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FILTERS IN ALMOST LATTICES

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Abstract. The concept of a final segment in an Almost Lattice (AL) is introduced and a set of identities for the set $F_n(L)$ of all final segments of an AL L to become a complete lattice is established. Moreover, the concept of a filter in an Almost Lattice(AL) is introduced and the smallest filter containing a given nonempty subset is described in an AL. Also, proved that the set $\mathcal{F}(L)$ of all filters of an AL L form a lattice with set inclusion and the set $P\mathcal{F}(L)$ of all principal filters of L is a sublattice of the lattice $\mathcal{F}(L)$. Further, a set of identities for the lattice $\mathcal{F}(L)$ to become a complete lattice is established. Finally, the concept of a prime filter in an AL L is introduced and a necessary and sufficient condition for a proper filter to become prime filter is derived.

1. Introduction

Filters in lattices were investigated by Tarski, Moisil and others many of whose results are found in Birkhoff's lattice theory [**1**]. In [**6**] U. M. Swamy and G. C. Rao introduced the concept of an Almost Distributive Lattice (ADL) as a common abstraction of all existing ring theoretic and lattice theoretic generalizations of a Boolean Algebra. They introduced the concepts of ideals and filters in an ADL and proved that the set of all filters in an ADL form a distributive lattice. Also, they established a set of identities that the lattice of all filters in an ADL to become a complete lattice. The concept of Almost Lattice (AL) was introduced by G. Nanaji Rao and Habtamu Tiruneh Alemu [**4**] as a common abstraction of almost all lattice theoretic generalizations of Boolean algebra like distributive lattices, almost distributive lattices and lattices. Further, they introduced the concept of ideals in an AL [**2**] and proved that the set of all ideals in L form a lattice. Also,

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they established a set of identities that the lattice of all ideals in an AL to become a complete lattice.

In this paper, we introduced the concept of a final segment in an AL L and a set of identities for the set $F_n(L)$ of all final segments of an AL L to become a complete lattice is established. Moreover, the concept of a filter in an AL is introduced and the smallest filter containing a given non empty subset of an AL L is described. It is proved that, the set $\mathcal{F}(L)$ of all filters of an AL L form a lattice with set inclusion and proved the set $P\mathcal{F}(L)$ of all principal filters of L is a sublattice of the lattice $\mathcal{F}(L)$. A set of identities that the lattice $\mathcal{F}(L)$ of all filters in an AL L to become a complete lattice is established. Finally, we introduced the concept of prime filters in an AL and a necessary and sufficient condition for a proper filter to become prime filter is derived.

2. Preliminaries

In this section, we collect a few important definitions and results which are already known and which will be used more frequently in this text. For the notions and notations used in this text, which are not predefined, a reader can refer to literature [**1, 2, 3, 4, 5, 6, 7**]

Definition 2.1. Let A and B be non empty sets. A relation R from A to B is a subset of $A \times B$. Relations from A to A are called relations on A.

Note that if R is a relation on a non empty set A, then for any $a, b \in A$ we write aRb instead of $(a, b) \in R$ and say that "a is in relation R to b". A relation R on the set A may have some of the following properties.

(1) R is reflexive if for all a in A, we have aRa.
(2) R is symmetric, if for all a, b in A, aRb implies bRa.
(3) R is antisymmetric, if for all a, b in A, aRb and bRa implies $a = b$.
(4) R is transitive if for all a, b, c in A, aRb and bRc imply aRc.

Definition 2.2. A relation R on a non empty set A is called a partial order relation if R is reflexive, antisymmetric and transitive. In this case, (A, R) is called a partially ordered set or poset.

Definition 2.3. Let $(P, \leqslant)$ be a poset and $S \subseteq P$. Then

(1) $a \in P$ is called a lower bound of S if and only if for all $x \in S; a \leqslant x$.
(2) $a \in P$ is called an upper bound of S if and only if for all $x \in S; x \leqslant a$.
(3) The greatest amongst the lower bounds, whenever if it exists is called the infimum of S and is denoted by inf S or $\bigwedge S$.
(4) The least amongst the upper bound of S whenever if it exists is called supremum of S and is denoted by sup S or $\bigvee S$.

Definition 2.4. Let $(P, \leqslant)$ be a poset. Then P is called a lattice ordered set if for every pair x, y of elements of P, the sup(x,y) and inf(x,y) exist.

Definition 2.5. An algebra $(L, \vee, \wedge)$ of type $(2, 2)$ is called a lattice if it satisfies the following axioms. For any $x, y, z \in L$,

(1) $x \vee y = y \vee x$ and $x \wedge y = y \wedge x$. (Commutative Law)
(2) $(x \vee y) \vee z = x \vee (y \vee z)$ and $(x \wedge y) \wedge z = x \wedge (y \wedge z)$. (Associative Law)
(3) $x \vee (x \wedge y) = x$ and $x \wedge (x \vee y) = x$. (Absorption Law)

Note that in any lattice $(L, \vee, \wedge)$, $x \vee x = x$ and $x \wedge x = x$ (Idempotent Law).

THEOREM 2.1. *Let $(L, \leqslant)$ be a lattice ordered set. If we define $x \wedge y$ is the inf (x, y) and $x \vee y$ is the sup(x, y), where $x, y \in L$, then $(L, \vee, \wedge)$ is a lattice.*

THEOREM 2.2. *Let $(L, \vee, \wedge)$ be a lattice. If we define a relation $\leqslant$ on L, by $x \leqslant y$ if and only if $x = x \wedge y$, or equivalently $x \vee y = y$. Then $(L, \leqslant)$ is a lattice ordered set.*

In view of the above two theorems, we can observe that, the concepts of lattice and lattice ordered set are equivalent. We refer to it as a lattice in future.

DEFINITION 2.6. Let $(P, \leqslant)$ be a poset. Then P is said to be complete lattice if every non empty subset of P has least upper bound and greatest lower bound in P.

THEOREM 2.3. *If $(P, \leqslant)$ is a poset bounded above each of whose nonempty subsets R has an infimum, then each nonempty subsets of P has a suprimum, too, and by the definition of $\bigwedge R = inf\ R$, $\bigvee R = sup\ R$, then P becomes a complete lattice.*

COROLLARY 2.1. *If a bounded lattice is complete with respect to one of the lattice operations, it is also complete with respect to the other.*

DEFINITION 2.7. An algebra $(L, \vee, \wedge)$ of type (2, 2) is called an Almost Lattice(AL) if it satisfies the following axioms. For any $a, b, c \in L$:

A_1. $(a \wedge b) \wedge c = (b \wedge a) \wedge c$
A_2. $(a \vee b) \wedge c = (b \vee a) \wedge c$
A_3. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
A_4. $(a \vee b) \vee c = a \vee (b \vee c)$
A_5. $a \wedge (a \vee b) = a$
A_6. $a \vee (a \wedge b) = a$
A_7 $(a \wedge b) \vee b = b$

LEMMA 2.1. *Let L be an AL. Then for any $a, b \in L$ we have the following:*

(1) $a \vee a = a$
(2) $a \wedge a = a$
(3) $a \wedge b = a$ *if and only if* $a \vee b = b$

DEFINITION 2.8. Let L be an AL and $a, b \in L$. Then we say that a is less than or equal to b and write as $a \leqslant b$ if and only if $a \wedge b = a$ or, equivalently $a \vee b = b$.

THEOREM 2.4. *Let L be an AL. For any $a, b, c \in L$, we have the following.*

(1) *The relation $\leqslant$ is a partial ordering on L and hence $(L, \leqslant)$ is a poset.*
(2) $a \leqslant b \Longrightarrow a \wedge b = b \wedge a$
(3) $a \wedge b = b \Longleftrightarrow a \vee b = a$

Definition 2.9. An AL L is said to be directed above if for any $a, b \in L$ there exists $c \in L$ such that $a \leqslant c$ and $b \leqslant c$.

Theorem 2.5. *Let L be an AL. Then the following are equivalent:*

(1) *L is directed above.*
(2) *$\wedge$ is commutative.*
(3) *$\vee$ is commutative.*
(4) *L is a lattice.*

Definition 2.10. Let L be an AL. Then a nonempty subset I of L is said to be an ideal of L if it satisfies the following:

(1) If $x, y \in I$, then there exists $d \in I$ such that $d \wedge x = x$ and $d \wedge y = y$.
(2) If $x \in I$ *and* $a \in L,$ *then* $x \wedge a \in I$.

Theorem 2.6. *Let L be an AL. Then the set $\mathcal{I}(L)$ of all ideals of L form a lattice under set inclusion in which the glb and lub for any $I, J \in \mathcal{I}(L)$ are respectively*

$$I \wedge J = I \cap J$$

and

$$I \vee J = \{x \in L|\ (a \vee b) \wedge x = x \ for\ some\ a \in I\ and\ b \in J\}.$$

Theorem 2.7. Let L be an AL. Then the following conditions are equivalent in L:

(1) The intersection of any family of ideals is nonempty.
(2) The intersections of any family of ideals is again an ideal.
(3) The lattice $\mathfrak{I}(L)$ has least element.
(4) The lattice $\mathfrak{I}(L)$ is complete.
(5) The class $P\mathfrak{I}(L)$ has least element.
(6) L has a minimal element.

Definition 2.11. A proper ideal P of an AL L is said to be prime if for any $x, y \in L$, $x \wedge y \in P$, then either $x \in P$ or $y \in P$.

3. Final segments

In this section, we introduce the concept of a final segment in an AL L and we give certain examples of final segments. Also, we establish a set of identities that the set $F_n(L)$ of all final segments of an AL L to become a complete lattice. First, we begin with the following definition:

Definition 3.1. Let L be an AL. A nonempty subset F of L is called a final segment of L if for any $a \in L,\ x \in F$ and $x \leqslant a$, implies $a \in F$.

Example 3.1. Let $L = \{a, b, c\}$. Define $\vee$ and $\wedge$ on L as follows:

$\vee$	a	b	c
a	a	b	c
b	b	b	b
c	c	c	c

and

$\wedge$	a	b	c
a	a	a	a
b	a	b	c
c	a	b	c

Then clearly, $(L, \vee, \wedge)$ is an AL. Put $F = \{b, c\}$. Then clearly, F is a final segment of L.

Example 3.2. Let $A = \{0, a\}$ and $B = \{0, b_1, b_2\}$ be two discrete ALs. Then $L = A \times B = \{(0,\ 0), (0,\ b_1), (0,\ b_2), (a,\ 0), (a,\ b_1), (a,\ b_2)\}$. Now, define $\vee$ and $\wedge$ on L by point wise operations as follows:

$\vee$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
$(0,0)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
$(0,b_1)$	$(0,b_1)$	$(0,b_1)$	$(0,b_1)$	(a,b_1)	(a,b_1)	(a,b_1)
$(0,b_2)$	$(0,b_2)$	$(0,b_2)$	$(0,b_2)$	(a,b_2)	(a,b_2)	(a,b_2)
$(a,0)$	$(a,0)$	(a,b_1)	(a,b_2)	$(a,0)$	(a,b_1)	(a,b_2)
(a,b_1)	(a,b_1)	(a,b_1)	(a,b_1)	(a,b_1)	(a,b_1)	(a,b_1)
(a,b_2)	(a,b_2)	(a,b_2)	(a,b_2)	(a,b_2)	(a,b_2)	(a,b_2)

and

$\wedge$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(0,0)$
$(0,b_1)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$
$(0,b_2)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(0,0)$	$(0,b_1)$	$(0,b_2)$
$(a,0)$	$(0,0)$	$(0,0)$	$(0,0)$	$(a,0)$	$(a,0)$	$(a,0)$
(a,b_1)	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)
(a,b_2)	$(0,0)$	$(0,b_1)$	$(0,b_2)$	$(a,0)$	(a,b_1)	(a,b_2)

Then clearly, $(L, \vee, \wedge, (0,0))$ is an AL with $(0,0)$ as its zero element. Put $F = \{(0,b_1), (0,b_2), (a,0), (a,b_1), (a,b_2)\}$. Then clearly, F is a final segment of L. But, F is not a sub AL, since $(0,b_1) \wedge (a,0) = (0,0) \notin F$. Also, in Example 3.1, clearly $F = \{a, b\}$ is a sub AL, but not a final segment, since $a \leqslant c$, but $c \notin F$.

In the following, we give a necessary and sufficient condition for a nonempty subset of an AL L to become a final segment of L.

Theorem 3.1. *Let L be an AL. Then a nonempty subset F of L is a final segment of L if and only if for all $x \in F$ and $a \in L$, $x \vee a \in F$.*

Proof. Suppose F is a final segment in L. Let $x \in F$ and $a \in L$. Then we have $x \leqslant x \vee a$ and hence $x \vee a \in F$. Conversely, assume the condition. Let F is a nonempty subset of L with $x \in F$ and $a \in L$ such that $x \leqslant a$. Then $a = x \vee a \in F$. Thus F is a final segment. □

Corollary 3.1. *Let L be an AL. If F is a nonempty subset of L such that $a \vee x \in F$ for all $x \in F$ and $a \in L$, then F is a final segment in L.*

Proof. Suppose F is a nonempty subset of an AL L. Let $x \in I$ and $a \in L$ such that $x \leqslant a$. Then $a = x \vee a = a \vee x \in F$. Therefore F is a final segment. □

But, the converse of the above corollary is not true in general. For, consider a discrete AL L with at least three elements. Let $x \neq y \in L - \{0\}$. Then clearly $\{x\}$ is a final segment in L, but, $y \vee x = y \notin \{x\}$.

Recall that an element m of an AL L is called maximal if for any $x \in L,\ m \leqslant x$, then $m = x$. Hence an element m of an AL L is maximal if and only if $m \wedge x = x$ forall x in L if and only if $m \vee x = m$ forall x in L.

It can be easily observed that the union of any class of final segments is a final segment, but, the intersection of any class of final segments need not be again a final segment. In the following, we give set of identities that intersection of any class of final segments is again a final segment in an AL L.

THEOREM 3.2. *Let L be an AL and $F_n(L)$ be the set of all final segments of L. Then the following are equivalent:*

(1) *The intersection of any class of final segments in $F_n(L)$ is a final segment.*
(2) *L has greatest element.*
(3) *L has unique maximal element.*
(4) *L is bounded above.*
(5) *$F_n(L)$ is a complete lattice with respect to set inclusion.*

PROOF. (1) $\Longrightarrow$ (2):- Assume (1). Let $a \in L$. Then clearly, $[a) = \{x \in L/a \leqslant x\}$ is a final segment. Therefore by (1), $\bigcap\limits_{a\in L} [a)$ is a final segment. Now, choose $m \in \bigcap\limits_{a\in L} [a)$. Then clearly, $a \leqslant m$ for all $a \in L$. Hence m is the greatest element of L.

(2) $\Longrightarrow$ (3) Suppose L has greatest element say m. Then clearly m is a maximal element. We shall prove that L has unique maximal element. Suppose m_1, m_2 are two maximal elements in L. Since m is greatest element in L, $m_1 \leqslant m$ and $m_2 \leqslant m$. It follows, $m_1 = m_2 \wedge m_1 = m_2 \wedge (m_1 \wedge m) = (m_2 \wedge m_1) \wedge m = (m_1 \wedge m_2) \wedge m = m_2 \wedge m = m_2$. Therefore L has unique maximal element.

(3) $\Longrightarrow$ (4):- Suppose L has unique maximal element say m. Then clearly $a \vee m$ is a maximal for all a in L. Hence by (3), we get $a \vee m = m$. Therefore $a \leqslant m$ for all $a \in L$. Thus L is bounded above by m.

(4) $\Longrightarrow$ (1): Suppose L is bounded above, say by m. Let $\{F_\alpha\}|\alpha \in \Delta$ be a class of final segments. Then clearly $m \in \bigcap\limits_{\alpha\in\Delta} F_\alpha$ and hence $\bigcap\limits_{\alpha\in\Delta} F_\alpha \neq \emptyset$. Now, let $x \in \bigcap\limits_{\alpha\in\Delta} F_\alpha$ and $x \leqslant a$. Then $x \in F_\alpha$ for all $\alpha \in \Delta$ and $x \leqslant a$. It follows that, $a \in F_\alpha$ for all $\alpha \in \Delta$ and hence $a \in \bigcap\limits_{\alpha\in\Delta} F_\alpha$. Therefore $\bigcap\limits_{\alpha\in\Delta} F_\alpha$ is a final segment of L.

Proof of (1) $\Leftrightarrow$ (5) is clear. □

COROLLARY 3.2. *Let L be an AL with 0 and $F_n(L)$ be the set of all final segments in L. Then the following are equivalent to each other.*

(1) *The intersection of any class of final segments is a final segment.*
(2) *L has largest element.*
(3) *L has unique maximal element.*
(4) *L is bounded.*
(5) *$F_n(L)$ is a complete lattice with respect to set inclusion.*

4. Filters in ALs

In this section, we introduce the concept of a filter in an AL L and describe the smallest filter containing a given nonempty subset of L. We prove that every filter F of an AL L is a final segment of L, but not converse and the set $\mathcal{F}(L)$ of all filters of an AL L form a lattice with respect to set inclusion and we prove the set $P\mathcal{F}(L)$ of all principal filters of L is a sublattice of the lattice $\mathcal{F}(L)$. Also, we establish a set of identities that the set $\mathcal{F}(L)$ all filters in an AL L to become a complete lattice. Finally, we introduce the concept of prime filters in an AL and we prove a necessary and sufficient condition for a proper filter to become a prime filter. First, we begin with the definition of a filter.

Definition 4.1. Let L be an AL. Then a nonempty subset F of L is said to be a filter if it satisfies the following:

(1) $x, y \in F$, implies $x \wedge y \in F$.
(2) $x \in F$ *and* $a \in L$, implies $a \vee x \in F$.

Example 4.1. Let $L = \{a, b, c\}$ be an AL given in Example 3.1. Then clearly, $F = \{b, c\}$ is a filter of L.

Example 4.2. Consider, an AL L in Example 3.2. Then clearly, $F = \{(a, b_1), (a, b_2)\}$ is a filter of L.

It can be easily seen that every filter of an AL L is a sub AL of L. But, converse need not be true.

For, consider the following example.

Example 4.3. Let L be an AL given in Example 3.1. Put $F = \{a, b\}$. Then clearly, F is sub AL of L. But, since $c \vee a = c \notin F$, F is not a filter.

Let us recall that, an element m of L is said to be maximal (minimal) if and only if $m \wedge x = x (x \wedge m = m)$ if and only if $m \vee x = m (x \vee m = x)$ for all $x \in L$. Therefore it can be easily seen that every filter of an AL L contains all maximal elements in L. Now, we prove the following.

Theorem 4.1. *Let L be an AL with a minimal element say m. If F is a filter in L such that $m \in F$, then $F = L$.*

Proof. Suppose F is a filter of L such that $m \in F$. We shall prove that $F = L$. Now, let $x \in L$. Then $x = x \vee m$. Therefore $x \in F$. Hence $L \subseteq F$. But, we have $F \subseteq L$. Thus $F = L$. □

Theorem 4.2. *Let L be an AL L with a maximal element. Then the set of all maximal elements of L form a filter.*

Proof. Suppose L has a maximal element say a. Now, put $F = \{m |\ m$ is a maximal element in $L\}$. Then clearly F is non empty, since $a \in F$. Let $x, y \in F$. Then for any $t \in L$, we have $x \wedge t = t$ *and* $y \wedge t = t$. Now, $(x \wedge y) \wedge t = x \wedge (y \wedge t) = x \wedge t = t$. Therefore $x \wedge y$ is a maximal element. Hence $x \wedge y \in F$. Also, let $x \in F$ *and* $t \in L$. Now, let $s \in L$. Then $(t \vee x) \vee s = t \vee (x \vee s) = t \vee x$. Thus $t \vee x$ is a maximal element of L. Hence $t \vee x \in F$. Therefore F is a filter of L. □

In the following, we describe a filter generated by a given nonempty subset of an AL L.

THEOREM 4.3. *Let L be an AL and S be a nonempty subset of L. Then*

$$[S) = \{x \vee (\bigwedge_{i=1}^{n} s_i) |\ x \in L, s_i \in S \ for\ 1 \leqslant i \leqslant n \ and\ n \in Z^+\}$$

is the smallest filter of L containing S.

PROOF. Let $S(\neq \emptyset) \subseteq L$. Put

$$T = \{\bigwedge_{i=1}^{n} s_i | s_i \in S \text{ for } 1 \leqslant i \leqslant n \text{ and } n \in Z^+\}$$

and let $H = \{x \in L | x \wedge t = t \ for\ some\ t \in T\}$. First, we prove that $[S) = H$. Let $a \in [S)$. Then $a = x \vee (\bigwedge_{i=1}^{n} s_i)$ where $x \in L, s_i \in S\ for\ 1 \leqslant i \leqslant n\ and\ n \in Z^+$. But, since $\bigwedge_{i=1}^{n} s_i \in T$ and $(x \vee (\bigwedge_{i=1}^{n} s_i)) \wedge (\bigwedge_{i=1}^{n} s_i) = \bigwedge_{i=1}^{n} s_i$, it follows that, $a \in H$. Therefore $[S) \subseteq H$. Conversely, suppose $a \in H$. Then $a \wedge t = t$ for some $t \in T$. Therefore we can write $t = \bigwedge_{i=1}^{n} s_i$ where $s_i \in S$ for $1 \leqslant i \leqslant n$ and $n \in Z^+$. Hence $a = a \vee t = a \vee (\bigwedge_{i=1}^{n} s_i)$. Thus $a \in [S)$. Therefore $H \subseteq [S)$. Thus $[S) = H$. Clearly, $S \subseteq T \subseteq [S)$. Now, let $x, y \in [S)$. Then there exists $t_1, t_2 \in T$, such that $x \wedge t_1 = t_1\ and\ y \wedge t_2 = t_2$. Then $t_1 \wedge t_2 = (x \wedge t_1) \wedge (y \wedge t_2) = ((x \wedge t_1) \wedge y) \wedge t_2 = (x \wedge (t_1 \wedge y)) \wedge t_2 = x \wedge ((t_1 \wedge y) \wedge t_2) = x \wedge ((y \wedge t_1) \wedge t_2) = x \wedge (y \wedge (t_1 \wedge t_2)) = (x \wedge y) \wedge (t_1 \wedge t_2)$. Thus $(x \wedge y) \wedge (t_1 \wedge t_2) = t_1 \wedge t_2$, since $t_1,\ t_2 \in T$. Therefore $x \wedge y \in [S)$. Again, let $x \in [S)\ and\ a \in L$. Then there exists $t \in T$ such that $x \wedge t = t$. Now, $(a \vee x) \vee t = a \vee (x \vee t) = a \vee x$. It follows that, $(a \vee x) \wedge t = t$. Therefore $a \vee x \in [S)$. Thus $[S)$ is a filter of L containing S. Suppose F is a filter of L such that $S \subseteq F$. We shall prove that $[S) \subseteq F$. Let $a \in [S)$. Then $a = x \vee (\bigwedge_{i=1}^{n} s_i)$ where $x \in L$, $s_i \in S$ for all i, $1 \leqslant i \leqslant n$ and $n \in Z^+$. It follows that, $s_i \in F$ for all i, $1 \leqslant i \leqslant n$ and hence $\bigwedge_{i=1}^{n} s_i \in F$. Thus $a = x \vee (\bigwedge_{i=1}^{n} s_i) \in F$. Hence $[S) \subseteq F$. Therefore $[S)$ is a smallest filter containing S. □

Note that, if $S = \{a\}$, then we write $[a)$ instead of $[\{a\})$.

COROLLARY 4.1. *Let L be an AL and $a \in L$. Then $[a) = \{x \vee a | x \in L\}$ is the smallest filter of L containing a and is called a principal filter generated by a.*

COROLLARY 4.2. *Let L be an AL. Then for any $a, b \in L$, $a \in [b)$ if and only if $a = a \vee b$.*

PROOF. Suppose $a, b \in L$ and suppose $a \in [b)$. Then $a = x \vee b$ for some $x \in L$. Now, $a \vee b = (x \vee b) \vee b = x \vee (b \vee b) = x \vee b = a$. Therefore $a = a \vee b$. Converse follows by the definition of $[b)$. □

COROLLARY 4.3. *Let L be an AL and $a, b \in L$. Then $a \in [b)$ if and only if $[a) \subseteq [b)$.*

PROOF. Suppose $a \in [b)$. Then $a = a \vee b$. Now, let $t \in [a)$. Then $t = t \vee a$. Now, $t \vee b = (t \vee a) \vee b = t \vee (a \vee b) = t \vee a = t$. Therefore $t \in [b)$. Thus $[a) \subseteq [b)$. Converse is trivial. □

Corollary 4.4. *Let L be an AL and F be a filter of L. Then for any* $x, y \in L$, $x \vee y \in F$ *if and only if* $y \vee x \in F$.

Proof. Suppose $x \vee y \in F$. Then $(y \vee x) \vee (x \vee y) \in F$. This implies $y \vee (x \vee (x \vee y)) \in F$. It follows that, $y \vee (x \vee y) \in F$. Hence $(y \vee x) \vee y \in F$. Therefore $y \vee x \in F$. Similarly, we can prove the converse. □

Corollary 4.5. *Let L be an AL. Then for any* $x, y \in L, [x \vee y) = [y \vee x)$.

Proof. We have $x \vee y \in [x \vee y)$ and hence $y \vee x \in [x \vee y)$. It follows that, $[y \vee x) \subseteq [x \vee y)$. Similarly, we can prove that $[x \vee y) \subseteq [y \vee x)$. Therefore $[x \vee y) = [y \vee x)$. □

Corollary 4.6. *Let L be an AL and F be a filter of L. Then for any* $x \in F$ *and* $a \in L, x \vee a \in F$ *and hence F is a final segment of L.*

But, converse of Corollary 4.6 is not true. For, in Example 3.2, $F = \{(0, b_1), (0, b_2), (a, 0), (a, b_1), (a, b_2)\}$ is a final segment but, not a filter, since $(a, 0) \wedge (0, b_1) = (0, 0) \notin F$. Next, we prove that the set of all filters of an AL L is a lattice.

Theorem 4.4. *Let L be an AL. Then the set* $\mathcal{F}(L)$ *of all filters of L is a lattice under set inclusion, in which the glb and lub of any F and G in* $\mathcal{F}(L)$ *are given respectively by:*

$$F \wedge G = F \cap G$$

and

$$F \vee G = \{x \in L | x \vee (a \wedge b) = x \ for\ some\ a \in F\ and\ b \in G\}$$

Proof. Clearly, $\mathcal{F}(L)$ is non empty and is a poset with respect to set inclusion. Let $F, G \in \mathcal{F}(L)$. Then clearly, $F \cap G$ is the glb of F and G in $\mathcal{F}(L)$. Now, we shall prove that $F \vee G = \{x \in L | x \vee (a \wedge b) = x \ for\ some\ a \in F\ and\ b \in G\}$ is the lub of F and G in $\mathcal{F}(L)$. We have F and G are nonempty. Since for any $x \in F$ and $y \in G$, $x = x \vee (x \wedge y)$ and $y = y \vee (x \wedge y)$, and hence $x, y \in F \vee G$, $F \vee G$ is nonempty. Let $x, y \in F \vee G$. Then $x \vee (a_1 \wedge b_1) = x$ and $y \vee (a_2 \wedge a_2) = y$ for some $a_1, a_2 \in F$ and $b_1, b_2 \in G$. Then $x \wedge (a_1 \wedge b_1) = a_1 \wedge b_1$ and $y \wedge (a_2 \wedge b_2) = a_2 \wedge b_2$. It follows that, $(x \wedge (a_1 \wedge b_1)) \wedge (y \wedge (a_2 \wedge b_2)) = a_1 \wedge b_1 \wedge a_2 \wedge b_2$. Hence $(x \wedge y) \wedge ((a_1 \wedge a_2) \wedge (b_1 \wedge b_2)) = (a_1 \wedge a_2) \wedge (b_1 \wedge b_2)$. Therefore $(x \wedge y) \vee ((a_1 \wedge a_2) \wedge (b_1 \wedge b_2)) = x \wedge y$ where $a_1 \wedge a_2 \in F$ and $b_1 \wedge b_2 \in G$. Therefore $x \wedge y \in F \vee G$. Also, let $x \in F \vee G$ and $t \in L$. Then $x \vee (a \wedge b) = x$ for some $a \in F$ and $b \in G$. Now, $(t \vee x) \vee (a \wedge b) = t \vee (x \vee (a \wedge b)) = t \vee x$. Hence $t \vee x \in F \vee G$. Therefore $F \vee G$ is a filter. Clearly, $F \vee G$ is an upper bound of F and G. Suppose K be a filter of L such that $F, G \subseteq K$. We shall prove that $F \vee G \subseteq K$. Now, let $x \in F \vee G$. Then $x \vee (a \wedge b) = x$ for some $a \in F$ and $b \in G$. Therefore $a, b \in K$ and hence $a \wedge b \in K$. It follows that, $x = x \vee (a \wedge b) \in K$. Therefore $F \vee G \subseteq K$. Thus $F \vee G$ is the lub of F and G. Therefore $\mathcal{F}(L)$ of all filters of L is a lattice. □

In the following, we prove the set $P\mathcal{F}(L)$ of all principal filters of an AL L is a sublattice of the lattice $\mathcal{F}(L)$ of all filters of L. For this, first we need the following.

Lemma 4.1. *Let L be an AL and $a, b \in L$. If $a \leqslant b$, then $[b) \subseteq [a)$.*

Proof. Suppose $a \leqslant b$. Then $b = a \vee b = b \vee a$. Now, let $t \in [b)$. Then $t = t \vee b$. Now, $t \vee b = t \vee (b \vee a) = (t \vee b) \vee a = t \vee a$. Hence $t \in [a)$. Therefore $[b) \subseteq [a)$. □

Lemma 4.2. *Let L be an AL and $a, b \in L$. Then we have the following:*

(1) $[a) \vee [b) = [a \wedge b) = [b \wedge a)$
(2) $[a) \cap [b) = [a \vee b) = [b \vee a)$

Proof. (1) Let $t \in [a \wedge b)$. Then $t = t \vee (a \wedge b)$. Since $a \in [a)$ and $b \in [b)$, $t \in [a) \vee [b)$. Therefore $[a \wedge b) \subseteq [a) \vee [b)$. Conversely, let $x \in [a) \vee [b)$. Then $x = x \vee (s \wedge t)$ for some $s \in [a)$ and $t \in [b)$. It follows that, $s = s \vee a$ and $t = t \vee b$ and hence $a = s \wedge a$ and $b = t \wedge b$. Now, $a \wedge b = (s \wedge a) \wedge (t \wedge b) = (s \wedge t) \wedge (a \wedge b)$. Consider, $x \vee (a \wedge b) = x \vee ((s \wedge t) \wedge (a \wedge b)) = (x \vee (s \wedge t)) \vee ((s \wedge t) \wedge (a \wedge b)) = x \vee ((s \wedge t) \vee ((s \wedge t) \wedge (a \wedge b)) = x \vee (s \wedge t) = x$. Therefore $x \in [a \wedge b)$ and hence $[a) \vee [b) \subseteq [a \wedge b)$. Thus $[a) \vee [b) = [a \wedge b)$. Now, let $t \in [b \wedge a)$. Then $t = t \vee (b \wedge a)$. Since $b \in [b)$, $a \in [a)$, it follows that, $t \in [b) \vee [a) = [a) \vee [b)$. Therefore $[b \wedge a) \subseteq [a) \vee [b)$. Conversely, let $t \in [a) \vee [b) = [b) \vee [a)$. Then $t = t \vee (x \wedge y)$ where $x \in [b)$ and $y \in [a)$ and hence $x = x \vee b$ and $y = y \vee a$. It follows that, $b = x \wedge b$ and $a = y \wedge a$. Now, $b \wedge a = (x \wedge b) \wedge (y \wedge a) = (x \wedge y) \wedge (b \wedge a)$. Consider, $t \vee (b \wedge a) = (t \vee (x \wedge y)) \vee ((x \wedge y) \wedge (b \wedge a)) = t \vee ((x \wedge y) \vee ((x \wedge y) \wedge (b \wedge a))) = t \vee (x \wedge y) = t$. Therefore $t \in [b \wedge a)$ and hence $[a) \vee [b) \subseteq [b \wedge a)$. Therefore $[a) \vee [b) = [b \wedge a)$.

(2) By Corollary 4.5, we have $[a \vee b) = [b \vee a)$. Now, let $t \in [a) \cap [b)$. Then $t \in [a)$ *and* $t \in [b)$ and hence $t = t \vee a$ *and* $t = t \vee b$. Hence $t = t \vee t = (t \vee a) \vee (t \vee b) = ((t \vee a) \vee t) \vee b = (t \vee a) \vee b = t \vee (a \vee b) \in [a \vee b)$. Hence $[a) \cap [b) \subseteq [a \vee b)$. Conversely, we have $a \leqslant a \vee b$ and $b \leqslant b \vee a$. Therefore $[a \vee b) \subseteq [a)$ and $[b \vee a) \subseteq [b)$. It follows that, $[a \vee b) \subseteq [a) \cap [b)$. Therefore $[a) \cap [b) = [a \vee b) = [b \vee a)$. □

Lemma 4.3. *Let L be an AL. Then for any $a, b \in L$, the following are equivalent:*

(1) $(a] \subseteq (b]$
(2) $b \wedge a = a$
(3) $b \vee a = b$
(4) $[b) \subseteq [a)$

Proof. (1) $\Longrightarrow$ (2):- Suppose $(a] \subseteq (b]$. Then $a \in (a] \subseteq (b]$. Hence $a \in (b]$. Therefore $a = b \wedge a$.

Proof of (2) $\Longrightarrow$ (3) is clear.

(3) $\Longrightarrow$ (4):- Suppose $b \vee a = b$. Let $t \in [b)$. Then $t = t \vee b$. Now, $t = t \vee b = t \vee (b \vee a) = (t \vee b) \vee a = t \vee a \in [a)$. Therefore $[b) \subseteq [a)$.

(4) $\Longrightarrow$ (1):- Suppose $[b) \subseteq [a)$. Let $t \in (a]$. Then $t = a \wedge t$. Now, $t = a \wedge t = (b \wedge a) \wedge t = b \wedge (a \wedge t) = b \wedge t \in (b]$. Therefore $(a] \subseteq (b]$. □

Corollary 4.7. *Let L be an AL. Then for any $a, b \in L$, $(a] = (b]$ if and only if $[a) = [b)$.*

Now, we have the following theorem whose proof follows by Lemmas 4.2, 4.3 and Corollary 4.7.

THEOREM 4.5. *Let L be an AL. The class $P(\mathcal{F}(L))$ of all principal filters of L is a sublattice of the lattice $\mathcal{F}(L)$ of all filters of L. Moreover, the lattice $P(\mathcal{F}(L))$ is dually isomorphic onto the lattice $P(\mathcal{F}(L))$.*

We have seen that the intersection of a finite family of filters is again a filter. But, the intersection of an arbitrary family of filters need not be a filter in an AL in general. In the following, we establish a set of identities that intersection of any family of filters is again a filter.

THEOREM 4.6. Let L be an AL. Then the following conditions are equivalent in L:

(1) The intersection of any family of ideals is nonempty.
(2) The intersection of any family of ideals is again an ideal.
(3) The lattice $\mathcal{I}(L)$ has least element.
(4) The lattice $\mathcal{I}(L)$ is complete.
(5) The lattice $P\mathcal{I}(L)$ has least element.
(6) The lattice $P\mathcal{F}(L)$ has greatest element.
(7) L has a minimal element.

PROOF. In view of Theorem 2.7, it is enough to prove that (5) $\Rightarrow$ (6) and (6) $\Rightarrow$ (7).

(5) $\Rightarrow$ (6): Suppose $P\mathcal{I}(L)$ has least element say $(a]$. Then $(a] \subseteq (x]$ for all $x \in L$. Therefore $[x) \subseteq [a)$ for all $x \in L$. Hence $[a)$ is the greatest element of $P\mathcal{F}(L)$.

(6) $\Rightarrow$ (7): Suppose $P\mathcal{F}(L)$ has greatest element say $[a)$. Then $[x) \subseteq [a)$ for all $x \in L$. Therefore $x \in [a)$ for all $x \in L$. Hence $x = x \vee a$ for all $x \in L$. Thus a is a minimal element of L. □

THEOREM 4.7. *Let L be an AL. Then the following conditions are equivalent in L:*

(1) *The intersection of any family of filters is nonempty.*
(2) *The intersection of any family of filters is again a filter.*
(3) *The lattice $\mathcal{F}(L)$ has least element.*
(4) *The lattice $\mathcal{F}(L)$ is complete.*
(5) *The lattice $P\mathcal{F}(L)$ has least element.*
(6) *The lattice $P\mathcal{I}(L)$ has greatest element.*
(7) *L has a maximal element.*

PROOF. (1) $\Rightarrow$ (2): Assume (1). Suppose $\{F_\alpha\}_{\alpha\in\Delta}$ is a family of filters in L. Then by (1), we get $F = \bigcap_{\alpha\in\Delta} F_\alpha$ is nonempty. Let $x, y \in F$. Then $x, y \in F_\alpha$ for all $\alpha \in \Delta$. Since each F_α is filter, $x \wedge y \in F_\alpha$ for all $\alpha \in \Delta$. It follows that, $x \wedge y \in \bigcap_{\alpha\in\Delta} F_\alpha = F$. Now, let $x \in F$ and $t \in L$. Then $x \in F_\alpha$ for all $\alpha \in \Delta$. Therefore $t \vee x \in F_\alpha$ for all $\alpha \in \Delta$. Hence $t \vee x \in F$. Thus F is an filter.

Proof of (2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (4): Assume (3). Then clearly $\mathcal{F}(L)$ is bounded below. Now, we shall prove that any nonempty subset of $\mathcal{F}(L)$ has suprimum. Let $\{F_\alpha\}_\alpha \in \Delta$ be a

nonempty subset of $\mathcal{F}(L)$. Now, define $\bigvee_{\alpha\in\Delta} F_\alpha = \{x \in L \mid (\exists\ \alpha_1, \alpha_2, ..., \alpha_n \in \Delta,\ n \in Z^+),\ (\exists\ a_i \in F_{\alpha_i})\ \textit{such that}\ x \vee (a_1 \wedge a_2 \wedge .... \wedge a_n) = x\}$. Now, we prove that $\bigvee_{\alpha\in\Delta} F_\alpha$ is the lub of $\{F_\alpha\}_{\alpha\in\Delta}$ in $\mathcal{F}(L)$. Clearly, each $\{F_\alpha\}_{\alpha\in\Delta}$ is contained in $\bigvee_{\alpha\in\Delta} F_\alpha$ and hence $\bigvee_{\alpha\in\Delta} F_\alpha$ is nonempty. Let $x, y \in \bigvee_{\alpha\in\Delta} F_\alpha$. Then there exists $\alpha_1, \alpha_2, ..., \alpha_n \in \Delta,\ n \in Z^+,\ \textit{and}\ a_i \in F_{\alpha_i}\ \textit{such that}\ x \vee (a_1 \wedge a_2 \wedge .... \wedge a_n) = x$, also there exists $\beta_1, \beta_2, ..., \beta_m \in \Delta,\ m \in Z^+,\ \textit{and}\ b_j \in F_{\beta_j}\ \textit{such that}\ y \vee (b_1 \wedge b_2 \wedge .... \wedge b_m) = y$. Now, consider $(x \wedge y) \wedge ((a_1 \wedge a_2 \wedge ... \wedge a_n) \wedge (b_1 \wedge b_2 \wedge ... \wedge b_m)) = (x \wedge (a_1 \wedge ... \wedge a_n)) \wedge (y \wedge (b_1 \wedge ... \wedge b_m)) = (a_1 \wedge a_2 \wedge ... a_n) \wedge (b_1 \wedge b_2 \wedge ... \wedge b_m)$. It follows that, $(x \wedge y) \vee ((a_1 \wedge a_2 \wedge ... \wedge a_n) \wedge (b_1 \wedge b_2 \wedge ... \wedge b_m)) = x \wedge y$. Therefore there exists $\gamma_1, \gamma_2, ..., \gamma_{n+m} \in \Delta,\ n+m \in Z^+,\ \textit{and}\ c_k \in F_{\gamma_k}\ \textit{such that}\ (x \wedge y) \vee (c_1 \wedge c_2 \wedge .... \wedge c_{n+m}) = x \wedge y$. Hence $x \wedge y \in \bigvee_{\alpha\in\Delta} F_\alpha$. Again, let $x \in \bigvee_{\alpha\in\Delta} F_\alpha$ and $t \in L$. Then there exists $\alpha_1, \alpha_2, ..., \alpha_n \in \Delta,\ n \in Z^+,\ \textit{and}\ a_i \in F_{\alpha_i}\ \textit{such that}\ x \vee (a_1 \wedge a_2 \wedge .... \wedge a_n) = x$. Now, consider $(t \vee x) \vee (a_1 \wedge a_2 \wedge ... \wedge a_n) = t \vee (x \vee (a_1 \wedge a_2 \wedge .... \wedge a_n)) = t \vee x$. Therefore $t \vee x \in \bigvee_{\alpha\in\Delta} F_\alpha$. Thus $\bigvee_{\alpha\in\Delta} F_\alpha$ is a filter of L. Clearly, $\bigvee_{\alpha\in\Delta} F_\alpha$ is an upper bound of $\{F_\alpha\}_{\alpha\in\Delta}$. Suppose $H \in \mathcal{F}(L)$ such that H is an upper bound of $\{F_\alpha\}_{\alpha\in\Delta}$. Then $F_\alpha \subseteq H$ for all $\alpha \in \Delta$. We shall prove that $\bigvee_{\alpha\in\Delta} F_\alpha \subseteq H$. Let $x \in \bigvee_{\alpha\in\Delta} F_\alpha$. Then there exists $\alpha_1, \alpha_2, ..., \alpha_n \in \Delta,\ n \in Z^+,\ \textit{and}\ a_i \in F_{\alpha_i}\ \textit{such that}\ x \vee (a_1 \wedge a_2 \wedge .... \wedge a_n) = x$. Since $a_i \in F_{\alpha_i} \subseteq H$ for all α_i, $a_i \in H$ for all i. It follows that, $a_1 \wedge a_2 \wedge ... \wedge a_n \in H$ and hence $x = x \vee (a_1 \wedge a_2 \wedge ... \wedge a_n) \in H$. Thus $\bigvee_{\alpha\in\Delta} F_\alpha \subseteq H$. Therefore $\bigvee_{\alpha\in\Delta} F_\alpha$ is the lub of $\{F_\alpha\}_{\alpha\in\Delta}$. Therefore by Theorem 2.3, $\mathcal{F}(L)$ is a complete lattice.

(4) $\Rightarrow$ (5): Suppose $\mathcal{F}(L)$ is complete. Since $P\mathcal{F}(L) \subseteq \mathcal{F}(L)$, $P\mathcal{F}(L)$ has a greatest lower bound say F. Now, it remains to prove that F is a principal filter. Let $a \in F$. Then $[a) \subseteq F$. On the other hand $F \subseteq [a)$ since F is the glb of $P\mathcal{F}(L)$. Therefore $[a) = F$. Thus F is the least element in $P\mathcal{F}(L)$.

(5) $\Rightarrow$ (6): Suppose $P\mathcal{F}(L)$ has least element say $[a)$. Then $[a) \subseteq [x)$ for all $x \in L$. It follows that, $(x] \subseteq (a]$ for all $x \in L$. Hence $(a]$ is the greatest element of $P\mathcal{I}(L)$.

(6) $\Longrightarrow$ (7): Suppose $P\mathcal{I}(L)$ has greatest element say $(a]$. Then $(x] \subseteq (a]$ for all $x \in L$. It follows that, $x \in (a]$ for all $x \in L$ and hence $a \wedge x = x$ for all $x \in L$. Therefore a is maximal.

(7) $\Rightarrow$ (1): Suppose L has a maximal element. Since every filter in L contains a maximal element, it follows that the intersections of any family of filters is nonempty. □

Next, we introduce the concept of prime filters in an AL L and we derive necessary and sufficient conditions for a proper filter to become a prime filter.

Definition 4.2. Let L be an AL. Then a proper filter P of L is said to be prime filter if and only if for any $x, y \in L,\ x \vee y \in P$ implies $x \in P$ or $y \in P$.

In Example 4.1, $F = \{b, c\}$ is a prime filter. But, in Example 4.2, $F = \{(a, b_1), (a, b_2)\}$ is a filter, but not a prime filter since $(a, 0) \vee (0, b_1) = (a, b_1) \in F$, but $(a, 0), (0, b_1) \notin F$. Now, we prove a necessary and sufficient condition for a proper filter to become a prime filter.

THEOREM 4.8. *Let L be an AL and P be a proper filter of L. Then P is a prime filter if and only if for any filters F and G of L, $F \cap G \subseteq P$ implies either $F \subseteq P$ or $G \subseteq P$.*

PROOF. Suppose P is a prime filter. Suppose F and G are filters of an AL L such that $F \cap G \subseteq P$. Suppose $F \nsubseteq P$. Then there exists $x \in F$ such that $x \notin P$. Let $y \in G$. Then $x \vee y \in G$. Also, $y \vee x \in F$ and hence $x \vee y \in F$. Therefore $x \vee y \in F \cap G \subseteq P$. Since P is prime filter and $x \notin P$, $y \in P$. Thus $G \subseteq P$. Conversely, assume the condition. We shall prove that P is a prime filter. Let $x, y \in L$ such that $x \vee y \in P$. Then $(x] \cap (y] = (x \wedge y] \subseteq P$. Therefore either $(x] \subseteq P$ or $(y] \subseteq P$. Hence $x \in P$ or $y \in P$. Thus P is prime. □

Finally, we establish a necessary and sufficient condition for a proper subset of an AL L to become a prime ideal in terms of prime filters.

THEOREM 4.9. *A subset P of an AL L is a prime ideal of L if and only if $L-P$ is a prime filter of L.*

PROOF. Suppose P is prime ideal of L. We shall to prove $L - P$ is a prime filter. Since P is proper and $P \neq \emptyset$, $L - P$ is a nonempty proper subset of L. Let $x, y \in L - P$. Then $x, y \notin P$. This implies, $x \wedge y \notin P$ and hence $x \wedge y \in L - P$. Now, let $x \in L - P$ and $t \in L$. Then $x \notin P$. If $t \vee x \in P$, then $x = (t \vee x) \wedge x \in P$, a contradiction to $x \notin P$. Therefore $t \vee x \notin P$ and hence $t \vee x \in L - P$. Therefore $L-P$ is a filter of L. Let $x, y \in L$ such that $x,\ y \notin L-P$. Then $x,\ y \in P$ and hence $x \vee y \in P$. It follows that, $x \vee y \notin L - P$. Thus $L - P$ is a prime filter. Conversely, suppose $L - P$ is a prime filter. Then clearly, P is a nonempty proper subset of L. Let $x, y \in P$. Then $x, y \notin L - P$. Since $L - P$ is a prime filter, $x \vee y \notin L - P$. Thus $x \vee y \in P$. Again, let $x \in P$ and $t \in L$. Then $x \notin L - P$. If $x \wedge t \in L - P$, then $x = x \vee (x \wedge t) \in L - P$, a contradiction. Therefore $x \wedge t \notin L - P$ and hence $x \wedge t \in P$. Thus P is an ideal of L. Now, let $x, y \in L$ such that $x, y \notin P$. Then $x, y \in L - P$. Since $L - P$ is a filter, $x \wedge y \in L - P$. Hence $x \wedge y \notin P$. Therefore P is a prime ideal of L. □

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