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ON $^{\#}g^{\#}$-CLOSED SETS IN IDEAL TOPOLOGICAL SPACES

Ilangovan Rajasekaran, Ochanan Nethaji and

Rajendran Premkumar

Abstract. In this paper, we introduce the notions of $^{\#}g^{\#}$-closed sets and $I_{^{\#}g^{\#}}$-closed sets. Characterizations and properties of $I_{^{\#}g^{\#}}$-closed sets and $I_{^{\#}g^{\#}}$-open sets are given. A characterization of normal spaces is given in terms of $I_{^{\#}g^{\#}}$-open sets.

1. Introduction

Levine [**6**] introduced generalized closed sets [briefly g-closed] and studied their basic properties. Veera Kumar [**15**] introduced $g^{\#}$-closed sets in topological spaces and studied their properties.

The aim of this paper, we introduce the notion of $^{\#}g^{\#}$-closed sets and $I_{^{\#}g^{\#}}$-closed sets. Characterizations and properties of $I_{^{\#}g^{\#}}$-closed sets and $I_{^{\#}g^{\#}}$-open sets are given. A characterization of normal spaces is given in terms of $I_{^{\#}g^{\#}}$-open sets.

2. Preliminaries

Definition 2.1. A subset A of a space (X, τ) is said to be

1. an α-open set [**11**] if $A \subseteq int(cl(int(A)))$;
2. regular open set [**13**] if $A = int(cl(A))$;
3. semi-open set [**5**] if $A \subseteq cl(int(A))$;
4. preopen set [**8**] if $A \subseteq int(cl(A)))$.

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The complements of the above mentioned sets are called their respective closed sets.

The family of all α-open sets in (X, τ), denoted by τ^{α}, is a topology on X finer than τ. The closure of A in (X, τ^{α}) is denoted by $cl_{\alpha}(A)$.

Definition 2.2. A subset A of a space (X, τ) is said to be

(1) g-closed [**6**] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.
    The complement of g-closed set is called g-open set;
(2) αg-closed [**7**] if $\alpha cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open.
    The complement of αg-closed set is called αg-open set;
(3) $g^{\#}$-closed [**15**] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is αg-open.
    The complement of $g^{\#}$-closed set is called $g^{\#}$-open set.

An ideal I on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies

(1) $A \in I$ and $B \subseteq A \Rightarrow B \in I$ and
(2) $A \in I$ and $B \in I \Rightarrow A \cup B \in I$ [**4**].

Given a topological space (X, τ) with an ideal I on X and if $\wp(X)$ is the set of all subsets of X, a set operator $(.)^{\star} : \wp(X) \to \wp(X)$, called a local function [**4**] of A with respect to τ and I is defined as follows: for $A \subseteq X$, $A^{\star}(I, \tau) = \{x \in X \mid U \cap A \notin I$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$.

We will make use of the basic facts about the local functions ([**3**], Theorem 2.3) without mentioning it explicitly.

A Kuratowski closure operator $cl^{\star}(.)$ for a topology $\tau^{\star}(I, \tau)$, called the $\star$-topology, finer than τ is defined by $cl^{\star}(A) = A \cup A^{\star}(I, \tau)$ [**14**]. When there is no chance for confusion, we will simply write $A^{\star}$ for $A^{\star}(I, \tau)$ and $\tau^{\star}$ for $\tau^{\star}(I, \tau)$. If I is an ideal on X, then (X, τ, I) is called an ideal space or an ideal topological space. $\mathcal{N}$ is the ideal of all nowhere dense subsets in (X, τ).

Definition 2.3. A subset A of an ideal space (X, τ, I) is called

(1) $\star$-closed set [**3**] if $A^{\star} \subseteq A$ (or) $A = cl^{\star}(A)$;
(2) $\star$-dense in itself [**2**] if $A \subseteq A^{\star}$.
(3) I_g-closed [**1**] if $A^{\star} \subseteq U$ whenever $A \subseteq U$ and U is open.

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If A $\subseteq$ X, cl(A) and int(A) will, respectively, denote the closure and interior of A in (X, τ) and int*(A) will denote the interior of A in (X, $\tau^{\star}$).

Definition 2.4. An ideal I is said to be

(1) codense [**12**] or τ-boundary [**10**] if $\tau \cap I = \{\phi\}$,
(2) completely codense [**12**] if $PO(X) \cap I = \{\phi\}$, where PO(X) is the family of all preopen sets in (X, τ).

Lemma 2.1 ([**12**]). *Every completely codense ideal is codense but not conversely.*

Lemma 2.2 ([**12**]). *Let (X, τ, I) be an ideal topological space and $A \subseteq X$. If $A \subseteq A^{\star}$, then $A^{\star} = cl(A^{\star}) = cl(A) = cl^{\star}(A)$.*

LEMMA 2.3 ([**12**]). *Let (X, τ, I) be an ideal topological space. Then I is codense if and only if $G \subseteq G^{\star}$ for every semi-open set G in X.*

LEMMA 2.4 ([**12**]). *Let (X, τ, I) be an ideal topological space. If I is completely codense, then $\tau^{\star} \subseteq \tau^{\alpha}$.*

LEMMA 2.5 ([**3**]). *Let (X, τ, I) be an ideal topological space and A, B subsets of X. Then the following properties hold:*

(1) $A \subseteq B \Rightarrow A^{\star} \subseteq B^{\star}$,
(2) $A^{\star} = cl(A^{\star}) \subseteq cl(A)$,
(3) $(A^{\star})^{\star} \subseteq A^{\star}$,
(4) $(A \cup B)^{\star} = A^{\star} \cup B^{\star}$,
(5) $(A \cap B)^{\star} \subseteq A^{\star} \cap B^{\star}$.

REMARK 2.1 ([**15**]). For a subset of a topological space, the following properties hold:

(1) every closed set is $g^{\#}$-closed but not conversely.
(2) every $g^{\#}$-closed set is g-closed but not conversely.

LEMMA 2.6 ([**9**]). *If (X, τ, I) is a T_I-space and A is an I_g-closed set, then A is a $\star$-closed set.*

LEMMA 2.7 ([**1**]). *Every g-closed set is I_g-closed but not conversely.*

3. On $^{\#}g^{\#}$-closed sets and $I_{^{\#}g^{\#}}$-closed sets

DEFINITION 3.1. A subset A of a topological space (X, τ) is called :
$^{\#}g^{\#}$-closed if $A \subseteq U$, $U \in g^{\#}$-open $\Longrightarrow cl(A) \subseteq U$.
The complement of $^{\#}g^{\#}$-closed set is called $^{\#}g^{\#}$-open set.

REMARK 3.1. In a space (X, τ, I), every closed set is $^{\#}g^{\#}$-closed but not conversely.

EXAMPLE 3.1. Let $X = \{e_1, e_2, e_3\}$ be a set with the topology $\tau = \{\phi, X, \{e_1\}, \{e_1, e_2\}\}$. Then $\{e_1, e_3\}$ is $^{\#}g^{\#}$-closed set but not closed set.

THEOREM 3.1. *In a space (X, τ, I), every $^{\#}g^{\#}$-closed set is g-closed but not conversely.*

PROOF. It follows from the fact that every open set is $g^{\#}$-open. □

DEFINITION 3.2. A subset A of an ideal space (X, τ, I) is called :
$I_{^{\#}g^{\#}}$-closed if $A \subseteq U$, $U \in g^{\#}$-open $\Longrightarrow A^{\star} \subseteq U$.
The complement of $I_{^{\#}g^{\#}}$-closed set is called $I_{^{\#}g^{\#}}$-open set.

REMARK 3.2. If (X, τ, I) is any ideal space, then every $I_{^{\#}g^{\#}}$-closed set is I_g-closed but not conversely.

PROOF. It follows from the fact that every open set is $g^{\#}$-open. □

The following Theorem gives characterizations of $I_{^{\#}g^{\#}}$-closed sets.

THEOREM 3.2. *If (X, τ, I) is any ideal space and $A \subseteq X$, then the following are equivalent.*

(1) *A is $I_{\#g^\#}$-closed.*
(2) *$cl^\star(A) \subseteq U$ whenever $A \subseteq U$ and U is $g^\#$-open in X.*
(3) *For all $x \in cl^\star(A)$, $g^\# cl(\{x\}) \cap A \neq \phi$.*
(4) *$cl^\star(A) - A$ contains no nonempty $g^\#$-closed set.*
(5) *$A^\star - A$ contains no nonempty $g^\#$-closed set.*

PROOF. (1) ⇒ (2) If A is $I_{\#g^\#}$-closed, then $A^\star \subseteq U$ whenever $A \subseteq U$ and U is $g^\#$-open in X and so $cl^\star(A) = A \cup A^\star \subseteq U$ whenever $A \subseteq U$ and U is $g^\#$-open in X. This proves (2).

(2) ⇒ (3) Suppose $x \in cl^\star(A)$. If $g^\# cl(\{x\}) \cap A = \phi$, then $A \subseteq X - g^\# cl(\{x\})$. By (2), $cl^\star(A) \subseteq X - g^\# cl(\{x\})$, a contradiction, since $x \in cl^\star(A)$.

(3) ⇒ (4) Suppose $F \subseteq cl^\star(A) - A$, F is $g^\#$-closed and $x \in F$. Since $F \subseteq X - A$, then $A \subseteq X - F$, $g^\# cl(\{x\}) \cap A = \phi$. Since $x \in cl^\star(A)$ by (3), $g^\# cl(\{x\}) \cap A \neq \phi$. Therefore $cl^\star(A) - A$ contains no nonempty $g^\#$-closed set.

(4) ⇒ (5) Since $cl^\star(A) - A = (A \cup A^\star) - A = (A \cup A^\star) \cap A^c = (A \cap A^c) \cup (A^\star \cap A^c) = A^\star \cap A^c = A^\star - A$, therefore $A^\star - A$ contains no nonempty $g^\#$-closed set.

(5) ⇒ (1) Let $A \subseteq U$ where U is $g^\#$-open set. Therefore $X - U \subseteq X - A$ and so $A^\star \cap (X - U) \subseteq A^\star \cap (X - A) = A^\star - A$. Therefore $A^\star \cap (X - U) \subseteq A^\star - A$. Since $A^\star$ is always closed set, so $A^\star \cap (X - U)$ is a $g^\#$-closed set contained in $A^\star - A$. Therefore $A^\star \cap (X - U) = \phi$ and hence $A^\star \subseteq U$. Therefore A is $I_{\#g^\#}$-closed. □

THEOREM 3.3. *In a space (X, τ, I), every $\star$-closed set is $I_{\#g^\#}$-closed but not conversely.*

PROOF. Let A be a $\star$-closed, then $A^\star \subseteq A$. Let $A \subseteq U$ where U is $g^\#$-open. Hence $A^\star \subseteq U$ whenever $A \subseteq U$ and U is $g^\#$-open. Therefore A is $I_{\#g^\#}$-closed. □

THEOREM 3.4. *Let (X, τ, I) be an ideal space. For every $A \in I$, A is $I_{\#g^\#}$-closed.*

PROOF. Let $A \subseteq U$ where U is $g^\#$-open set. Since $A^\star = \phi$ for every $A \in I$, then $cl^\star(A) = A \cup A^\star = A \subseteq U$. Therefore, by Theorem 3.2, A is $I_{\#g^\#}$-closed. □

THEOREM 3.5. *If (X, τ, I) is an ideal space, then $A^\star$ is always $I_{\#g^\#}$-closed for every subset A of X.*

PROOF. Let $A^\star \subseteq U$ where U is $g^\#$-open. Since $(A^\star)^\star \subseteq A^\star$ [**3**], we have $(A^\star)^\star \subseteq U$ whenever $A^\star \subseteq U$ and U is $g^\#$-open. Hence $A^\star$ is $I_{\#g^\#}$-closed. □

THEOREM 3.6. *Let (X, τ, I) be an ideal space. Then every $I_{\#g^\#}$-closed, $g^\#$-open set is $\star$-closed set.*

PROOF. Since A is $I_{\#g^\#}$-closed and $g^\#$-open, then $A^\star \subseteq A$ whenever $A \subseteq A$ and A is $g^\#$-open. Hence A is $\star$-closed. □

COROLLARY 3.1. *If (X, τ, I) is a T_I ideal space and A is an $I_{\#g^\#}$-closed set, then A is $\star$-closed set.*

COROLLARY 3.2. *Let (X,τ,I) be an ideal space and A be an $I_{\#g^{\#}}$-closed set. Then the following are equivalent.*

(1) *A is a $\star$-closed set.*
(2) *$cl^{\star}(A)-A$ is a $g^{\#}$-closed set.*
(3) *$A^{\star}-A$ is a $g^{\#}$-closed set.*

PROOF. (1) $\Rightarrow$ (2) If A is $\star$-closed, then $A^{\star}\subseteq A$ and so $cl^{\star}(A)-A=(A\cup A^{\star})-A=\phi$. Hence $cl^{\star}(A)-A$ is $g^{\#}$-closed set.

(2) $\Rightarrow$ (3) Since $cl^{\star}(A)-A=A^{\star}-A$ and so $A^{\star}-A$ is $g^{\#}$-closed set.

(3) $\Rightarrow$ (1) If $A^{\star}-A$ is a $g^{\#}$-closed set, since A is $I_{\#g^{\#}}$-closed set, by Theorem 3.2, $A^{\star}-A=\phi$ and so A is $\star$-closed. □

THEOREM 3.7. *Let (X,τ,I) be an ideal space. Then every ${}^{\#}g^{\#}$-closed set is an $I_{\#g^{\#}}$-closed set but not conversely.*

PROOF. Let A be a ${}^{\#}g^{\#}$-closed set. Then $cl(A)\subseteq U$ whenever $A\subseteq U$ and U is $g^{\#}$-open. We have $cl^{\star}(A)\subseteq cl(A)\subseteq U$ whenever $A\subseteq U$ and U is $g^{\#}$-open. Hence A is $I_{\#g^{\#}}$-closed. □

THEOREM 3.8. *If (X,τ,I) is an ideal space and A is a $\star$-dense in itself, $I_{\#g^{\#}}$-closed subset of X, then A is ${}^{\#}g^{\#}$-closed.*

PROOF. Suppose A is a $\star$-dense in itself, $I_{\#g^{\#}}$-closed subset of X. Let $A\subseteq U$ where U is $g^{\#}$-open. Then by Theorem 3.2(2), $cl^{\star}(A)\subseteq U$ whenever A $\subseteq$ U and U is $g^{\#}$-open. Since A is $\star$-dense in itself, by Lemma 2.2, $cl(A)=cl^{\star}(A)$. Therefore $cl(A)\subseteq U$ whenever $A\subseteq U$ and U is $g^{\#}$-open. Hence A is ${}^{\#}g^{\#}$-closed. □

COROLLARY 3.3. *If (X,τ,I) is any ideal space where $I=\{\phi\}$, then A is $I_{\#g^{\#}}$-closed if and only if A is ${}^{\#}g^{\#}$-closed.*

PROOF. From the fact that for I=$\{\phi\}$, $A^{\star}$=cl(A) $\supseteq$ A. Therefore A is $\star$-dense in itself. Since A is $I_{\#g^{\#}}$-closed, by Theorem 3.8, A is ${}^{\#}g^{\#}$-closed. Conversely, by Theorem 3.7, every ${}^{\#}g^{\#}$-closed set is $I_{\#g^{\#}}$-closed set. □

COROLLARY 3.4. *If (X,τ,I) is any ideal space where I is codense and A is a semi-open, $I_{\#g^{\#}}$-closed subset of X, then A is ${}^{\#}g^{\#}$-closed.*

PROOF. By Lemma 2.3, A is $\star$-dense in itself. By Theorem 3.8, A is ${}^{\#}g^{\#}$-closed. □

REMARK 3.3. We have the following implications for the subsets stated above.

$$\begin{array}{ccccc}
\textit{closed} & \longrightarrow & {}^{\#}g^{\#}\textit{-closed} & \longrightarrow & g\textit{-closed} \\
\downarrow & & \downarrow & & \downarrow \\
\star\textit{-closed} & \longrightarrow & I_{\#g^{\#}}\textit{-closed} & \longrightarrow & I_{g}\textit{-closed}
\end{array}$$

THEOREM 3.9. *Let (X,τ,I) be an ideal space and $A\subseteq X$. Then A is $I_{\#g^{\#}}$-closed if and only if $A=F-N$ where F is $\star$-closed and $\mathcal{N}$ contains no nonempty $g^{\#}$-closed set.*

PROOF. If A is $I_{\#_g\#}$-closed, then by Theorem 3.2 (5), $\mathcal{N} = A^\star - A$ contains no nonempty $g^\#$-closed set. If $F = cl^\star(A)$, then F is $\star$-closed such that $F - N = (A \cup A^\star) - (A^\star - A) = (A \cup A^\star) \cap (A^\star \cap A^c)^c = (A \cup A^\star) \cap ((A^\star)^c \cup A) = (A \cup A^\star) \cap (A \cup (A^\star)^c) = A \cup (A^\star \cap (A^\star)^c) = A$.

Conversely, suppose $A = F - \mathcal{N}$ where F is $\star$-closed and $\mathcal{N}$ contains no nonempty $g^\#$-closed set. Let U be a $g^\#$-open set such that $A \subseteq U$. Then $F - \mathcal{N} \subseteq U$ and $F \cap (X - U) \subseteq \mathcal{N}$. Now $A \subseteq F$ and $F^\star \subseteq F$ then $A^\star \subseteq F^\star$ and so $A^\star \cap (X - U) \subseteq F^\star \cap (X - U) \subseteq F \cap (X - U) \subseteq \mathcal{N}$. By hypothesis, since $A^\star \cap (X - U)$ is $g^\#$-closed, $A^\star \cap (X - U) = \phi$ and so $A^\star \subseteq$ U. Hence A is $I_{\#_g\#}$-closed. □

THEOREM 3.10. *Let (X, τ, I) be an ideal space and $A \subseteq X$. If $A \subseteq B \subseteq A^\star$, then $A^\star = B^\star$ and B is $\star$-dense in itself.*

PROOF. Since $A \subseteq B$, then $A^\star \subseteq B^\star$ and since $B \subseteq A^\star$, then $B^\star \subseteq (A^\star)^\star \subseteq A^\star$. Therefore $A^\star = B^\star$ and $B \subseteq A^\star \subseteq B^\star$. Hence proved. □

THEOREM 3.11. *Let (X, τ, I) be an ideal space. If A and B are subsets of X such that $A \subseteq B \subseteq cl^\star(A)$ and A is $I_{\#_g\#}$-closed, then B is $I_{\#_g\#}$-closed.*

PROOF. Since A is $I_{\#_g\#}$-closed, then by Theorem 3.2(4), $cl^\star(A) - A$ contains no nonempty $g^\#$-closed set. Since $cl^\star(B) - B \subseteq cl^\star(A) - A$ and so $cl^\star(B) - B$ contains no nonempty $g^\#$-closed set. Hence B is $I_{\#_g\#}$-closed. □

COROLLARY 3.5. *Let (X, τ, I) be an ideal space. If A and B are subsets of X such that $A \subseteq B \subseteq A^\star$ and A is $I_{\#_g\#}$-closed, then A and B are ${}^\#g^\#$-closed sets.*

PROOF. Let A and B be subsets of X such that $A \subseteq B \subseteq A^\star \Rightarrow A \subseteq B \subseteq A^\star \subseteq cl^\star(A)$ and A is $I_{\#_g\#}$-closed. By the above Theorem, B is $I_{\#_g\#}$-closed. Since $A \subseteq B \subseteq A^\star$, then $A^\star = B^\star$ and so A and B are $\star$-dense in itself. By Theorem 3.8, A and B are ${}^\#g^\#$-closed. □

The following Theorem gives a characterization of $I_{\#_g\#}$-open sets.

THEOREM 3.12. *Let (X, τ, I) be an ideal space and $A \subseteq X$. Then A is $I_{\#_g\#}$-open if and only if $F \subseteq int^\star(A)$ whenever F is $g^\#$-closed and $F \subseteq A$.*

PROOF. Suppose A is $I_{\#_g\#}$-open. If F is $g^\#$-closed and $F \subseteq A$, then $X - A \subseteq X - F$ and so $cl^\star(X - A) \subseteq X - F$ by Theorem 3.2 (2). Therefore $F \subseteq X - cl^\star(X - A) = int^\star(A)$. Hence $F \subseteq int^\star(A)$.

Conversely, suppose the condition holds. Let U be a $g^\#$-open set such that $X - A \subseteq U$. Then $X - U \subseteq A$ and so $X - U \subseteq int^\star(A)$. Therefore $cl^\star(X - A) \subseteq U$. By Theorem 3.2 (2), $X - A$ is $I_{\#_g\#}$-closed. Hence A is $I_{\#_g\#}$-open. □

COROLLARY 3.6. *Let (X, τ, I) be an ideal space and $A \subseteq X$. If A is $I_{\#_g\#}$-open, then $F \subseteq int^\star(A)$ whenever F is closed and $F \subseteq A$.*

THEOREM 3.13. *Let (X, τ, I) be an ideal space and $A \subseteq X$. If A is $I_{\#_g\#}$-open and $int^\star(A) \subseteq B \subseteq A$, then B is $I_{\#_g\#}$-open.*

PROOF. Since A is $I_{\#g^{\#}}$-open, then $X-A$ is $I_{\#g^{\#}}$-closed. By Theorem 3.2 (4), $cl^{\star}(X-A)-(X-A)$ contains no nonempty $g^{\#}$-closed set. Since $int^{\star}(A)\subseteq int^{\star}(B)$ which implies that $cl^{\star}(X-B)\subseteq cl^{\star}(X-A)$ and so $cl^{\star}(X-B)-(X-B)\subseteq cl^{\star}(X-A)-(X-A)$. Hence B is $I_{\#g^{\#}}$-open. □

The following Theorem gives a characterization of $I_{\#g^{\#}}$-closed sets in terms of $I_{\#g^{\#}}$-open sets.

THEOREM 3.14. *Let (X,τ,I) be an ideal space and $A\subseteq X$. Then the following are equivalent.*

(1) *A is $I_{\#g^{\#}}$-closed.*
(2) *$A\cup(X-A^{\star})$ is $I_{\#g^{\#}}$-closed.*
(3) *$A^{\star}-A$ is $I_{\#g^{\#}}$-open.*

PROOF. (1) ⇒ (2) Suppose A is $I_{\#g^{\#}}$-closed. If U is any $g^{\#}$-open set such that $A\cup(X-A^{\star})\subseteq U$, then $X-U\subseteq X-(A\cup(X-A^{\star}))=X\cap(A\cup(A^{\star})^{c})^{c}=A^{\star}\cap A^{c}=A^{\star}-A$. Since A is $I_{\#g^{\#}}$-closed, by Theorem 3.2 (5), it follows that $X-U=\phi$ and so $X=U$. Now $A\cup(X-A^{\star})\subseteq X$ and so $(A\cup(X-A^{\star}))^{\star}\subseteq X^{\star}\subseteq X=U$. Hence $A\cup(X-A^{\star})$ is $I_{\#g^{\#}}$-closed.

(2) ⇒ (1) Suppose $A\cup(X-A^{\star})$ is $I_{\#g^{\#}}$-closed. If F is any $g^{\#}$-closed set such that $F\subseteq A^{\star}-A$, then $F\subseteq A^{\star}$ and $F\nsubseteq A$. Hence $X-A^{\star}\subseteq X-F$ and $A\subseteq X-F$. Therefore $A\cup(X-A^{\star})\subseteq A\cup(X-F)=X-F$ and $X-F$ is $g^{\#}$-open. Since $(A\cup(X-A^{\star}))^{\star}\subseteq X-F\Rightarrow A^{\star}\cup(X-A^{\star})^{\star}\subseteq X-F$ and so $A^{\star}\subseteq X-F\Rightarrow F\subseteq X-A^{\star}$. Since $F\subseteq A^{\star}$, it follows that $F=\phi$. Hence A is $I_{\#g^{\#}}$-closed.

(2) ⇔ (3) Since $X-(A^{\star}-A)=X\cap(A^{\star}\cap A^{c})^{c}=X\cap((A^{\star})^{c}\cup A)=(X\cap(A^{\star})^{c})\cup(X\cap A)=A\cup(X-A^{\star})$, it is obvious. □

THEOREM 3.15. *Let (X,τ,I) be an ideal space. Then every subset of X is $I_{\#g^{\#}}$-closed if and only if every $g^{\#}$-open set is $\star$-closed.*

PROOF. Suppose every subset of X is $I_{\#g^{\#}}$-closed. If $U\subseteq X$ is $g^{\#}$-open, then U is $I_{\#g^{\#}}$-closed and so $U^{\star}\subseteq U$. Hence U is $\star$-closed. Conversely, suppose that every $g^{\#}$-open set is $\star$-closed. If U is $g^{\#}$-open set such that $A\subseteq U\subseteq X$, then $A^{\star}\subseteq U^{\star}\subseteq U$ and so A is $I_{\#g^{\#}}$-closed. □

The following Theorem gives a characterization of normal spaces in terms of $I_{\#g^{\#}}$-open sets.

THEOREM 3.16. *Let (X,τ,I) be an ideal space where I is completely codense. Then the following are equivalent.*

(1) *X is normal.*
(2) *For any disjoint closed sets A and B, there exist disjoint $I_{\#g^{\#}}$-open sets U and V such that $A\subseteq U$ and $B\subseteq V$.*
(3) *For any closed set A and open set V containing A, there exists an $I_{\#g^{\#}}$-open set U such that $A\subseteq U\subseteq cl^{\star}(U)\subseteq V$.*

PROOF. (1) $\Rightarrow$ (2) The proof follows from the fact that every open set is $I_{\#g\#}$-open.

(2) $\Rightarrow$ (3) Suppose A is closed and V is an open set containing A. Since A and $X-V$ are disjoint closed sets, there exist disjoint $I_{\#g\#}$-open sets U and W such that $A \subseteq U$ and $X-V \subseteq W$. Since $X-V$ is $g^{\#}$-closed and W is $I_{\#g\#}$-open, $X-V \subseteq int^{\star}(W)$ and so $X - int^{\star}(W) \subseteq V$. Again $U \cap W = \phi \Rightarrow U \cap int^{\star}(W) = \phi$ and so

$$U \subseteq X - int^{\star}(W) \Rightarrow cl^{\star}(U) \subseteq X - int^{\star}(W) \subseteq V.$$

U is the required $I_{\#g\#}$-open sets with $A \subseteq U \subseteq cl^{\star}(U) \subseteq V$.

(3) $\Rightarrow$ (1) Let A and B be two disjoint closed subsets of X. By hypothesis, there exists an $I_{\#g\#}$-open set U such that $A \subseteq U \subseteq cl^{\star}(U) \subseteq X - B$. Since U is $I_{\#g\#}$-open, $A \subseteq int^{\star}(U)$. Since I is completely codense, by Lemma 2.4, $\tau^{\star} \subseteq \tau^{\alpha}$ and so $int^{\star}(U)$ and $X - cl^{\star}(U)$ are in τ^{α}. Hence

$$A \subseteq int^{\star}(U) \subseteq int(cl(int(int^{\star}(U)))) = G$$

and

$$B \subseteq X - cl^{\star}(U) \subseteq int(cl(int(X - cl^{\star}(U)))) = H.$$

G and H are the required disjoint open sets containing A and B respectively, which proves (1). □

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Department of Mathematics, Tirunelveli Dakshina Mara Nadar Sangam College, T. Kallikulam - 627 113, Tirunelveli District, Tamil Nadu, India.

E-mail address: sekarmelakkal@gmail.com.

Research Scholar, Department of Mathematics, School of Mathematics, Madurai Kamaraj University, Madurai, Tamil Nadu, India.

E-mail address: jionetha@yahoo.com.

Department of Mathematics, Senthamarai College of Arts and Science, Vadapalanji, Madurai - 21, Tamil Nadu, India.

E-mail address: prem.rpk27@gmail.com.