# SPECTRA OF THE UPPER TRIANGULAR BAND MATRIX $U(r ; 0 ; s)$ ON THE HAHN SPACE 

## Nuh Durna


#### Abstract

The aim of this paper is to obtain subdivision of the spectrum which is formed point spectrum, continuous spectrum and residual spectrum of the operator $U(r ; 0 ; s)$, is defined as $U(r ; 0 ; s)\left(x_{k}\right)=\left(r x_{k}+s x_{k+2}\right)$, on the Hahn space $h$ of all $x=\left(x_{k}\right)$ null sequences such that $\sum_{k=0}^{\infty} k\left|x_{k+1}-x_{k}\right|$ is finite.


## 1. Introduction

Spectral theory is one of the most useful tools in science. There are many applications in some branches of science such as control theory, matrix theory, function theory, quantum physics, and complex analysis. For example, atomic energy levels are determined and therefore the frequency of a laser or the spectral signature of a star are obtained by it in quantum mechanics. The resolvent set of the band operators is important for solving in above explanations problems. Band matrices emerge in the applications of mathematics. Banded matrices are used in telecommunication system analysis, finite difference methods for solving partial differential equations, linear recurrence systems with non-constant coefficients, etc (see $[\mathbf{2 1}]$ ), so, it is natural to ask the question of whether one can obtain some results about the spectral decomposition of $U(r ; 0 ; s)$ matrix.

[^0]Hahn [8] introduced the space $h$ of all sequences $x=\left(x_{k}\right) \in c_{0}$ such that $\sum_{k=0}^{\infty} k\left|x_{k+1}-x_{k}\right|$ is finite. The norm $\|x\|_{h}=\sum_{k=1}^{\infty} k\left|x_{k+1}-x_{k}\right|+\sup _{k}\left|x_{k}\right|$ was defined on the space $h$ by Hahn [8]. Rao [17] defined a new norm of $h$ given by $\|x\|_{h}=\sum_{k=1}^{\infty} k\left|x_{k+1}-x_{k}\right|$. The dual space of $h$ is norm isomorphic to the Banach space $\sigma_{\infty}=\left\{x=\left(x_{k}\right) \in w: \sup _{n} \frac{1}{n}\left|\sum_{k=1}^{n} x_{k}\right|<\infty\right\}$.

In this paper, we will calculate the point spectrum, the continuous spectrum, and the residual spectrum of $U(r ; 0 ; s)$ matrix on the Hahn sequence space.

Let $X$ and $Y$ be the Banach spaces, and $L: X \rightarrow Y$ be a bounded linear operator. By

$$
R(L)=\{y \in Y: y=L x, x \in X\}
$$

we denote the range of $L$ and by $B(X)$, we show the set of all bounded linear operators on $X$ into itself.

Let $L: D(L) \rightarrow X$ be a linear operator, defined on $D(L) \subset X$, where $D(L)$ denote the domain of $L$ and $X$ is a complex normed space. Let $L_{\lambda}:=\lambda I-L$ for $L \in B(X)$ and $\lambda \in \mathbb{C}$ where $I$ is the identity operator. $L_{\lambda}^{-1}$ is known as the resolvent operator of $L$.

The resolvent set of $L$ is the set of complex numbers $\lambda$ of $L$ such that $L_{\lambda}^{-1}$ exists, is bounded and, is defined on a set which is dense in $X$, denoted by $\rho(L, X)$. Its complement is given by $\mathbb{C} \backslash \rho(L ; X)$ is called the spectrum of $L$, denoted by $\sigma(L, X)$.

The spectrum $\sigma(L, X)$ is union of three disjoint sets as follows: The point spectrum $\sigma_{p}(L, X)$ is the set such that $L_{\lambda}^{-1}$ does not exist. If the operator $L_{\lambda}^{-1}$ is defined on a dense subspace of $X$ and is unbounded then $\lambda \in \mathbb{C}$ belongs to the continuous spectrum $\sigma_{c}(L, X)$ of $L$. Furthermore, we say that $\lambda \in \mathbb{C}$ belongs to the residual spectrum $\sigma_{r}(L, X)$ of $L$ if the operator $L_{\lambda}^{-1}$ exists, but its domain of definition (i.e. the range $R(\lambda I-L)$ of $(\lambda I-L)$ is not dense in $X$ than in this case $L_{\lambda}^{-1}$ may be bounded or unbounded. From above definitions we have

$$
\begin{equation*}
\sigma(L, X)=\sigma_{p}(L, X) \cup \sigma_{c}(L, X) \cup \sigma_{r}(L, X) \tag{1.1}
\end{equation*}
$$

and

$$
\sigma_{p}(L, X) \cap \sigma_{c}(L, X)=\emptyset, \sigma_{p}(L, X) \cap \sigma_{r}(L, X)=\emptyset, \sigma_{r}(L, X) \cap \sigma_{c}(L, X)=\emptyset
$$

By $w$, we denote the space of all sequences. Well-known examples of Banach sequence spaces are the spaces $\ell_{\infty}, c, c_{0}$ and $b v$ of bounded, convergent, null and bounded variation sequences, respectively. Also by $\ell_{p}, b v_{p}$ we denote the spaces of all $p$-absolutely summable sequences and $p$-bounded variation sequences, respectively.

Many researchers investigated the spectrum and the fine spectrum of linear operators defined by some determined matrices over certain sequence spaces. For instance, in 2010, fine spectra of upper triangular double-band matrices were studied by Karakaya and Altun ([9]). In 2013, the spectrum of the Cesàro matrix considered as a bounded operator on $\overline{b v_{0}} \cap \ell_{\infty}$ were studied by Tripathy and

Saikia ([13]), and Tripathy and Paul ([14]) examined the spectra of the operator $D(r, 0,0, s)$ on sequence spaces $c_{0}$ and $c$. Also in [11], they investigated the spectrum of the operator $D(r, 0,0, s)$ over the sequence spaces $\ell_{p}$ and $b v_{p}$. In 2014, the spectrum of the Rhaly operator on the sequence space $\overline{b v_{0}} \cap \ell_{\infty}$ was determined by Tripathy and Das ([15]). In 2015, Tripathy and Das determined the spectrum and fine spectrum of the upper triangular matrix $U(r, s)$ on the sequence space $c s=\left\{x=\left(x_{n}\right) \in w: \lim _{n \rightarrow \infty} \sum_{i=0}^{n} x_{i}\right.$ exists $\}$. Also they determined the subdivisions of the spectrum of the operator $U(r, s)$ on the same space and, Paul and Tripathy ([12]) investigated the fine spectrum of the operator $D(r, 0,0, s)$ over a sequence space $b v_{0}$. In 2016, Das and Tripathy ([1]) studied the spectra and fine spectra of the matrix $B(r, s, t)$ on the sequence space $c s$, and Yeşilkayagil and Kirişci calculated the fine spectrum of the forward difference operator on the Hahn space in [18]. In 2017, Yildirim and Durna ([19]) examined the spectrum and some subdivisions of the spectrum of discrete generalized Cesàro operators on $\ell_{p}$, $(1<p<\infty)$, El-Shabrawy and Abu-Janah ([6]) determined spectra and the fine spectra of generalized difference operator $B(r, s)$ on the sequence spaces $b v_{0}$ and $h$, Durna ([3]) studied subdivision of the spectra for the generalized difference operator $\Delta_{a, b}$ on the sequence space $\ell_{p}(1<p<\infty)$, Karakaya et al. ([10]) examined the fine spectra and subspectrum of operator with periodic coefficients, and the fine spectrum of the lower triangular matrix $B(r, s)$ over the Hahn sequence space was investigated by Das ([2]). In 2018, Durna et al. ([4]) studied partition of the spectra for the generalized difference operator $B(r, s)$ on the sequence space $c s$, Durna ([5]) studied subdivision of spectra for some lower triangular doule-band matrices as operators on $c_{0}$ and Yildirim et al. ([20] ) studied the spectrum and fine spectrum of generalized Rhaly-Cesàro matrices on $c_{0}$ and $c$.

## 2. Spectrum and Fine Spectrum of $U(r ; 0 ; s)$

The operator $U(r ; 0 ; s)$ is defined as $U(r ; 0 ; s)\left(x_{k}\right)=\left(r x_{k}+s x_{k+2}\right)$ and its matrix representation is upper triangular matrix $U(r ; 0 ; s)$ is an infinite matrix with the non-zero diagonals are the entries of an oscillatory sequence of form

$$
U(r ; 0 ; s)=\left[\begin{array}{cccccccc}
r & 0 & s & 0 & 0 & 0 & 0 & \cdots  \tag{2.1}\\
0 & r & 0 & s & 0 & 0 & 0 & \cdots \\
0 & 0 & r & 0 & s & 0 & 0 & \cdots \\
0 & 0 & 0 & r & 0 & s & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right] \quad(s \neq 0)
$$

In this paper, we will calculate spectral decomposition of above matrix.
Lemma 2.1 (Rao [17], Proposition 10). The matrix $A=\left(a_{n k}\right)$ gives rise to a bounded linear operator $T \in B(h)$ from $h$ to itself if and only if
(i) $\sum_{n=1}^{\infty} n\left|a_{n k}-a_{n+1, k}\right|$ converges, for each $k$,
(ii) $\sup _{k} \frac{1}{k} \sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}\left(a_{n v}-a_{n+1, v}\right)\right|<\infty$,
(iii) $\lim _{n \rightarrow \infty} a_{n k}=0$, for each $k$.

Theorem 2.1. $U(r ; 0 ; s): h \rightarrow h$ is a bounded linear operator.
Proof. Let us use Lemma 2.1 for proof.
(i) $\sum_{n=1}^{\infty} n\left|a_{n k}-a_{n+1, k}\right|=\left\{\begin{array}{cl}|r| & , \quad k=1 \\ 3|r| & , \quad k=2 \\ (2 k-1)|r|+(2 k-5)|s| & , \quad k \geqslant 3\end{array}\right.$ is convergent.
(ii)

$$
\sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}\left(a_{n v}-a_{n+1, v}\right)\right|=\left\{\begin{array}{cc}
(k-2)|s|+k|r| & , \quad k \geqslant 2 \\
|r| & , \quad k=1
\end{array} .\right.
$$

Therefore $\frac{1}{k} \sum_{n=1}^{\infty} n\left|\sum_{v=1}^{k}\left(a_{n v}-a_{n+1, v}\right)\right|$ is convergent.
(iii) For each $k$, it is clear that $\lim _{n \rightarrow \infty} a_{n k}=0$.

Thus the Lemma 2.1's assertion is hold.
Lemma 2.2 (Golberg [7, p.59]). $T$ has a dense range if and only if $T^{*}$ is 1-1.
Lemma 2.3 (Golberg [7, p.60]). T has a bounded inverse if and only if $T^{*}$ is onto.

Theorem 2.2. $\sigma_{p}(U(r ; 0 ; s), h)=\{\lambda \in \mathbb{C}:|\lambda-r|<|s|\}$.
Proof. Let $U(r ; 0 ; s) x=\lambda x$ for $x \neq \theta=(0,0,0, \ldots)$ in $h$. That is, let $\lambda$ be an eigenvalue of the operator $U(r ; 0 ; s)$. Then we have

$$
\left\{\begin{array}{ll}
x_{2 n} & =\left(\frac{\lambda-r}{s}\right)^{n} x_{0} \\
x_{2 n+1} & =\left(\frac{\lambda-r}{s}\right)^{n} x_{1}
\end{array}, n=0,1,2, \ldots\right.
$$

Thus we get

$$
\begin{aligned}
\sum_{n=1}^{\infty} n\left|x_{n}-x_{n+1}\right| & =\sum_{k=1}^{\infty}(2 k-1)\left|x_{2 k-1}-x_{2 k}\right|+\sum_{k=1}^{\infty} 2 k\left|x_{2 k}-x_{2 k+1}\right| \\
& =\left|x_{1}-\frac{\lambda-r}{s} x_{0}\right| \sum_{k=1}^{\infty}(2 k-1)\left|\frac{\lambda-r}{s}\right|^{k-1} \\
& +\left|x_{0}-x_{1}\right| \sum_{k=1}^{\infty} 2 k\left|\frac{\lambda-r}{s}\right|^{k}
\end{aligned}
$$

Since

$$
\lim _{k \rightarrow \infty} \frac{(2 k+2)\left|\frac{\lambda-r}{s}\right|^{k}\left|\frac{\lambda-r}{s}\right|}{2 k\left|\frac{\lambda-r}{s}\right|^{k}}=\left|\frac{\lambda-r}{s}\right|
$$

from D'Alembert's ratio test, the series $\sum_{k=1}^{\infty}(2 k-1)\left|\frac{\lambda-r}{s}\right|^{k-1}$ is convergent if and only if $\left|\frac{\lambda-r}{s}\right|<1$ and hence, $x=\left(x_{n}\right) \in h$ if and only if $|\lambda-r|<|s|$. Therefore, $\sigma_{p}(U(r ; 0 ; s), h)=\{\lambda \in \mathbb{C}:|\lambda-r|<|s|\}$.

Let $T: h \longmapsto h$ is a bounded linear operator and $A$ is the matrix representation of $T$. Then we know that the transpose $A^{t}$ of the matrix $A$ is the matrix representation of adjoint operator $T^{*}: h^{*} \longmapsto h^{*}$.

We noticed that the dual space $h^{*}$ of $h$ is isometrically isomorphic to the Banach space

$$
\sigma_{\infty}=\left\{x=\left(x_{k}\right) \in w: \sup _{n} \frac{1}{n}\left|\sum_{k=1}^{n} x_{k}\right|<\infty\right\} .
$$

Theorem 2.3. $\sigma_{p}\left(U(r ; 0 ; s)^{*}, h^{*} \cong \sigma_{\infty}\right)=\emptyset$.
Proof. Then there exists $x \neq \theta=(0,0,0, \ldots)$ in $\sigma_{\infty}$ such that $U(r ; 0 ; s)^{*} x=$ $\lambda x$ if $\lambda$ is an eigenvalue of the operator $U(r ; 0 ; s)^{*}$.
Then, we have

$$
\begin{gather*}
r x_{0}=\lambda x_{0}  \tag{2.2}\\
r x_{1}=\lambda x_{1}  \tag{2.3}\\
s x_{k}+r x_{k+2}=\lambda x_{k+2}, k=0,1,2, \ldots \tag{2.4}
\end{gather*}
$$

Let $x_{0} \neq 0$ then we obtain that $\lambda=r$ from (2.2). Then we have $s x_{0}+r x_{2}=r x_{2}$ for $k=0$ from (2.4). Since $s \neq 0$, we get $x_{0}=0$. This is a contradiction.

Let $x_{0}=0$ and $x_{1} \neq 0$ then we obtain that $\lambda=r$ from (2.3). Then $s x_{1}+r x_{3}=$ $r x_{3}$ for $k=1$ from (2.4). Since $s \neq 0$, we get $x_{1}=0$. This is a contradiction.

Let $x_{0}=0, x_{1}=0$ and $x_{2} \neq 0$ then we obtain that $s x_{0}+r x_{2}=\lambda x_{2}$ for $k=0$ from (2.4). Hence we get $\lambda=r$. Then $s x_{2}+r x_{4}=r x_{4}$ for $k=2$ from (2.4). Since $s \neq 0$, we get $x_{2}=0$. This is a contradiction.

Finally, let $x_{k}$ be the first non-zero term of the sequence $\left(x_{n}\right)$. Then we obtain that $s x_{k-2}+r x_{k}=\lambda x_{k}$ for $k-2$ from (2.4). Hence we get $\lambda=r$. Then $s x_{k}+r x_{k+2}=r x_{k+2}$ from (2.4). Since $s \neq 0$, we get $x_{k}=0$. This is a contradiction.

Therefore $U(r ; 0 ; s)^{*} x=\lambda x$ for $x \in \sigma_{\infty}$ implies to $x=(0,0,0, \ldots)$. Thus, $\sigma_{p}\left(U(r ; 0 ; s)^{*}, c_{0}^{*} \xlongequal{=} \ell_{1}\right)=\emptyset$.

Theorem 2.4. $\sigma_{r}(U(r ; 0 ; s), h)=\emptyset$.
Proof. Since, $\sigma_{r}(U(r ; 0 ; s), h)=\sigma_{p}\left(U^{*}(r ; 0 ; s), \sigma_{\infty}\right) \backslash \sigma_{p}(U(r ; 0 ; s), h)$, Theorems 2.2 and 2.3 give us required result.

Lemma 2.4 .

$$
\sum_{k=1}^{n}\left(\sum_{i=0}^{k-1} a_{i} b_{k i}\right)=\sum_{i=0}^{n-1} a_{i}\left(\sum_{k=i+1}^{n} b_{k i}\right)
$$

where $\left(a_{k}\right)$ and $\left(b_{n k}\right)$ are real numbers.
Proof. It is clear.
Theorem 2.5. We have

$$
\sigma_{c}(U(r ; 0 ; s), h)=\{\lambda \in \mathbb{C}:|\lambda-r|=|s|\}
$$

and

$$
\sigma(U(r ; 0 ; s), h)=\{\lambda \in \mathbb{C}:|\lambda-r| \leqslant|s|\}
$$

Proof. Let $y=\left(y_{n}\right) \in \sigma_{\infty}$ be such that $(U(r ; 0 ; s)-\lambda I)^{*} x=y$ for some $x=\left(x_{n}\right)$. Then we get system of linear equations:

$$
\begin{array}{rccc}
(r-\lambda) x_{0} & = & y_{0} & \\
(r-\lambda) x_{1} & = & y_{1} & \\
& \vdots & & , n \geqslant 0 \\
s x_{n}+(r-\lambda) x_{n+2} & = & y_{n+2} &
\end{array}
$$

Solving these equations, we have

$$
\begin{gathered}
x_{0}=\frac{1}{r-\lambda} y_{0}, \quad x_{1}=\frac{1}{r-\lambda} y_{1}, \quad x_{2}=\frac{1}{r-\lambda} y_{2}-\frac{s}{(r-\lambda)^{2}} y_{0}, \quad x_{3}=\frac{1}{r-\lambda} y_{3}-\frac{s}{(r-\lambda)^{2}} y_{1} \\
x_{4}=\frac{1}{r-\lambda} y_{4}-\frac{s}{(r-\lambda)^{2}} y_{2}+\frac{s^{2}}{(r-\lambda)^{3}} y_{0} \\
x_{5}=\frac{1}{r-\lambda} y_{5}-\frac{s}{(r-\lambda)^{2}} y_{3}+\frac{s^{2}}{(r-\lambda)^{3}} y_{1}
\end{gathered}
$$

Thus we get

$$
x_{2 n+t}=\frac{1}{r-\lambda}\left[y_{2 n+t}+\sum_{k=0}^{n-1}(-1)^{n-k}\left(\frac{s}{r-\lambda}\right)^{n-k} y_{2 k+t}\right], \quad t=0,1 ; n=1,2, \ldots
$$

Therefore we have

$$
\begin{aligned}
& \frac{1}{2 n+t}\left|\sum_{k=0}^{2 n+t} x_{k}\right|=\frac{1}{2 n+t}\left|x_{0}+x_{1}+x_{2}+x_{3}+\cdots+x_{2 n+t}\right| \\
& =\frac{1}{2 n+t}\left|x_{0}+x_{1}+\sum_{k=1}^{n} x_{2 k+t}\right| \\
& \leqslant \frac{1}{2 n+t}\left|\frac{y_{0}}{r-\lambda}+\frac{y_{1}}{r-\lambda}\right| \\
& +\frac{1}{2 n+t}\left|\sum_{k=1}^{n} \frac{1}{r-\lambda}\left[y_{2 k+t}+\sum_{i=0}^{k-1}(-1)^{k-i}\left(\frac{s}{r-\lambda}\right)^{k-i} y_{2 i+t}\right]\right| \\
& \leqslant \frac{1}{2 n+t}\left|\frac{y_{0}}{r-\lambda}+\frac{y_{1}}{r-\lambda}\right| \\
& +\frac{1}{\mid r-\lambda} \frac{1}{2 n+t}\left|\sum_{k=1}^{n} y_{2 k+t}\right|+\frac{1}{2 n+t}\left|\sum_{k=1}^{n} \frac{1}{r-\lambda} \sum_{i=0}^{k-1}(-1)^{k-i}\left(\frac{s}{r-\lambda}\right)^{k-i} y_{2 i+t}\right| \\
& \leqslant \frac{1}{2 n+t}\left|\frac{y_{0}}{r-\lambda}+\frac{y_{1}}{r-\lambda}\right| \\
& +\frac{1}{|r-\lambda|}\left[\frac{1}{2 n+t}\left|\sum_{k=1}^{n} y_{2 k+t}\right|+\frac{1}{2 n+t}\left|\sum_{k=1}^{n} \sum_{i=0}^{k-1}(-1)^{k-i}\left(\frac{s}{r-\lambda}\right)^{k-i} y_{2 i+t}\right|\right]
\end{aligned}
$$

Thus

$$
\begin{equation*}
\frac{1}{2 n+t}\left|\sum_{k=0}^{2 n+t} x_{k}\right| \leqslant \frac{1}{|r-\lambda|}\left[2\|y\|_{\sigma_{\infty}}+\frac{1}{2 n+t}\left|\sum_{k=1}^{n} \sum_{i=0}^{k-1}(-1)^{k-i}\left(\frac{s}{r-\lambda}\right)^{k-i} y_{2 i+t}\right|\right] \tag{2.5}
\end{equation*}
$$

Now, we consider the sum $\frac{1}{2 n+t}\left|\sum_{k=1}^{n} \sum_{i=0}^{k-1}(-1)^{k-i}\left(\frac{s}{r-\lambda}\right)^{k-i} y_{2 i+t}\right|$. In Lemma 2.4 if we take $a_{i}=y_{2 i+t}$ and $b_{k i}=(-1)^{k-i}\left(\frac{s}{r-\lambda}\right)^{k-i}$ then we have

$$
\begin{align*}
& \frac{1}{2 n+t}\left|\sum_{k=1}^{n} \sum_{i=0}^{k-1}(-1)^{k-i}\left(\frac{s}{r-\lambda}\right)^{k-i} y_{2 i+t}\right| \\
= & \frac{1}{2 n+t}\left|\sum_{i=0}^{n-1} y_{2 i+t} \sum_{k=i+1}^{n}(-1)^{k-i}\left(\frac{s}{r-\lambda}\right)^{k-i}\right| \tag{2.6}
\end{align*}
$$

Hence

$$
\begin{align*}
& \frac{1}{2 n+t}\left|\sum_{k=1}^{n} \sum_{i=0}^{k-1}(-1)^{k-i}\left(\frac{s}{r-\lambda}\right)^{k-i} y_{2 i+t}\right| \\
& =\frac{1}{2 n+t}\left|\left(\frac{-s}{r+s-\lambda}\right) \sum_{i=0}^{n-1} y_{2 i+t}\left[1-\left(\frac{-s}{r-\lambda}\right)^{n-i}\right]\right|  \tag{2.7}\\
& \leqslant\left|\frac{s}{r+s-\lambda}\right|\left(\|y\|_{\sigma_{\infty}}+\left|\frac{s}{r-\lambda}\right|^{n+1} \frac{1}{2 n+t}\left|\sum_{i=0}^{n-1} y_{2 i+t}\left(\frac{r-\lambda}{-s}\right)^{i}\right|\right)
\end{align*}
$$

If we take $a_{i}=y_{2 i+t}, b_{i}=\left(\frac{r-\lambda}{-s}\right)^{i}$ and apply Abel's partial summation formula to $\operatorname{sum} \sum_{i=0}^{n-1} y_{2 i+t}\left(\frac{r-\lambda}{-s}\right)^{i}$, we obtain

$$
\sum_{i=0}^{n-1} y_{2 i+t}\left(\frac{r-\lambda}{-s}\right)^{i}=\left(\frac{r-\lambda}{-s}\right)^{n} \sum_{i=0}^{n} y_{2 i+t}+\frac{s-r+\lambda}{s} \sum_{i=0}^{n-2}\left(\frac{r-\lambda}{-s}\right)^{i} \sum_{k=0}^{i} y_{2 k+t}
$$

since $s_{n}=\sum_{i=0}^{n} y_{2 i+t}, \Delta b_{i}=\frac{s-r+\lambda}{s}\left(\frac{r-\lambda}{-s}\right)^{i}$. Thus

$$
\begin{aligned}
& \left.\left.\left|\frac{s}{r-\lambda}\right|^{n+1} \frac{1}{2 n+t}\right|_{i=0} ^{n-1} y_{2 i+t}\left(\frac{r-\lambda}{-s}\right)^{i} \right\rvert\, \\
& =\left|\frac{s}{r-\lambda}\right|^{n+1} \frac{1}{2 n+t}\left|\left(\frac{r-\lambda}{-s}\right)^{n} \sum_{i=0}^{n} y_{2 i+t}+\frac{s-r+\lambda}{s} \sum_{i=0}^{n-2}\left(\frac{r-\lambda}{-s}\right)^{i} \sum_{k=0}^{i} y_{2 k+t}\right| \\
& \leqslant\left|\frac{s}{r-\lambda}\right|\|y\|_{\sigma_{\infty}}+\left|\frac{s}{r-\lambda}\right|^{n+1}\left|\frac{s-r+\lambda}{s}\right|_{i=0}^{n-2}\left|\frac{r-\lambda}{s}\right|^{i} \frac{1}{2 n+t}\left|\sum_{k=0}^{i} y_{2 k+t}\right| \\
& \leqslant\left[\left|\frac{s}{r-\lambda}\right|+\left|\frac{s}{r-\lambda}\right|^{n+1}\left|\frac{s-r+\lambda}{s}\right| \sum_{i=0}^{n-2}\left|\frac{r-\lambda}{s}\right|^{i}\right]\|y\|_{\sigma_{\infty}} \\
& =\left[\left|\frac{s}{r-\lambda}\right|+\left|\frac{s}{r-\lambda}\right|^{n+1}\left|\frac{s-r+\lambda}{s}\right| \frac{1-\left|\frac{r-\lambda}{s}\right|^{n-1}}{1-\left|\frac{r-\lambda}{s}\right|}\right]\|y\|_{\sigma_{\infty}}
\end{aligned}
$$

and we get

$$
\begin{align*}
& \left|\frac{s}{r-\lambda}\right|^{n+1} \frac{1}{2 n+t}\left|\sum_{i=0}^{n-1} y_{2 i+t}\left(\frac{r-\lambda}{-s}\right)^{i}\right| \\
\leqslant & {\left[\left|\frac{s}{r-\lambda}\right|+\frac{|s-r+\lambda|}{|s|-|r-\lambda|}\left(\left|\frac{s}{r-\lambda}\right|^{n+1}-\left|\frac{s}{r-\lambda}\right|^{2}\right)\right]\|y\|_{\sigma_{\infty}} } \tag{2.8}
\end{align*}
$$

Replacing (2.8) in (2.7), we have

$$
\begin{align*}
& \frac{1}{2 n+t}\left|\sum_{k=1}^{n} \sum_{i=0}^{k-1}(-1)^{k-i}\left(\frac{s}{r-\lambda}\right)^{k-i} y_{2 i+t}\right|  \tag{2.9}\\
& \leqslant\left|\frac{s}{r+s-\lambda}\right|\left[1+\left|\frac{s}{r-\lambda}\right|+\frac{|s-r+\lambda|}{|s|-|r-\lambda|}\left(\left|\frac{s}{r-\lambda}\right|^{n+1}-\left|\frac{s}{r-\lambda}\right|^{2}\right)\right]\|y\|_{\sigma_{\infty}}
\end{align*}
$$

Finally replacing (2.9) in (2.5), we get

$$
\begin{aligned}
& \frac{1}{2 n+t}\left|\sum_{k=0}^{2 n+t} x_{k}\right| \\
& \leqslant \frac{1}{|r-\lambda|}\left\{2\left|\frac{s}{r+s-\lambda}\right|\left[1+\left|\frac{s}{r-\lambda}\right|+\frac{|s-r+\lambda|}{|s|-|r-\lambda|}\left(\left|\frac{s}{r-\lambda}\right|^{n+1}-\left|\frac{s}{r-\lambda}\right|^{2}\right)\right]\right\}\|y\|_{\sigma_{\infty}}
\end{aligned}
$$

Since $y=\left(y_{n}\right) \in \sigma_{\infty}, x=\left(x_{n}\right) \in \sigma_{\infty}$ if $\left|\frac{s}{r-\lambda}\right|<1$. Consequently, if for $\lambda \in \mathbb{C}$, $|r-\lambda|>|s|$, then $\left(x_{n}\right) \in \sigma_{\infty}$. Therefore, the operator $(U(r ; 0 ; s)-\lambda I)^{*}$ is onto if $|r-\lambda|>|s|$. Then by Lemma 2.3, $U(r ; 0 ; s)-\lambda I$ has a bounded inverse if $|r-\lambda|>|s|$. So,

$$
\sigma_{c}(U(r ; 0 ; s), h) \subseteq\{\lambda \in \mathbb{C}:|\lambda-r| \leqslant|s|\}
$$

Since $\sigma(L, h)$ is the disjoint union of $\sigma_{p}(L, h), \sigma_{r}(L, h)$ and $\sigma_{c}(L, h)$, therefore

$$
\sigma(U(r ; 0 ; s), h) \subseteq\{\lambda \in \mathbb{C}:|\lambda-r| \leqslant|s|\}
$$

By Theorem 2.2, we get

$$
\{\lambda \in \mathbb{C}:|\lambda-r|<|s|\}=\sigma_{p}(U(r ; 0 ; s), h) \subset \sigma(U(r ; 0 ; s), h)
$$

Since, $\sigma(L, h)$ is closed and thus

$$
\overline{\{\lambda \in \mathbb{C}:|\lambda-r|<|s|\}} \subset \quad \overline{\sigma(U(r ; 0 ; s), h)}=\sigma(U(r ; 0 ; s), h)
$$

and

$$
\{\lambda \in \mathbb{C}:|\lambda-r| \leqslant|s|\} \subset \sigma(U(r ; 0 ; s), h)
$$

Hence,

$$
\sigma(U(r ; 0 ; s), h)=\{\lambda \in \mathbb{C}:|\lambda-r| \leqslant|s|\}
$$

and so

$$
\sigma_{c}(U(r ; 0 ; s), h)=\{\lambda \in \mathbb{C}:|\lambda-r|=|s|\}
$$

## References

[1] R. Das and B. C. Tripathy. The spectrum and fine spectrum of the lower triangular matrix $B(r, s, t)$ on the sequence space cs. Songklanakarin J. Sci. Technol., 38(3)(2016), 265-274.
[2] R. Das. On the fine spectrum of the lower triangular matrix $B(r, s)$ over the Hahn sequence space. Kyungpook Math. J., $\mathbf{5 7}(3)(2017), 441-455$.
[3] N. Durna. Subdivision of the spectra for the generalized difference operator $\Delta_{a, b}$ on the sequence space $\ell_{p},(1<p<\infty)$. CBU J. Sci., 13(2)(2017), 359-364.
[4] N. Durna, M. Yildirim and R. Kıliç. Partition of the spectra for the generalized difference operator $B(r, s)$ on the sequence space cs. Cumhuriyet Sci. J., $\mathbf{3 9}(1)(2018), 7-15$.
[5] N. Durna. Subdivision of spectra for some lower triangular doule-band matrices as operators on $c_{0}$. Ukr. Math. J., $\mathbf{7 0}(7)(2018), 1052-1062$.
[6] S. R. El-Shabrawy and S. H. Abu-Janah. Spectra of the generalized difference operator on the sequence spaces $b v_{0}$ and $h$. Linear and Multilinear Algebra, 66(8)(2017), 1691-1708.
[7] S. Goldberg. Unbounded Linear Operators, McGraw Hill, New York, 1966.
[8] H. Hahn. Über Folgen linearer operationen. Monatsh Math Phys., 32(1922), 3-88.
[9] V. Karakaya and M. Altun. Fine spectra of upper triangular double-band matrices. J. Comput. Appl. Math., 234(2010), 1387-1394.
[10] V. Karakaya, M. Dzh. Manafov and N. Şimşek. On fine spectra and subspectrum (approximate point, defect and compression) of operator with periodic coefficients. Journal of Nonlinear and Convex Analysis, 18(4)(2017), 709-717.
[11] A. Paul and B. C. Tripathy. The spectrum of the operator $D(r, 0,0, s)$ over the sequence spaces $\ell_{p}$ and $b v_{p}$. Hacet. J. Math. Stat., 43(3) (2014), 425-434.
[12] A. Paul and B. C. Tripathy. The Spectrum of the operator $D(r, 0,0, s)$ over the sequence space $b v_{0}$. Georgian Math. J., $22(3)(2015), 421-426$.
[13] B. C. Tripathy and P. Saikia. On the spectrum of the Cesàro operator $C_{1}$ on $\overline{b v_{0}} \cap \ell_{\infty}$. Math. Slovaca, 63(3)(2013), 563-572.
[14] B. C. Tripathy and A. Paul. The Spectrum of the operator $D(r, 0,0, s)$ over the sequence spaces $c_{0}$ and $c$. Kyungpook Math. J., $\mathbf{5 3}(2)(2013), 247-256$.
[15] B. C. Tripathy and R. Das. Spectra of the Rhaly operator on the sequence space $\overline{b v_{0}} \cap \ell_{\infty}$. Bol. Soc. Parana. Math., 32(1)(2014), 263-275.
[16] B. C. Tripathy and R. Das. Spectrum and fine spectrum of the upper triangular matrix $U(r, s)$ over the sequence space cs. Proyecciones J. Math., 34(2)(2015), 107-125.
[17] K. C. Rao. The Hahn sequence spaces I. Bull. Calcutta. Math. Soc., 82(1990), 72-78.
[18] M. Yeşilkayagil and M. Kirişci. On the fine spectrum of the forward difference operator on the Hahn space. Gen. Math. Notes., 33(2)(2016), 1-16.
[19] M. Yildirim and N. Durna. The spectrum and some subdivisions of the spectrum of discrete generalized Cesàro operators on $\ell_{p},(1<p<\infty)$. J. Inequal. Appl., 193(2017), 1-13.
[20] M. Yildirim, M. Mursaleen and Ç. Doğan. The spectrum and fine spectrum of generalized Rhaly-Cesàro matrices on $c_{0}$ and c. Operators and Matrices, $12(4)(2018)$, 955-975.
[21] J. M. Varah. On the solution of block-tridiagonal systems arising from certain finite-difference equations. Mathematics of Computation, 26(1)(1972), 859-868.

Received by editors 27.12.2018; Revised version 13.05.2019; Available online 20.05.2019.
Faculty of Science, Department of Mathematics, Sivas Cumhuriyet Universitiy, Sivas, Turkey

E-mail address: ndurna@cumhuriyet.edu.tr


[^0]:    2010 Mathematics Subject Classification. Primary 47A10; Secondary 47B37.
    Key words and phrases. Hahn space, point spectrum, continuous spectrum and residual spectrum of the operator $U(r ; 0 ; s)$.

    This work is supported by the Scientific Research Project Fund of Sivas Cumhuriyet University under the project number F-583.

