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ON GENERALIZATION OF QUASI IDEALS IN SEMIRINGS

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Abstract. Ideals have played an important role in studies of semirings, and related systems. Their generalization is in the form of one-sided ideals. One-sided ideals are generalized to quasi ideals and quasi ideals are further generalized to bi ideals. In this article, we generalize the quasi ideals through an index m called the m-quasi ideals, and study their important properties in semirings. We introduce the idea of m-regular semirings and study their important properties through m-quasi ideals.

1. Introduction and Preliminaries

A semiring is a nonempty set A together with two binary operations addition $+$ and multiplication $\cdot$ usually denoted by an ordered triple $(A, +, \cdot)$ if $(A, +)$ is a commutative semigroup, $(A, \cdot)$ is semigroup and right and left distributive laws i.e., $a(b + c) = ab + ac$ and $(a + b)c = ac + bc \quad \forall \quad a, b, c \in A$, hold. A nonempty subset H of A is called its subsemiring if it is itself a semiring under the operations of A, that is, $H^2 \subset H$. A subsemiring L/R of A is called a left-ideal/right-ideal of A if $AL \subseteq L/LA \subseteq R$. A subsemiring I is called a two-sided or simply an ideal of A if it is both a left and right ideal. By generalizing one-sided ideals, we can define a quasi-ideal as a subsemiring Q such that $QA \cap AQ \subseteq Q$. A further generalization of *quasi ideal* results in defining a *bi ideal* as a subsemiring B of A such that $BAB \subseteq A$.

All ideals and one-sided ideals are *quasi ideals*, but the converse is not true. The quasi ideals are bi ideals, but the converse is not true. A detailed study of the quasi ideals and bi ideals is found in [**5**] and [**7**].

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If X and Y are two non-empty subsets of a semiring $(S, +, \cdot)$, then the *sum* $X+Y$ respectively the *product* XY of X and Y are defined by $X+Y = \{x+y : x \in X \quad \text{and} \quad y \in Y\}$, and $XY = \{\sum_{\text{finite}} x_i y_i : x_i \in X \quad \text{and} \quad y_i \in Y\}$. The undefined terms and notations can be followed in [**4**] and [**3**].

In this paper, we generalize quasi-ideals by an index m, where m is a positive integer. In Section 2, we summarize some results about the m-left/ m-right ideals and m-bi ideals from [**2**] and [**6**], and describe their properties in association with the left and right ideals. In Section 3, we introduce the idea of the m-quasi ideal in semirings. In Section 4, we present the new ideal of the m-regular semirings. The conclusion of the paper is given in Section 5.

2. One-sided and m-bi ideals

Definition 2.1. For a semiring A, and a positive integer m, we have $A^m = AAA...A$(m-times) [**6**].

Now $A^2 = AA \subseteq A$; as A is a semiring. Therefore, $A^3 = AAA \subseteq A^2 \subseteq A$, i.e., $A^3 \subseteq A^2$, and $A^3 \subseteq A$. So, we conclude that $A^s \subseteq A^m$ for all positive integers s and m, such that $s \geqslant m$. Consequently $A^m \subseteq A$, for all m.

Definition 2.2. A subsemiring L of a semiring $(A, +, \cdot)$ is called m-left ideal of A if $A^m L \subseteq L$, where m is a positive integer [**2**]. The least positive such integer m is called the *left-potency* of L. Similarly, the subsemiring R of A is said to be m-right ideal of A if $RA^m \subseteq R$, where m is a positive integer. The least positive such integer m is called the *right-potency* of R.

A subsemiring I of A is called an m-two-sided ideal or simply an m-ideal of A if it is both m-left ideal and m-right ideal of A i.e., $A^m I A^m \subseteq I$, where m is a positive integer. The least positive such integer m is called the *potency* of I.

Proposition 2.1. *Every left/right ideal is an m-left / m- right ideal*

Proof. Let L be a left ideal of A, then $LA^mL \subseteq LAL \subseteq L$. That is $LA^mL \subseteq L$. So, L is an m for any positive integer m.
The proof for right ideal R is similar. □

Corollary 2.1. *Every m-left ideal (m-right ideal) of A is an n-left ideal (n-right ideal) of A, for all* $n \geqslant m$

Proof. Let L be an m-left ideal of A, then $LA^nL \subseteq LA^mL \subseteq L$. This gives that L is n-left ideal of A. The proof for m-right ideal is similar. □

Example 3.2 shows that the m-left/m-right ideals need not be the *left* or *right* ideals.

Theorem 2.3. *Let A be a semiring.*

(1) *For an m-left ideal,* L_i *of A,* $i \in I$*, we have* $\bigcap_{i \in I} L_i$ *is an m-left ideal of A.*
(2) *Similarly, if* R_i *is an n-right ideal of A for any* $i \in I$*, then* $\bigcap_{i \in I} R_i$ *is an n-right ideal of A.*

PROOF. (1) Let $\{L_\lambda : \lambda \in \wedge\}$ be a family of m-left ideals of semiring A, then $L = \bigcap_{\lambda\in\wedge} L_\lambda$, being the intersection of subsemirings of A is a subsemiring of A. Since

$$A^{m_\lambda} L_\lambda \subseteq L_\lambda \quad \forall \quad \lambda \in \wedge,$$

and $L \subseteq L_\lambda \quad \forall \quad \lambda \in \wedge$, therefore

$$A^{\max\{m_\lambda:\lambda\in\wedge\}} L \subseteq A^{m_\lambda} L_\lambda \subseteq L_\lambda \quad \forall \quad \lambda \in \wedge.$$

That is, $A^{\max\{m_\lambda:\lambda\in\wedge\}} L \subseteq L_\lambda \quad \forall \quad \lambda \in \wedge$. This gives

$$A^{\max\{m_\lambda:\lambda\in\wedge\}} L \subseteq \bigcap_{\lambda\in\wedge} L_\lambda = L.$$

So, $A^{\max\{m_\lambda:\lambda\in\wedge\}} L \subseteq L$. Thus L is an m-left ideal with *bipotency* $\max\{m_1, m_2, ...\}$.
(2) Analogously. □

THEOREM 2.4. *Let A be a semiring,*

(1) *The m-left ideal generated by a subsemiring H of A is $H + A^m H$,*
(2) *The m-right ideal generated by a non-empty subset H of A is $H + HA^m$.*

PROOF. (1) Let $< H >_m = H + A^m H$. We need to show that $< H >_m$ is the minmal m-left ideal of A which contains H. $< H >_m$ is clearly closed under addition. Consider, $(H+A^m A)(H+A^m H) = H^2+HA^m H+A^m HH+A^m HA^m H \subseteq H + AA^m H + A^m AH + A^m AA^m H \subseteq H + A^{m+1}H + A^{m+1}H + A^{2m+1}H \subseteq H + AH + AH + AH \subseteq H + AH$. So $< H >_m$ is a subsemiring of A. Next, we need to show that $A^m < H >_m \subseteq < H >_m$. Consider $A^m < H >_m = A^m(H + A^m H) = A^m H + A^m A^m H = A^m H + A^{2m} H \subseteq \{0\} + A^m H \subseteq H + A^m H$. Therefore, $A^m < H >_m \subseteq < H >_m$. That is, $A^m < H >_m$ is an m-left ideal containing H. That is, $A^m H \subseteq H$. To show that $< H >_m$ is the minmal m-left ideal of A which contains H, let H' be any other m-left ideal of A containing H. Then $H + A^m H \subseteq H' + A^m H' \subseteq H' + H' \subseteq H'$. Therefore, $< H >_m = H + A^m H \subseteq H'$. Hence, $< H >_m$ is the minimal m-left ideal of A which contains H.
(2) Analogously. □

Now, we summarize some results about the m-bi ideals from [**6**].

DEFINITION 2.5. Let $(A, +, \cdot)$ be a semiring. An *m-bi ideal* B of A is a subsemiring of A such that $BA^m B \subseteq B$ where m is a least positive integer , not necessarily 1. The least positive such integer m is called the *bipotency* of the *bi ideal* B.

REMARK 2.1. $BA^m B \subseteq B$ is called the *bipotency condition.* Every bi ideal of a semiring is its 1-bi ideal(bi ideal of bipotency 1). All the so-called 1-bi ideals are simply the bi ideals, whereas those with bipotency $m > 1$ are to be specified with the value of m. For every $m \geqslant 1$, every bi-ideal is an m-bi ideal. Every m-bi ideal of the semiring A is an n-bi right ideal of A, for all $n \geqslant m$. The converse of this statement is not true [**6**]. Left ideal L and the right ideal R of the semiring A are its 1-bi ideals.

PROPOSITION 2.2. *The product of any number of m-bi ideals of a semiring A, with identity e, is an m-bi ideal.*

PROOF. It is sufficient to prove the result for two m-bi ideals of A. Suppose B_1 and B_2 be bi ideals of A with bipotencies m_1 and m_2 respectively, that is, $B_1 A^{m_1} B_1 \subseteq B_1$ and $B_2 A^{m_1} B_2 \subseteq B_2$, m_1 and m_2 are any positive integers. Then $B_1 B_2$ being the finite sum of the product is obviously closed under addition. Now we have,

$$\begin{aligned}(B_1B_2)^2 &= (B_1B_2)(B_1B_2) = (B_1AB_1)B_2 = (B_1Ae...eB_1)B_2 \\ &\subseteq (B_1AA...AB_1)B_2 \subseteq (B_1A^mB_1)B_2 \subseteq B_1B_2.\end{aligned}$$

That is, $(B_1B_2)^2 \subseteq B_1B_2$. So, B_1B_2 is closed under multiplication. B_1B_2 is a subsemiring of A. Moreover,

$$\begin{aligned}&B_1B_2(A^{\max(m_1,m_2)})B_1B_2 \subseteq B_1AA^{\max(m_1,m_2)}B_1B_2 \\ &= B_1A^{1+\max(m_1,m_2)}B_1B_2 \subseteq B_1A^{m_1}B_1B_2 \subseteq B_1B_2.\end{aligned}$$

We used the result $A^{1+\max(m_1,m_2)} \subseteq A^{m_1}$ as is evident by Definition 2.1. So, $B_1B_2(A^{\max(m_1,m_2)})B_1B_2 \subseteq B_1B_2$. Thus, B_1B_2 is m-bi ideal of A with bipotency $\max(m_1, m_2)$. □

PROPOSITION 2.3. *Let T be an arbitrary subset of a semiring A with identity e, and B be an m- bi ideal of A, m not necessarily* 1. *Then the product BT is also m-bi ideal of A.*

PROOF. The product BT as defined in Section 1 is closed under addition. Next, $(BT)^2 = (BT)(BT) = (BTB)T \subseteq (BAB) \subseteq BAe...eB \subseteq BAA...AB \subseteq (BA^mB)T \subseteq BT$. So, $BT^2 \subseteq BT$ making it a subsemiring of A. Moreover, $BT(A^m)BT \subseteq BAA^mBT \subseteq BA^{1+m}BT \subseteq BA^mBT \subseteq BT$. Therefore BT is a bi ideal of A with bipotency m. □

Similarly, we can prove that TB is also m-bi ideal of A.

PROPOSITION 2.4. *The intersection of a family of bi ideals of semiring A with bipotencies $m_1, m_2, ...,$ is also a bi ideal with bipotency $max\{m_1, m_2, ...\}$.*

PROOF. See [**6**]. □

Sum of two m-bi ideals of a semiring is not an m-bi ideals. See example in [**6**]. The following theorem tells about the intersection of m-left and right ideal of a semiring A.

THEOREM 2.6. *Let $L(R)$ be an m-left ideal(n-right ideal) of a semiring A, then their intersection, $L \cap R$, is a t-bi ideal of A, where $t = max(m, n)$.*

PROOF. $L \cap R$ is clearly a subsemiring of A. Since L is m-bi ideal and R is n-bi ideals of A, their intersection becomes $max(m, n)$-bi ideals from the result 2.4. Similarly, we can show that $L \cap R(A^{max\{m,n\}})L \cap R \subseteq R$. Consequently, $L \cap RA^{max\{m,n\}}L \cap R \subseteq L \cap R$ □

REMARK 2.2. The integer m for any ideal specifies the number of times of pre or post-multiplication of the semiring A with a subsemiring H so that it becomes an ideal. A right/left ideal is the 1-left/1-right ideals because one needs to multiply A on right/left side of H to make it right/left ideal. Similarly, a bi ideal B is a 1-bi ideal, and a quasi ideal is a 1-quasi ideal in the sense that $BAB = BA^1B \subseteq B$ and $QA^1 \cap A^1Q = QA \cap AQ \subseteq Q$ respectively.

3. m-quasi ideals

Moin et al., gave the idea of (m, n)-quasi ideals in semigroups [**1**]. In this section, we generalize the quasi ideals, through a single index m, in semirings, where m is a positive integer.

DEFINITION 3.1. A subsemiring Q of a semiring $(A, +, \cdot)$ is called m-quasi ideal of A if $QA^m \cap A^mQ \subseteq Q$, where m is a positive integer called the *quasi-potency of* Q.

PROPOSITION 3.1. *For any $m \geqslant 1$, a quasi ideal is an m-quasi ideal.*

PROOF. If Q is a quasi ideal of A, then $QA^m \cap A^mQ \subseteq QA \cap AQ \subseteq Q$. That is $QA^m \cap A^mQ \subseteq Q$. So, Q is m-ideal. □

COROLLARY 3.1. *Every m-quasi ideal of A is an n-quasi ideal of A, for all $n \geqslant m$*

PROOF. For an m-quasi ideal Q, we have $QA^n \cap A^nQ \subseteq QA^m \cap A^mQ \subseteq Q$. That is $QA^n \cap A^nQ \subseteq Q$. So, Q is n-quasi ideal. □

Every m-quasi ideal is not a quasi ideal. This is evident from the following example.

EXAMPLE 3.2. Let

$$A = \left\{ \begin{bmatrix} 0 & l & m & n \\ 0 & 0 & o & p \\ 0 & 0 & 0 & q \\ 0 & 0 & 0 & 0 \end{bmatrix} : l, m, n, o, p, q \text{ are any positive real numbers} \right\}$$

and

$$A^0 = A \cup \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

then $(A, +, \cdot)$ is a semiring under the usual operations of addition $+$ and multiplication $\cdot$ of matrices. Let

$$H = \left\{ \begin{bmatrix} 0 & l & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & q \\ 0 & 0 & 0 & 0 \end{bmatrix} : l, q \text{ are any positive real numbers} \right\} \cup \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this case, H is not a quasi ideal of A, but it is a 3-quasi ideal of A as $A^3H \cap HA^3 \subseteq H$. Moreover, H is 3-left ideal of A as $A^3H \subseteq H$, but it is not a left ideal of A because $AH \nsubseteq H$. H is a 3-right ideal of A, but not a right ideal of A.

PROPOSITION 3.2. *Every m-left ideal/m-right ideal and hence every m-ideal is a quasi ideal with quasi-potency m.*

PROOF. Let L be an m-left ideal. Then $LA^m \cap A^mL \subseteq L \cap L \subseteq L$. So, $LA^m \cap A^mL \subseteq L$. Thus L is m-quasi ideal. $\square$

The converse of the above theorem is not true. That is, every m-quasi ideal is not always m-right/m-left ideals.

EXAMPLE 3.3. Let A be the semiring as given in Example 3.2, and

$$T = \left\{ \begin{bmatrix} l & 0 & 0 & 0 \\ 0 & 0 & 0 & q \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} : l, q \text{ are any positive real numbers} \right\} \cup \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this case, T is a 2-quasi ideal of A as $A^2T \cap TA^2 \subseteq T$, but T is not a 2-right ideal of A because $TA^2 \nsubseteq T$.

PROPOSITION 3.3. *The intersection of any family of m-quasi ideals of a semiring A is its m-quasi ideal.*

PROOF. Let $\{Q_\lambda : \lambda \in I\}$ be a family of m-quasi ideals of a semiring A, then

$$A^m \left(\bigcap_{\lambda \in I} Q_\lambda \right) \bigcap \left(\bigcap_{\lambda \in I} Q_\lambda \right) A^m \subseteq Q_\lambda \text{ for all } \lambda \in I.$$

This gives

$$A^m \left(\bigcap_{\lambda \in I} Q_\lambda \right) \bigcap \left(\bigcap_{\lambda \in I} Q_\lambda \right) A^m \subseteq \bigcap_{\lambda \in I} Q_\lambda.$$

Thus $\bigcap_{\lambda \in I} Q_\lambda$ is an m-quasi ideal of A. $\square$

COROLLARY 3.2. *For an m-right ideal R and an m-left ideal L of a semiring A, their intersection is an m-quasi ideal of A.*

PROOF. L and R being the m-left and m-ideals of A are also its m-quasi ideals, so by above theorem, the intersection, $L \cap R$, is m-quasi ideal of A. $\square$

The m-quasi Q has m-intersection property if Q is the intersection of an m-left ideal and an m-right ideal of A. In this case, every m-left ideal and every m-right ideal have the m intersection property. The following theorem characterizes m-quasi ideals having the m-intersection property.

THEOREM 3.4. *A m-quasi ideal Q of a semiring A has the m-intersection property if and only if*

$$(Q + A^mQ) \cap (Q + QA^m) = Q.$$

Proof. $(i) \Rightarrow (ii)$. Let Q has the m-intersection property. Now we show that

$$(Q + A^m Q) \cap (Q + QA^m) = Q.$$

It is very obvious that

$$Q \subseteq (Q + A^m Q) \cap (Q + QA^m).$$

Since Q has the m- intersection property, so we write $Q = L \cap R$ for some m-left ideal L and some m-right ideal R of A. Thus $Q \subseteq L$ and $Q \subseteq R$. Moreover, $A^m Q \subseteq A^m L \subseteq L$, and $QA^m \subseteq RA^m \subseteq R$. This implies that $Q + QA^m \subseteq R$ and $Q + A^m Q \subseteq L$. Therefore,

$$(Q + QA^m) \cap (Q + A^m Q) \subseteq Q.$$

Consequently,

$$(Q + QA^m) \cap (Q + A^m Q) = Q.$$

Next we show that $(ii) \Rightarrow (i)$. Consider,

$$(Q + QA^m) \cap (Q + A^m Q) = Q.$$

Since it is clear that both $(Q + QA^m)$ and $(Q + A^m Q)$ are respectively m-right and m-left ideals of A as $A^m Q$ and QA^m both are m-right and m-left ideals of A. Therefore, Q has m-intersection property. □

Theorem 3.5. *For m-quasi-ideal Q of A, if $A^m Q \subseteq QA^m$ or $QA^m \subseteq A^m Q$, then Q has m-intersection property.*

Proof. Without of loss of generality, suppose that $A^m Q \subseteq QA^m$, then $A^m Q = A^m Q \cap QA^m \subseteq Q$. That is, $A^m Q \subseteq Q$. So, Q is m-left ideal of A. Thus, Q has the m-intersection property. □

The sum and the product of m-quasi ideals both need not be m-quasi ideal as is evident from the following two examples.

Example 3.6. Let A be the semiring as given in Example 3.2, $Q = H + T$; H as given in Example 3.2 and T as given in Example 3.3. Then H and T are respectively 3-quasi and 2-quasi ideals of A as explained in Examples 3.2 and 3.3, but Q is not m-quasi ideal of A. Indeed

$$Q = \left\{ \begin{bmatrix} m & l & 0 & 0 \\ 0 & 0 & 0 & p \\ 0 & 0 & 0 & q \\ 0 & 0 & 0 & 0 \end{bmatrix} : m, l, p, q \text{are any positive real numbers} \right\} \cup \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and even $Q^2 \nsubseteq Q$ i.e., Q is not a subsemiring of A. So, sum of m-quasi ideals is not an m-quasi ideal.

Example 3.7. Let A be the semiring as given in Example 3.2, $Q = HT$; H as given in Example 3.2 and

$$T = \left\{ \begin{bmatrix} 0 & 0 & m & 0 \\ 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 \\ q & 0 & 0 & 0 \end{bmatrix} : m, p, q \text{are any positive real numbers} \right\} \cup \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then H and T are respectively 3-quasi and 2-quasi ideals of A, but Q is not an m-quasi ideal of A. Actually,

$$Q = \left\{ \begin{bmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & y & 0 & 0 \end{bmatrix} : x, y \text{ are any positive real numbers} \right\} \cup \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and even $Q^2 \nsubseteq Q$ i.e., Q is not a subsemiring of A. So, product of m-quasi ideals is not an m-quasi ideal.

Proposition 3.4. *Every m-quasi ideal Q of a semiring A is its m-bi ideal.*

Proof. As Q is a m-quasi ideal of A, then

$$QA^mQ \subseteq QA^mA \cap A^mAQ = QA^{m+1} \cap A^{m+1}Q \subseteq QA^m \cap A^mQ \subseteq Q,$$

i.e., $QA^mQ \subset Q$, i.e., Q is m-bi ideal of A. □

Proposition 3.5. *For two m-quasi ideals of a semiring A with identity e, their product is m-bi ideal of A.*

Proof. Let Q_1 and Q_2 be two quasi ideals of a semiring A with quasi-potencies m_1 and m_2 respectively, that is, $Q_1A^{m_1} \cap A^{m_1}Q_1 \subseteq Q_1$ and $Q_2A^{m_1} \cap A^{m_2}Q_2 \subseteq Q_2$, m_1 and m_2 are any positive integers. Then Q_1Q_2 being the finite sum of the product is closed under addition. Using the result that every m-quasi ideal is m- bi ideal, we have,

$$\begin{aligned}(Q_1Q_2)^2 &= (Q_1Q_2)(Q_1Q_2) = (Q_1AQ_1)Q_2 = (Q_1Ae...eQ_1)Q_2 \\ &\subseteq (Q_1AA...AQ_1)Q_2 \subseteq (Q_1A^{m_1}Q_1)Q_2 \subseteq Q_1Q_2.\end{aligned}$$

That is, $(Q_1Q_2)^2 \subseteq Q_1Q_2$. So, Q_1Q_2 is closed under multiplication. Q_1Q_2 is a subsemiring of A. Moreover,

$$\begin{aligned}&Q_1Q_2(A^{\max(m_1,m_2)})Q_1Q_2 \subseteq Q_1AA^{\max(m_1,m_2)}Q_1Q_2 \\ &= Q_1A^{1+\max(m_1,m_2)}Q_1Q_2 \subseteq Q_1A^{m_1}Q_1Q_2 \subseteq Q_1Q_2.\end{aligned}$$

We have used $A^{1+\max(m_1,m_2)} \subseteq A^{m_1}$ as is evident by Definition 2.1. So,

$$Q_1Q_2(A^{\max(m_1,m_2)})Q_1Q_2 \subseteq Q_1Q_2.$$

Thus, Q_1Q_2 is an m-bi ideal of A with quasi-potency $\max(m_1, m_2)$. □

Remark 3.1. Every m-bi ideal may not be an m-quasi ideal. In Example 3.2, H is a 2-bi ideals as $HA^2H \subseteq H$, but H is not a 2-quasi ideal of A as $HA^2 \cap A^2H \nsubseteq H$.

Theorem 3.8. *Suppose Q be an m-quasi ideal of A and H be a subsemiring of A, then $H \cap Q$ is either empty or an m-quasi ideal of H.*

PROOF. If $H \cap Q$ is not empty, then since $H \cap Q \subseteq H$, therefore

$$H^m(H \cap Q) \subseteq H^m Q \text{ and } (H \cap Q)H^m \subseteq QH^m.$$

So,

$$H^m(H \cap Q) \cap (H \cap Q)H^m \subseteq H^m Q \cap QH^m \subseteq A^m Q \cap QA^m \subseteq Q.$$

Thus $H \cap Q$ is an m-quasi ideal of H. □

4. m-Regular Semirings

An element a of a semiring A is called regular if $axa = a$ for some $x \in A$. Semiring A is called regular if every element of A is regular. If a is a regular element of A, the ax and xa are idempotent; $ax \cdot ax = (axa)x = ax, \quad xa \cdot xa = x(axa) = xa$.

DEFINITION 4.1. An element a of a semiring A is called m-regular if $aya = a$ for some $y \in A^m$. Semiring A is called m-regular if every element of A is m-regular. A is m-regular if $a \in aA^m a$ for all $a \in A$.

REMARK 4.1. Every regular (1-regular) semiring is an m-regular semiring, but the converse is not true.

Otto Steinfeld characterized the rings and semigroups through the properties of their quasi ideals in [**8**]. We characterize the semirings through the properties of the m-quasi ideals in the theorems 4.2 and 4.3 given below with the courtesy to Otto Steinfeld.

THEOREM 4.2. *The following conditions for a semiring A are equivalent:*
1. *A is m-regular with identity e,*
2. *For every m- right ideal R and m-left ideal L, $RL = R \cap L$,*
3. *For every m-right ideal R and m-left ideal L,*
   (a) *$R^2 = R$,*
   (b) *$L^2 = L$,*
   (c) *RL is a Quasi-ideal of A,*
4. *The set of m-quasi ideals of A is a regular(multiplicative) semigroup,*
5. *Every m-quasi ideal Q has the form $QA^m Q = Q$.*

PROOF. (1) $\Rightarrow$ (2): Let R and L be the m-right and the m-left ideals of A respectively, then $RL \subseteq A.e.e...eL \subseteq A^m L \subseteq L$. That is, $RL \subseteq L$. Similarly, $RL \subseteq R$. Thus $RL \subseteq R \cap L$. For the reverse inclusion, let $x \in R \cap L$, then $x \in A$ and as A is m-regular, so for some $y \in A^m$, we have $x = xyx = (xy)x \in RL$, because R is m-right ideal . Thus $R \cap L = RL$.

(2) $\Rightarrow$ (3): Let $RL = R \cap L$, then by Corollary 3.2, RL is an m-quasi ideal of A. Now, if A is a semiring, then the m-right ideal generated by R is $R + A^m R$, so by (2), we have

$$R = R \cap (R + A^m R) = R(R + A^m R) = R^2 + RA^m R = R^2 + RR \subseteq R^2 + R^2 \subseteq R^2.$$

i.e., $R \subseteq R^2$, i.e., $R^2 = R$. Similarly, we can prove that $L^2 = L$.

(3) $\Rightarrow$ (4): Suppose that (3) holds and let K be the set of m-quasi ideals of A, then $Q + A^m Q$ is the m-left ideal of A generated by Q. So by (3), we have

$$Q \subseteq Q + A^mQ = (Q + A^mQ)^2 = (Q + A^mQ)(Q + A^mQ)$$
$$= Q^2 + QA^mQ + A^mQQ + A^mQA^mQ \subseteq Ae.e...Q + A^{m+1}Q + A^{m+1}Q + A^{2m+1}Q$$
$$\subseteq A^mQ + A^{m+1}Q + A^{m+1}Q + A^{2m+1}Q \subseteq A^mQ \text{ i.e., } Q \subseteq A^mQ.$$

In a similar way, we can prove that $Q \subseteq QA^m$. So, $Q \subseteq A^mQ \cap QA^m$. Since Q is m-quasi ideal, $A^mQ \cap QA^m \subseteq Q$ i.e.,

$$A^mQ \cap QA^m = Q \tag{4.1}$$

Now using 3(c) and Equation (4.1), we get

$$A^mRL \cap RLA^m = RL \tag{4.2}$$

for every m-right ideal R and m-left ideal L of A. Now, we shall prove that the product Q_1Q_2 of two m-quasi ideals Q_1 and Q_2 is an m-quasi ideal of A. By properties 3(a) and 3(b), we have

$$A^mQ_1Q_2 = (A^mQ_1Q_2)(A^mQ_1Q_2) = (A^mQ_1Q_2)(A^mA^mQ_1Q_2)$$

and so $Q_1Q_2A^m = (Q_1Q_2A^mA^m)(Q_1Q_2A^m)$. Thus, the Equation (4.2) gives

$(Q_1Q_2A^m) \cap (A^mQ_1Q_2) =$
$(Q_1Q_2A^m)(A^mQ_1Q_2)A^m \cap A^m(Q_1Q_2A^m)(A^mQ_1Q_2) =$
$(Q_1Q_2A^m)(A^mQ_1Q_2) \subseteq Q_1(Q_2A^mQ_2) \subseteq Q_1Q_2.$

i.e., $(Q_1Q_2)A^m \cap A^m(Q_1Q_2) \subseteq Q_1Q_2$. i.e., Q_1Q_2 is an m-quasi ideal of A. Since the multiplication defined in K is associative, so K is a semigroup.

Finally, we shall show that K is a regular semigroup. If Q is an arbitrary m-quasi ideal of A, then the properties 3(a), 3(b) and the relations (4.1) and (4.2) imply that

$$Q = QA^m \cap A^mQ = (QA^m.A^mQ)A^m \cap A^m(QA^m.A^mQ) = QA^m.A^mQ = QA^mQ \subseteq Q.$$

Hence $Q = QA^mQ$. This means that K is a regular semigroup.

(4) $\Rightarrow$ (5): Let Q be an m-quasi ideal of A, then by (4) above, we can find an m-quasi ideal X of A so that

$$Q = QX^mQ \subseteq QA^mQ \subseteq A^mQ \cap QA^m \subseteq Q,$$

i.e., $Q = QA^mQ$.

(5) $\Rightarrow$ (1): Let $a \in A$ and $< a >_l$ and $< a >_r$ be the principal m-left ideal and the principal m-right ideal of A generated by a, then by Proposition 3.2, $< a >_l \cap < a >_r$ is an m-quasi ideal of A. So by (5), we have $< a >_l \cap < a >_r = (< a >_l \cap < a >_r)A^m(< a >_r \cap < a >_r) \subseteq < a >_r A^m < a >_l$. Since $a \in < a >_r \cap < a >_l$, it follows that $a \in < a >_r A^m < a >_l$. But $< a >_r A^m = aA^m$ and $A^m < a >_l = A^ma$, therefore $a \in aA^m < a >_l = aA^ma$ i.e., $a \in aA^ma$ i.e., A is m-regular. $\square$

THEOREM 4.3. *Let A be a semiring, then the following assertions hold:*

(1) *Every m-quasi ideal Q of A can be written in the form $Q = R \cap L = RL$, where R is the m-right and L is the m-left ideal,*

(2) *For an m-quasi ideal Q of A, then $Q^2 = Q^3$,*
(3) *Every m-bi ideal of A is its m-quasi ideal,*
(4) *Every m-bi ideal of any two-sided ideal of A is an m-quasi ideal of A.*

PROOF. (1) Since Q is an m-quasi ideal of a semiring A, therefore

$$R =< Q >_r= Q + QA^m = QA^m \text{ and } L =< Q >_l= Q + A^mQ = A^Q.$$

Obviously

$$Q \subseteq R \cap L = QA^m \cap A^mQ \subseteq Q$$

i.e., $Q = R \cap L$. But A is a regular semiring, therefore $Q = R \cap L = RL$ by Theorem 4.2.

(2) $Q^3 \subseteq Q^2$ always holds, we have to show that $Q^2 \subseteq Q^3$. By Theorem 4.2, Q^2 is an m-quasi ideal of A. Furthermore,

$$Q^2 = Q^2A^mQ^2 = QQA^mQQ \subseteq QQQ = Q^3$$

i.e., $Q^2 \subseteq Q^3$.

(3) Let B be an m-bi ideal of A, then A^mB is m-left ideal and BA^m is an m-right ideal of A. By Theorem 4.2, we have,

$$BA^m \cap A^mB = BA^mA^mB = B(A^2)^mB \subseteq BA^mB \subseteq B$$

i.e., $BA^m \cap A^mB \subseteq B$ i.e., B is an m-quasi ideal of A.

(4) Finally, let C be two-sided ideal of A, and B be an m-bi ideal of C. Then obviously C is a regular subsemiring of A. By theorem (3), B is m-quasi ideal of C. Now $BA^mB \subseteq BA^mC$ and $BA^mB \subseteq CA^mB$, so

$$BA^mB \subseteq BA^mC \cap CA^mB \subseteq BC \cap CB \subseteq B$$

i.e., $BA^mB \subseteq B$ i.e., B is an m-bi ideal of A. Again by (3), B is an m-quasi ideal of A. □

5. Conclusion

We have reviewed the ideas of m-left and m-right ideals in semirings. Then, we have introduced the idea of the m-quasi ideals in the semirings theory; of which the already defined m-bi ideals are the generalized forms. We have studied the important properties of m-quasi ideals from algebraic point of view, and also in comparison with the m-left, m-right ideals and m-bi ideals. Along with the concept of m-quasi ideals, we have also introduced the new idea of m-regular semirings. With the help of these two new concepts, new dimensions of studies of semirings have been discovered. These new concepts will have more applications in discovering the hidden properties of semirings.

References

[1] M. A. Ansari, M. R. Khan, and J. Kaushik. A note on (m, n) quasi-ideals in semigroups. *Int. J. Math. Anal.*, **3**(38)(2009), 1853–1858.
[2] R. Chinram. A note on (m, n)-quasi-ideals in semirings. *Int. J. Pure Appl. Math*, **49**(1)(2008), 45–52.

[3] J. S. Golan. *The theory of semirings with applications in mathematics and theoretical computer science.* Addison-Wesley Longman Ltd. Essex, UK, 1992.

[4] J. M. Howie. *An introduction to semigroup theory.* Academic Press, New York-London, 1976.

[5] M. Munir and M. Habib. Characterizing semirings using their quasi and bi-ideals. *Proceedings of the Pakistan Academy of Sciences*, **53**(4)(2016), 469–475.

[6] M. Munir and A. Shafiq. A generalization of bi ideals in semirings. *Bull. Int. Math. Virtual Inst.*, **8**(1)(2018), 123–133.

[7] M. Shabir, A. Ali, and S. Batool. A note on quasi-ideals in semirings. *Southeast Asian Bull. Math.*, **27**(5)(2004), 923–928.

[8] O. Steinfeld. *Quasi-ideals in rings and semigroups.* Akadémiai Kiadó, Budapest, 1978.

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