)$ be an abelian group. $A$ is called a left $M$-module over $M$ $\Gamma$-ring with respect to a mapping $\cdot : M \times \Gamma \times A \to A$ if for all $m, m', M, x, y \in A$ and $\alpha, \beta \in \Gamma$,

1. $ma(x + y) = max + may$,
2. $(m + m') \alpha x = max + may$,
3. $ma(m'\beta x) = (mam')\beta x$.

Similarly, one can define a right $M$-module.

Let $M$ be a $\Gamma$-ring and $A, B$ be two left $M$-modules. A additive mapping $f : A \to B$ is called a left $M$-module homomorphism if $f(max) = maf(x)$ for all $m \in M, x \in A$ and $\alpha \in \Gamma$. 


Let $M$ be a prime $\Gamma$-ring such that $M \Gamma M \neq M$. Denote
$$\mathcal{M} := \{(U, f) : U (\neq 0) \text{ is an ideal of } M \text{ and } f : U \rightarrow M \text{ is a right } M\text{-module homomorphism}\}.$$ Define a relation $\sim$ on $\mathcal{M}$ by
$$(U, f) \sim (V, g) \Leftrightarrow \exists W (\neq 0) \subset U \cap V \text{ such that } f = g \text{ on } W.$$
Since $M$ is prime $\Gamma$-ring, it is possible to find a non-zero $W$ and so "$\sim$" is an equivalent relation. This gives a chance for us to get a partition of $\mathcal{M}$. We denote the equivalence class by $Cl(U, f) = f$, where
$$\hat{f} := \{ g : V \rightarrow M \mid (U, f) \sim (V, g) \},$$
and denote by $Q$ the set of all equivalence classes. Then $Q$ is a $\Gamma$-ring, which is called the quotient $\Gamma$-ring of $M$ (see [11], [12], and [14]).

Let $M$ be a $\Gamma$-ring with unity. An element $u$ in $M$ is called a unity of $M$ if it has a multiplicative inverse in $M$. If every nonzero element of $M$ is a unity, we say that $M$ is a $\Gamma$-division ring. A $\Gamma$-ring $M$ is a $\Gamma$-field if it is a commutative $\Gamma$-division ring.

**Lemma 2.1 ([12, Lemma 3.3]).** Let $M$ be a prime $\Gamma$-ring such that $M \Gamma M \neq M$. Then the extended centroid $C_\Gamma$ of $M$ is a $\Gamma$-field.

Let $M$ be a $\Gamma$-ring and $A$ be a $M$-module. A subset $B = \{b_i : i \in I\}$ of $A$ is called linearly independent, if for every distinct $b_1, b_2, ..., b_n \in B$, $m_1, m_2, ..., m_n \in M$ and $\beta_1, \beta_2, ..., \beta_n \in \Gamma$ such that $\sum_{i=0}^n m_i \beta_i b_i = 0$ implies that $m_1 = m_2 = ... = m_n = 0$.

**Lemma 2.2 ([11, p. 476]).** Let $M$ be a prime $\Gamma$-ring such that $M \Gamma M \neq M$ and the extended centroid $C_\Gamma$ of $M$. If $a_i$ and $b_i$ are non-zero elements of $M$ such that $\sum a_i \gamma_i x \beta_i b_i = 0$ for all $x \in M$ and $\gamma_i, \beta_i \in \Gamma$, then the $a_i$'s (also $b_i$'s) are linearly independent over $C_\Gamma$. Moreover, if $a \gamma x \beta b = b \gamma x \beta a$ for all $x \in M$ and $\gamma, \beta \in \Gamma$ where $a (\neq 0)$, $b \in M$ are fixed, then there exists $\lambda \in C_\Gamma$ such that $b = \lambda a$ for all $\alpha \in \Gamma$.

**Lemma 2.3 ([16, Lemma 2]).** Let $M$ be a prime $\Gamma$-ring, $U$ a non-zero right (resp. left) ideal of $M$ and $a \in M$. If $U \Gamma a = \{0\}$ (resp. $a \Gamma U = \{0\}$), then $a = 0$.

**Lemma 2.4 ([10, Lemma 1]).** Let $M$ be a semi-prime $\Gamma$-ring and $U$ a non-zero ideal of $M$. Then $Ann_U M = Ann_U U$.

Let $M$ be a semi-prime $\Gamma$-ring and $U$ a non-zero ideal of $M$. In this case, we will write $Ann_U U = Ann_U M = Ann_U$ by Lemma 2.4. Let us denote by $F$ a set of all ideals of $M$ which have zero annihilator in $M$. In this case, the set $F$ is closed under multiplication by Lemma 2.4.

**Theorem 2.2 ([14, Theorem 3.5]).** Let $M$ be a semi-prime $\Gamma$-ring and $Q$ the quotient $\Gamma$-ring of $M$. Then the $\Gamma$-ring $Q$ satisfies the following properties:

1) For any element $q \in Q$, there exists an ideal $U_q \in F$ such that $q(U_q) \subseteq M$ (or $qU_q \subseteq M$ for all $\gamma \in \Gamma$).
ii) If $q \in Q$ and $q(U) = \langle 0 \rangle$ for some $U \in F$ (or $q\gamma U_q = \langle 0 \rangle$ for some $U \in F$ and for all $\gamma \in \Gamma$), then $q = 0$.

iii) If $U \in F$ and $\Psi : U \rightarrow M$ is a right $M$-module homomorphism, then there exists an element $q \in Q$ such that $\Psi(u) = q(u)$ for all $u \in U$ (or $\Psi(u) = q\gamma u$ for all $u \in U$ and $\gamma \in \Gamma$).

iv) Let $W$ be a submodule (an $(M,M)$-subbimodule) in $Q$ and $\Psi : W \rightarrow Q$ a right $M$-module homomorphism. If $W$ contains the ideal $U$ ideal of the $\Gamma$-ring $M$ such that $\Psi(U) \subseteq M$ and $\text{Ann}U = \text{Ann}W$, then there is an element $q \in Q$ such that $\Psi(b) = q(b)$ for any $b \in W$ (or $\Psi(b) = q\gamma b$ for any $b \in W$ and $\gamma \in \Gamma$) and $q(a) = 0$ for any $a \in \text{Ann}W$ (or $q\gamma a = 0$ for any $a \in \text{Ann}W$ and $\gamma \in \Gamma$).

**Lemma 2.5 ([16, Lemma 1]).** Let $M$ be a prime $\Gamma$-ring and $Z$ the center of $M$. If $a, b, c \in M$ and $\beta, \gamma \in \Gamma$, then

$$[a\gamma b, c]_\beta = a\gamma [b, c]_\beta + [a, c]_\beta \gamma b + a\gamma (c\beta b) - a\beta (c\gamma b)$$

where $[a, b]_\gamma$ is a $\gamma$-ring epimorphism and $d$ be a $\theta$-derivation if $d(x\gamma y) = d(x)\alpha\theta(y) + x\alpha d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

**Definition 2.1.** Let $M$ be a $\Gamma$-ring and $\theta : M \rightarrow M$ be a function. An additive mapping $\phi : M \rightarrow M$ is called $\theta$-derivation if $\phi(a\gamma b) = \phi(a)\theta(b)$ for all $a, b \in M$ and $\gamma \in \Gamma$.

**Definition 2.2.** Let $M$ be a $\Gamma$-ring, $\theta : M \rightarrow M$ and $\varphi : M \rightarrow M$ be functions. An additive mapping $d : M \rightarrow M$ is called $(\theta, \varphi)$-derivation if $d(x\gamma y) = d(x)\alpha\theta(y) + \varphi(x)d(y)$ for all $x, y \in M$ and $\alpha \in \Gamma$.

### 3. $\theta$-derivation on prime $\Gamma$-ring

In this section, $M$ is a prime $\Gamma$-ring such that $M\Gamma M \neq M$, $Z$ is the center of $M$, $C_\Gamma$ is the extended centroid of $M$ and $[a, b]_\gamma = a\gamma b - b\gamma a$ for all $a, b \in M$ and $\gamma \in \Gamma$.

**Lemma 3.1.** Let $M$ be a prime $\Gamma$-ring, $U$ be a non-zero ideal of $M$, $\theta : M \rightarrow M$ be a $\Gamma$-ring epimorph and $d$ be a $\theta$-derivation of $M$. If $a\Gamma d(U) = \langle 0 \rangle$ ($d(U)\Gamma a = \langle 0 \rangle$) for all $a \in M$, then $a = 0$ or $d = 0$.

**Proof.** Let $a\Gamma d(U) = \langle 0 \rangle$. For all $u \in U$, $x \in M$ and $\alpha, \beta \in \Gamma$, we get

$$0 = a\alpha d(u\beta x) = d(u\alpha x)\beta \theta(x) + a\alpha \beta d(x)$$

$$= a\alpha \beta d(x).$$

From Lemma 2.3, we obtain that $a = 0$ or $d = 0$, since $M$ is a prime $\Gamma$-ring. $\Box$

**Lemma 3.2.** Let $M$ be a prime $\Gamma$-ring, $U$ be a non-zero ideal of $M$, $\theta, \varphi : M \rightarrow M$ be $\Gamma$-ring epimorphisms and $d$ be a $(\theta, \varphi)$-derivation of $M$. If $a\Gamma d(U) = \langle 0 \rangle$ ($d(U)\Gamma a = \langle 0 \rangle$) for all $a \in M$, then $a = 0$ or $d = 0$.

**Proof.** Let $a\Gamma d(U) = \langle 0 \rangle$. For all $u \in U$, $x \in M$ and $\alpha, \beta \in \Gamma$, we get

$$0 = a\alpha d(u\beta x) = d(u\alpha x)\beta \theta(x) + a\alpha \varphi(u) d(x)$$

$$= a\alpha \varphi(u) d(x).$$
From Lemma 2.3, we obtain that $a = 0$ or $d = 0$, since $M$ is a prime $\Gamma$-ring and $\varphi$ is a $\Gamma$-ring epimorphism.

**Theorem 3.1.** Let $M$ be a prime $\Gamma$-ring with $\text{char} M = 2$, $\theta : M \to M$ be a $\Gamma$-ring epimorphism and $0 \neq d_1, d_2$ be $\theta$-derivations on $M$ such that $d_1 \theta = \theta d_1$ and $d_2 \theta = \theta d_2$. If for all $x \in M$,

$$d_1 d_2 (x) = 0$$

then there exists $\lambda \in C_\Gamma$ such that $d_2 (x) = \lambda \alpha d_1 (x)$ for all $x \in M$ and $\alpha \in \Gamma$.

**Proof.** Let $\alpha \in \Gamma$ and $x, y \in M$. Replacing $x$ by $x \circ y$ in (3.1) and using (3.1), we get

$$0 = d_2 (x) \alpha d_1 (\theta (y)) + d_1 (x) \alpha d_2 (\theta (y)),$$

since $\text{char} M = 2$ and $d_2 \theta = \theta d_2$.

Replacing $x$ by $x \beta z$ in (3.2) and using (3.2), we get

$$d_2 (x) \beta \theta (z) \alpha d_1 (\theta (y)) = d_1 (x) \beta \theta (z) \alpha d_2 (\theta (y)),$$

for all $x, y, z \in M$ and $\alpha, \beta \in \Gamma$. Since $\theta$ is $\Gamma$-ring epimorphism, for all $x, w, m \in M$ and $\alpha, \beta \in \Gamma$, we get

$$d_2 (x) \beta m \alpha d_1 (w) = d_1 (x) \beta m \alpha d_2 (w).$$

Now if we replace $w$ by $x$ in (3.4), then we obtain

$$d_2 (x) \beta m \alpha d_1 (x) = d_1 (x) \beta m \alpha d_2 (x),$$

for all $x, m \in M$ and $\beta, \alpha \in \Gamma$. If $d_1 (x) \neq 0$, then there exists $\lambda (x) \in C_\Gamma$, such that $d_2 (x) = \lambda (x) \gamma d_1 (x)$ for all $x \in M$ and $\gamma \in \Gamma$. Thus, if $d_1 (x) \neq 0 \neq d_1 (w)$, then (3.4) implies that

$$(\lambda (y) - \lambda (x)) \gamma d_1 (x) \beta m \alpha d_1 (w) = 0.$$ 

Since $M$ is a prime $\Gamma$-ring, we conclude by using Lemma 3.1 that $\lambda (y) = \lambda (x)$ for all $x, y \in M$. Hence we prove that there exists $\lambda \in C_\Gamma$ such that $d_2 (x) = \lambda \gamma d_1 (x)$ for all $x \in M$ and $\gamma \in \Gamma$ with $d_1 (x) \neq 0$. On the other hand, if $d_1 (x) = 0$, then $d_2 (x) = 0$ as well. Therefore, $d_2 (x) = \lambda \gamma d_1 (x)$ for all $x \in M$ and $\gamma \in \Gamma$. This completes the proof. \hfill \Box

**Proposition 3.1.** Let $M$ be a prime $\Gamma$-ring with $\text{char} M = 2$, $\theta : M \to M$ be a $\Gamma$-ring epimorphism and $0 \neq d$ be a $\theta$-derivation of $M$ such that $d \theta = \theta d$. If for all $x \in M$,

$$d (x) \in Z,$$

then there exists $\lambda (m) \in C_\Gamma$ such that $d (m) = \lambda (m) \alpha d (z)$ for all $m, z \in M$ and $\alpha \in \Gamma$ or $M$ is commutative.

**Proof.** From (3.7), we have

$$[d (x), y]_\beta = 0,$$

for all $x, y \in M$ and $\beta \in \Gamma$. 

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Replacing $x$ by $x\gamma z$ in (3.8), we get

$$d(x\gamma z), y, \beta = d(x) \gamma (\beta y) + x\gamma d(z) \beta y - y\beta d(x) \gamma \theta (z) - y\beta x y d(z).$$

(3.9)

Since $\theta$ is $\Gamma$-ring epimorphism, we get

$$d(x) \gamma [m, y, \beta] + d(z) \gamma [x, y, \beta],$$

(3.10)

for all $x, y, z, m \in M$ and $\gamma, \beta \in \Gamma$. Replacing $x$ by $d(x)$ in (3.10), we get

$$0 = d^2(x) \gamma [m, y, \beta] + d(z) \gamma [d(x), y, \beta],$$

(3.11)

for all $x, y, z, m \in M$. Using (3.8) in (3.11), we get

$$d^2(x) \gamma [m, y, \beta] = 0,$$

(3.12)

for all $x, y, z, m \in M$ and $\gamma, \beta \in \Gamma$.

Now, substituting $x\alpha z$ for $x$ in (3.12), it follows from $\text{char} M = 2$ that

$$0 = d^2(x) \alpha \theta^2(z) \gamma [m, y, \beta],$$

(3.13)

for all $x, y, z, m \in M$ and $\gamma, \beta \in \Gamma$. Since $M$ is prime $\Gamma$-ring and $\theta$ is $\Gamma$-ring epimorphism, we obtain

$$d^2(x) = 0 \text{ for all } x \in M \text{ or } [m, y, \beta] = 0 \text{ for all } y, m \in M \text{ and } \beta \in \Gamma.$$

From (3.14), if $d^2(x) = 0$ for all $x \in M$, then replacing $x$ by $x\gamma y$ in this last relation, it follows from $d(x) \in Z$ that

$$d(x) \gamma d(m) = d(m) \gamma d(x) \text{ for all } x, m \in M \text{ and } \gamma \in \Gamma.$$

(3.15)

Replacing $x$ by $x\alpha z$ in (3.15), it follows from (3.15) that for all $x, z, m \in M$ and $\alpha \in \Gamma$,

$$d(x) \alpha \gamma d(m) = d(m) \alpha y d(x),$$

since $\theta$ is $\Gamma$-ring epimorphism.

If $d(m) \neq 0$, then there exists $\lambda (m) \in C_T$ such that $d(x) = \lambda (x) \alpha d(m)$ for all $x, m \in M$ and $\alpha \in \Gamma$ by Lemma 2.2. On the other hand, it follows from (3.14) that if $[m, y, \beta] = 0$ for all $y, m \in M$ and $\beta \in \Gamma$, then $M$ is commutative. This completes the proof.

**Theorem 3.2.** Let $M$ be a prime $\Gamma$-ring with $\text{char} M = 2$, $U$ be a non-zero ideal of $M$, $\theta : M \rightarrow M$ be a $\Gamma$-ring epimorphism and $0 \neq d_1, d_2$ be $\theta$-derivation on $M$ such that $d_1 \theta = \theta d_1$ and $d_2 \theta = \theta d_2$. If $d_2(U) \subseteq U$ and for all $u \in U$,

$$d_1 d_2 (u) = 0$$

(3.16)

then there exists $\lambda \in C_T$ such that $d_2(x) = \lambda \alpha d_1(x)$ for all $x \in M$ and $\alpha \in \Gamma$.

**Proof.** Let $\gamma \in \Gamma$ and $u, v \in U$. Replacing $u$ by $d_2(u) \gamma v$ in (3.16) and using hypothesis, we have

$$d_2^2 (u) \gamma d_1 (w) = 0,$$

(3.17)
Also, replacing $y$ for all $(3.22)$ which implies from Lemma 2.3 that

$$d_2(u) \gamma d_1(d_2(x)) = 0.$$  

Since $d_2 \neq 0$, $d_1(d_2(x)) = 0$ for all $x \in M$ from Lemma 3.1. From here, there exists $\lambda \in C_\Gamma$ such that $d_2(x) = \lambda a d_1(x)$ for all $x \in M$ and $\alpha \in \Gamma$. \hfill $\square$

**Theorem 3.3.** Let $M$ be a prime $\Gamma$-ring, $U$ be a non-zero right ideal of $M$, $\theta : M \to M$ be $\Gamma$-ring epimorphism and $0 \neq d$ be a $\theta$-derivation of $M$ such that $d\theta = \theta d$. If

$$d(u) \gamma a = 0 \text{ for all } u \in U \text{ and } \gamma \in \Gamma,$$

where $a$ is a fixed element of $M$, then there exists an element of $Q$ such that $q\gamma a = 0$ implies $q\gamma u = 0$ for all $u \in U$ and $\gamma \in \Gamma$.

**Proof.** Let $u \in U$, $x \in M$ and $\beta \in \Gamma$. Since $U$ is a right ideal of $M$, we have $u\beta x \in U$. Replacing $u$ by $u\beta x$ in (3.19), we get

$$d(u) \beta \theta(x) \gamma a + u\beta d(x) \gamma a = 0$$

for all $u \in U$, $x \in M$ and $\gamma, \beta \in \Gamma$. And so,

$$d(u) \beta \left( \sum \theta(x) \gamma a m \right) = - \left( u\beta \left( \sum d(x) \gamma a m \right) \right)$$

for all $m \in M$, $\alpha \in \Gamma$. Therefore, for any $v \in V = M\Gamma a \Gamma M$ which is a non-zero ideal of $M$, we have

$$d(u) \beta v = u\beta f(v)$$

for all $u \in U$. $f(v)$ is independent of $u$ but it is dependent on $v$. Since $M$ is a prime $\Gamma$-ring, $f(v)$ is well-defined. Note that $v a y \in V$ for any $y \in M$, $v \in V$ and $\alpha \in \Gamma$. Replacing $v$ by $v a y$ in (3.21), we get

$$d(u) \beta v a y = u\beta f(v a y),$$

$$u\beta f(v) a y = u\beta f(v a y)$$

$$u\beta (f(v) a y - f(v a y)) = 0$$

which implies from Lemma 2.3 that

$$f(v) a y = f(v a y)$$

for all $y \in M$, $v \in V$ and $\alpha \in \Gamma$. It follows from (3.22) that $f : V \to M$ is a right $M$-module homomorphism. In this case, $q = Cl(V,f) \in Q$. Moreover, $f(v) = q\beta v$ for all $v \in V$ and $\alpha \in \Gamma$ by Theorem 2.2. Let $x \in M$, $v \in V$, $u \in U$ and $\gamma, \beta \in \Gamma$. Replacing $v$ by $\theta(x) \gamma v$ in (3.21), we get

$$d(u) \beta \theta(x) \gamma v = u\beta f(\theta(x) \gamma v) = u\beta q\beta \theta(x) \gamma v. $$

Also, replacing $u$ by $u\gamma x$ in (3.21), we get

$$d(u\gamma x) \beta v = u\gamma x \beta f(v)$$
which yields prime-ring, replacing and so we have
\[ u \beta q \gamma \theta (x) \gamma v = u \beta x \gamma q \gamma v - u \beta d (x) \gamma v \]
for all \( x \in M, v \in V, u \in U \) and \( \gamma, \beta \in \Gamma \). Hence \( d (x) = x \gamma q - q \beta \theta (x) \) for all \( x \in M \) and \( \gamma, \beta \in \Gamma \) by Lemma 2.3. Let \( u \in U \) and \( x \in M \), \( d (u) = q \alpha u - \theta (u) \beta q \) and \( d (x) = q \beta x - \theta (x) \alpha q \). Then we have
\[ 0 = d (u \beta x) \gamma a = (q \alpha (u \beta x) - \theta (u \beta x) \beta q) \gamma a. \]
Thus, \( q \alpha u \beta x \gamma a = \theta (u \beta x) \beta q a \). If \( q \gamma a = 0 \), then \( q \alpha u \beta x \gamma a = 0 \), and so since \( M \) is prime \( \Gamma \)-ring, we get \( q \alpha U = \{0\} \).

**Theorem 3.4.** Let \( M \) be a prime \( \Gamma \)-ring with \( \text{char}M \neq 2 \), \( U \) a non-zero right ideal of \( M \) and \( 0 \neq d \) be a \( \theta \)-derivation of \( M \) such that \( d \theta = \theta d \) and \( \theta (U) \subseteq U \). Then the subring of \( M \) generated by \( d (U) \) contains no nonzero right ideals of \( M \) if and only if \( d (U) \Gamma \theta (U) = \{0\} \).

**Proof.** Let \( A \) be the subring generated by \( d (U) \). Let \( S = A \cap U, u \in U, s \in S \) and \( \gamma, \alpha \in \Gamma \). Then \( d (s \gamma u) = d (s) \gamma \theta (u) + s \gamma d (u) \), and so we have \( d (s) \gamma \theta (u) \in S \). Thus \( d (S) \Gamma \theta (U) \) is a right ideal of \( M \). In this case, \( d (S) \Gamma \theta (U) = \{0\} \) by hypothesis. \( d (u \gamma a) = d (u) \gamma \theta (a) + u \gamma d (a) \in S \) and \( d (u) \gamma \theta (a) \in S \) where \( u \in U, a \in A \). Thus, we have \( u \gamma d (a) \in S \). Therefore, \( 0 = d (u \gamma d (a)) \gamma \theta (u) = (d (u) \gamma \theta (d (a)) + u \gamma d^2 (a)) \gamma \theta (u) \). Since \( M \) is prime \( \Gamma \)-ring, it follows from Lemma 2.3 that
\[ d (u) \gamma \theta (d (a)) + u \gamma d^2 (a) = 0 \]
for all \( u \in U, \gamma \in \Gamma \) and \( a \in A \). Replacing \( u \) by \( u \beta v \) where \( v \in U, \beta \in \Gamma \) in (3.26), we get, for all \( u, v \in U, \beta, \gamma \in \Gamma \) and \( a \in A \)
\[ d (u) \beta v \gamma \theta (d (a)) = 0. \]
Since \( M \) is prime \( \Gamma \)-ring, we get \( d (U) \Gamma U = \{0\} \) or \( d (A) \Gamma U = \{0\} \). If \( d (A) \Gamma U = \{0\} \), then \( d^2 (U) \Gamma U = \{0\} \). Let \( u, v \in U \) and \( \beta \in \Gamma \). Then
\[ 0 = d (d (u \beta v)) = d^2 (u) \beta \theta^2 (v) + 2d (u) \beta d (\theta (v)) + u \beta d^2 (v), \]
and so we have \( d (u) \beta d (\theta (v)) = 0 \) for all \( u, v \in U \) and \( \beta \in \Gamma \) by \( \text{char}M \neq 2 \). Replacing \( u \) by \( w \gamma w \) where \( w \in U, \gamma \in \Gamma \) in last relation, we have
\[ d (u) \gamma \theta (w) \beta d (\theta (v)) = 0 \]
which yields \( d (u) \gamma \theta (w) = 0 \) for all \( u, w \in U \) and \( \gamma \in \Gamma \).

Conversely, assume that \( d (U) \Gamma \theta (U) = \{0\} \). Then \( A \Gamma d (U) = \{0\} \). Since \( M \) is a prime \( \Gamma \)-ring, \( A \) contains no non-zero right ideals. \( \square \)
4. $(\theta, \varphi)$-derivation on prime $\Gamma$-ring

**Theorem 4.1.** Let $M$ be a prime $\Gamma$-ring with $\text{char} M = 2$, $\theta : M \to M$ and $\varphi : M \to M$ be $\Gamma$-ring epimorphisms and $0 \neq d_1, d_2$ be $(\theta, \varphi)$-derivations on $M$ such that $d\theta = \theta d_1$ and $d\varphi = \varphi d_1$, $i = 1, 2$. If for all $x \in M$,

\begin{equation}
\tag{4.1}
d_1 d_2 (x) = 0
\end{equation}

then there exists $\lambda \in C_\Gamma$ such that $d_2 (x) = \lambda ad_1 (x)$ for all $x \in M$ and $\alpha \in \Gamma$.

**Proof.** Let $\alpha \in \Gamma$ and $x, y \in M$. Replacing $x$ by $xy$ in (4.1) and using (4.1), we get

\begin{equation}
\tag{4.2}
d_2 (\varphi (x)) ad_1 (\theta (y)) = d_1 (\varphi (x)) ad_2 (\theta (y)),
\end{equation}

since $\text{char} M = 2$ and $d_1 \theta = \theta d_1$, $d_1 \varphi = \varphi d_1$ for $i = 1, 2$.

Since $\varphi$ is $\Gamma$-ring epimorphism, for all $m, y \in M$ and $\alpha \in \Gamma$, we get

\begin{equation}
\tag{4.3}
d_2 (m) ad_1 (\theta (y)) = d_1 (m) ad_2 (\theta (y)).
\end{equation}

Replacing $m$ by $m\beta z$ in (4.2) and using (4.2), we get

\begin{equation}
\tag{4.4}
d_2 (m) \beta \theta (z) ad_1 (\theta (y)) = d_1 (m) \beta \theta (z) ad_2 (\theta (y)),
\end{equation}

for all $m, y, z \in M$ and $\alpha, \beta \in \Gamma$. Since $\theta$ is $\Gamma$-ring epimorphism, for all $m, n, k \in M$ and $\alpha, \beta \in \Gamma$, we get

\begin{equation}
\tag{4.5}
d_2 (m) \beta n ad_1 (k) = d_1 (m) \beta n ad_2 (k).
\end{equation}

Now if we replace $k$ by $m$ in (4.5), then we obtain

\begin{equation}
\tag{4.6}
d_2 (m) \beta n ad_1 (m) = d_1 (m) \beta n ad_2 (m),
\end{equation}

for all $n, m \in M$ and $\beta, \alpha \in \Gamma$. If $d_1 (m) \neq 0$, then there exists $\lambda (m) \in C_\Gamma$ such that $d_2 (m) = \lambda (m) \gamma d_1 (m)$ for all $x \in M$ and $\gamma \in \Gamma$. Thus, if $d_1 (m) \neq 0 \neq d_1 (k)$, then (4.5) implies that

\begin{equation}
\tag{4.7}
(\lambda (k) - \lambda (m)) \gamma d_1 (m) \beta z ad_1 (k) = 0.
\end{equation}

Since $M$ is a prime $\Gamma$-ring, we conclude by using Lemma 2.2 that $\lambda (k) = \lambda (m)$ for all $k, m \in M$. Hence we prove that there exists $\lambda \in C_\Gamma$ such that $d_2 (m) = \lambda (m) \gamma d_1 (m)$ for all $m \in M$ and $\gamma \in \Gamma$ with $d_1 (m) \neq 0$. On the other hand, if $d_1 (m) = 0$, then $d_2 (m) = 0$ as well. Therefore, $d_2 (m) = \lambda \gamma d_1 (m)$ for all $m \in M$ and $\gamma \in \Gamma$. This completes the proof. \qed

**Proposition 4.1.** Let $M$ be a prime $\Gamma$-ring with $\text{char} M = 2$, $\theta : M \to M$ and $\varphi : M \to M$ be $\Gamma$-ring epimorphisms and $0 \neq d$ be a $(\theta, \varphi)$-derivation of $M$ such that $d\theta = \theta d$, $d\varphi = \varphi d$. If for all $x \in M$,

\begin{equation}
\tag{4.8}
d (x) \in Z,
\end{equation}

then there exists $\lambda (m) \in C_\Gamma$ such that $d (m) = \lambda (m) ad (z)$ for all $m, z \in M$ and $\alpha \in \Gamma$ or $M$ is commutative.
Proof. From (4.8), we have
\[ [d(x), y]_\beta = 0, \]
for all \( x, y \in M \) and \( \beta \in \Gamma \).

Replacing \( x \) by \( x\gamma z \) in (4.9), we get
\[ [d(x\gamma z, y)]_\beta = d(x) \gamma \theta(z) \beta y + \varphi(x) \gamma d(z) \beta y - y \beta d(x) \gamma \theta(z) - y \beta \varphi(x) \gamma d(z). \]

Since \( \theta \) is \( \Gamma \)-ring epimorphism, we get
\[ 0 = d(x) \gamma (m \beta y - y \beta m) + d(z) \gamma (\varphi(x) \beta y - y \beta \varphi(x)) \]
\[ = d(x) \gamma [m, y]_\beta + d(z) \gamma [\varphi(x), y]_\beta, \]
for all \( x, y, z, m \in M \) and \( \gamma, \beta \in \Gamma \). Replacing \( x \) by \( d(x) \) in (4.11), we get
\[ 0 = d^2(x) \gamma [m, y]_\beta + d(z) \gamma [\varphi(d(x)), y]_\beta, \]
for all \( x, y, z, m \in M \). Since \( \varphi \) is \( \Gamma \)-ring epimorphism, we get
\[ 0 = d^2(x) \gamma [m, y]_\beta + d(z) \gamma [d(m), y]_\beta, \]
for all \( x, y, z, m, n \in M \) and \( \gamma, \beta \in \Gamma \).

Using (4.9) in (4.13), we get
\[ d^2(x) \gamma [m, y]_\beta = 0, \]
for all \( x, y, m \in M \) and \( \gamma, \beta \in \Gamma \).

Now, substituting \( x\alpha z \) for \( x \) in (4.14), we get
\[ 0 = (d^2(x) \alpha \theta^2(z) + \varphi(d(x)) \alpha d(\theta(z)) + d(\varphi(x)) \alpha \theta d(z)) + \varphi^2(x) \alpha d^2(z) \gamma [m, y]_\beta, \]
for all \( x, y, z, m \in M \) and \( \alpha, \gamma, \beta \in \Gamma \).

Using (4.13) in last relation, we have
\[ 0 = d^2(x) \alpha \theta^2(z) \gamma [m, y]_\beta, \]
for all \( x, y, z, m \in M \) and \( \alpha, \gamma, \beta \in \Gamma \). Since \( M \) is prime \( \Gamma \)-ring and \( \theta \) is \( \Gamma \)-ring epimorphism, we obtain
\[ d^2(x) = 0 \]
for all \( x \in M \) or \( [m, y]_\beta = 0 \) for all \( y, m \in M \) and \( \beta \in \Gamma \).

From (4.16), if \( d^2(x) = 0 \) for all \( x \in M \), then replacing \( x \) by \( x\gamma y \) in this last relation, it follows from \( d(x) \in Z \) that
\[ d(x) \gamma d(m) = d(m) \gamma d(x) \]
for all \( x, m \in M \) and \( \gamma \in \Gamma \).

Replacing \( x \) by \( x\alpha n \) in (4.17), it follows from (4.17) that for all \( x, n, m \in M \) and \( \alpha, \gamma \in \Gamma \),
\[ d(x) \alpha \theta(n) \gamma d(m) = d(m) \gamma d(x) \alpha \theta(n). \]

Since \( \theta \) is \( \Gamma \)-ring epimorphism, we have
\[ d(x) \alpha k \gamma d(m) = d(m) \gamma d(x) \alpha k, \]
for all \( x, m, k \in M \) and \( \alpha, \gamma \in \Gamma \).
If \( d(x) \neq 0 \), then there exists \( \lambda(m) \in C_\Gamma \) such that \( d(x) = \lambda(x) od(m) \) for all \( x, m \in M \) and \( \alpha \in \Gamma \) by Lemma 2.2. On the other hand, it follows from (4.16) that if \( [m, y]_\beta = 0 \) for all \( y, m \in M \) and \( \beta \in \Gamma \), then \( M \) is commutative. This completes the proof.

**Theorem 4.2.** Let \( M \) be a prime \( \Gamma \)-ring with \( \text{char} \, M = 2 \), \( U \) be a non-zero ideal of \( M \), \( \theta : M \to M \) and \( \varphi : M \to M \) be \( \Gamma \)-ring epimorphisms and \( 0 \neq d_1, d_2 \) be \((\theta, \varphi)\)-derivations on \( M \) such that \( d_i\theta = \theta d_i \) and \( d_i\varphi = \varphi d_i \), \( i = 1, 2 \). If \( d_2(U) \subseteq U \) and for all \( u \in U 
\begin{align*}
(4.19) \quad d_1d_2(u) = 0
\end{align*}
then there exists \( \lambda \in C_\Gamma \) such that \( d_2(x) = \lambda od_1(x) \) for all \( x \in M \) and \( \alpha \in \Gamma \).

**Proof.** Let \( \gamma \in \Gamma \) and \( u, v \in U \). Replacing \( u \) by \( d_2(u) \gamma v \) in (4.19) and using hypothesis, we have
\begin{align*}
(4.20) \quad \varphi(d_1^2(u)) \gamma d_1(\theta(v)) = 0,
\end{align*}
for all \( u, v \in U \) and \( \gamma \in \Gamma \). Since \( \theta \) and \( \varphi \) are \( \Gamma \)-ring epimorphisms, we get
\begin{align*}
(4.21) \quad d_2^2(y) \gamma d_1(z) = 0,
\end{align*}
for all \( y, z \in M \) and \( \gamma \in \Gamma \). Since \( d_1 \neq 0 \), for all \( y \in M \), \( d_2^2(y) = 0 \) from Lemma 3.2. Replacing \( u \) by \( uvx \) in (4.19) and using hypothesis, we get
\begin{align*}
(4.22) \quad 0 = d_2(u) \gamma d_1(\theta(x)) + d_1(u) \gamma d_2(x) + u \gamma d_1(d_2(x)),
\end{align*}
for all \( u \in U \), \( x \in M \) and \( \gamma \in \Gamma \). Replacing \( u \) by \( d_2(u) \) in (4.22) and using (4.21), we get
\begin{align*}

\end{align*}
Since \( d_2 \neq 0 \), \( d_1(d_2(x)) = 0 \) for all \( x \in M \) from Lemma 3.2. From here, there exists \( \lambda \in C_\Gamma \) such that \( d_2(x) = \lambda od_1(x) \) for all \( x \in M \) and \( \alpha \in \Gamma \).

**References**


Received by editors 05.04.2019; Revised version 14.09.2019; Available online 23.09.2019.

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