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IDEALS IN ALMOST LATTICES

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Abstract. The concepts of an initial segment and an ideal in an Almost Lattice AL are introduced and proved that the set of all initial segments of an AL L with 0 form a complete lattice with respect to set inclusion. Described the smallest ideal containing a given nonempty subset of an AL L. Also, proved that the set of all ideals $\mathcal{I}(L)$ of an AL L form a lattice with respect to set inclusion and the set $P\mathcal{I}(L)$ of all principal ideals of L is a sublattice of the lattice $\mathcal{I}(L)$. A set of identities for $\mathcal{I}(L)$ to become a complete lattice is established. Moreover, the concept of a prime ideal in an AL L is introduced and a necessary and sufficient condition for a proper ideal to become a prime ideal is obtained. An isomorphism between the lattice of all ideals $\mathcal{I}(L)$ of an AL L and the lattice of all ideals $I(P\mathcal{I}(L))$ of the lattice $P\mathcal{I}(L)$ is obtained and a one-to-one correspondence between prime ideals of L and those of $P\mathcal{I}(L)$ is established. Further, an isomorphism between amicable sets M (as a lattice) of an AL L and the lattice $P\mathcal{I}(L)$ of all principal ideals of an AL L is derived.

1. Introduction

Ideals were first studied by Dedekind, who defined the concept for rings of algebraic integers. Later the concept of ideal was extended to rings in general. M. H. Stone investigated ideals in Boolean rings, which are lattice of a special kind. There is already a well-developed theory of ideals in lattice. There are only one reasonable way of defining what is to be meant by an ideal in a lattice. Recall that Dedekind's definition of an ideal in a ring R is that it is a collection J of elements of R which (1) contains the difference $a-b$, and hence the sum $a+b$, of any two of its elements a and b of J, and (2) contains all multiples such as ax or ya of any of $x, y \in R$ and $a \in J$. By analogy, a collection J of elements of a lattice L is called

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an ideal if (1) it contains the lattice sum $a \vee b$ of any two of its elements a and b, and (2) it contains all multiples $a \wedge x$ of any $x \in L$ and $a \in J$. The analogy is that the greatest lower bound, or lattice meet $a \wedge b$ corresponds to product in a ring, and the least upper bound, or lattice join $a \vee b$ corresponds to the sum of two elements in a ring. The concept of Almost Lattice (AL) was introduced by G. Nanaji Rao and Habtamu Tiruneh Alemu [**3**] as a common abstraction of almost all lattice theoretic generalizations of Boolean algebra like distributive lattices, almost distributive lattices and lattices and established necessary and sufficient conditions for an AL L to become a lattice. Also, the concept of amicable sets in almost lattices (AL) [**4**] was introduced by G. Nanaji Rao, Habtamu Tiruneh Alemu and Terefe Getachew Beyene and proved that every maximal set in L is embedded in an amicable set in L and also proved that every amicable set in L is embedded in a maximal set with unielement.

In this paper, the concept of an initial segment in an AL L is introduced and proved that the set of all initial segments of L with 0 form a complete lattice with respect to set inclusion. Also, the concept of an ideal in an $AL\ L$ is introduced and described the smallest ideal containing a given nonempty subset of an $AL\ L$. Also, proved that the set of all ideals $\mathcal{I}(L)$ of an $AL\ L$ form a lattice and derived a set of identities that the lattice $\mathcal{I}(L)$ to become a complete lattice. We proved that the set $P\mathcal{I}(L)$ of all principal ideals of an $AL\ L$ is a sublattice of the lattice $\mathcal{I}(L)$. Further, the concept of a prime ideal in an $AL\ L$ is introduced and a necessary and sufficient condition for a proper ideal to become prime ideal is established. Also, an isomorphism between the lattice of all ideals $\mathcal{I}(L)$ of an AL L and the lattice of all ideals $I(P\mathcal{I}(L))$ of the lattice $P\mathcal{I}(L)$ is obtained and a one-to-one correspondence between prime ideals of L and those of $P\mathcal{I}(L)$ is established. Finally, an isomorphism between amicable sets M (as a lattice) of an AL L and the lattice $P\mathcal{I}(L)$ of all principal ideals of an AL L is derived.

2. Preliminaries

In the text, we are using the two references [**3**] and [**4**] most frequently, that was published by us (the authors of this article). Also, the major work of the text is based on these two papers. But, the other reference ([**1, 2, 5, 6, 7**]) are used as a basic foundation for the text.

In this section, we collect a few important definitions and results which are already known and which will be used more frequently in the text.

Definition 2.1. An algebra $(L, \vee, \wedge)$ of type (2, 2) is called an Almost Lattice(AL) if it satisfies the following axioms. For any $a, b, c \in L$:

A_1. $(a \wedge b) \wedge c = (b \wedge a) \wedge c$
A_2. $(a \vee b) \wedge c = (b \vee a) \wedge c$
A_3. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
A_4. $(a \vee b) \vee c = a \vee (b \vee c)$
A_5. $a \wedge (a \vee b) = a$
A_6. $a \vee (a \wedge b) = a$
A_7 $(a \wedge b) \vee b = b$

Lemma 2.1. *Let L be an AL. Then for any $a, b \in L$ we have the following:*

(1) $a \vee a = a$
(2) $a \wedge a = a$
(3) $a \wedge b = a$ *if and only if* $a \vee b = b$

Definition 2.2. Let L be an AL and $a, b \in L$. Then we say that a is less than or equal to b and write as $a \leqslant b$ if and only if $a \wedge b = a$ or, equivalently $a \vee b = b$.

Theorem 2.1. *Let L be an AL. For any $a, b, c \in L$, we have the following.*

(1) *The relation $\leqslant$ is a partial ordering on L and hence $(L, \leqslant)$ is a poset.*
(2) $a \leqslant b \Longrightarrow a \wedge b = b \wedge a$
(3) $a \wedge b = b \Longleftrightarrow a \vee b = a$

Definition 2.3. An AL L is said to be directed above if for any $a, b \in L$ there exists $c \in L$ such that $a \leqslant c$ and $b \leqslant c$.

Theorem 2.2. *Let L be an AL. Then the following are equivalent:*

(1) *L is directed above.*
(2) $\wedge$ *is commutative.*
(3) $\vee$ *is commutative.*
(4) *L is a lattice.*

Definition 2.4. For any $a, b \in L$ of an AL L, a is said to be compatible with b, written as $a \sim b$ if and only if $a \wedge b = b \wedge a$ or, equivalently, $a \vee b = b \vee a$.

Definition 2.5. A subset S of an AL L is said to be compatible if $a \sim b$ for all $a, b \in S$.

Note that for any a in an AL L, $\{a\}$ is a compatible set of L and the set $\mathcal{F}$ of all compatible sets of L is a poset with respect to set inclusion.

Definition 2.6. Let L be an AL. Then by a maximal set in L is a maximal element in the poset $(\mathcal{F}, \subseteq)$.

Definition 2.7. An algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ is called an AL with 0 if it satisfying the following axioms. For any $a, b, c \in L$:

(A_1) $(a \wedge b) \wedge c = (b \wedge a) \wedge c$
(A_2) $(a \vee b) \wedge c = (b \vee a) \wedge c$
(A_3) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$
(A_4) $(a \vee b) \vee c = a \vee (b \vee c)$
(A_5) $a \wedge (a \vee b) = a$
(A_6) $a \vee (a \wedge b) = a$
(A_7) $(a \wedge b) \vee b = b$
(0_1) $0 \wedge a = 0$

Definition 2.8. Let L be an AL. Then an element $a \in L$ is maximal if for any $x \in L$, $a \leqslant x$ implies $a = x$.

Proposition 2.1. *For any $m \in L$ of an AL L, the following are equivalent:*

(1) *m is maximal.*

(2) $m \vee x = m, \ \forall x \in L$.
(3) $m \wedge x = x, \ \forall x \in L$.

THEOREM 2.3. *Let M is a maximal set in an AL L. Then M is a lattice with respect to induced operations.*

PROPOSITION 2.2. *Let M be a maximal set of an AL L and $a \in M$. Then for any $x \in L, \ x \wedge a \in M$.*

DEFINITION 2.9. Let M be a maximal set in an AL L. Then an element $x \in L$ is said to be M-amicable if there exists $a \in M$ such that $a \wedge x = x$.

THEOREM 2.4. *Let M be a maximal set in an AL L and $x \in L$ be M-amicable. Then there is a smallest element $a \in M$ with the property $a \wedge x = x$.*

Note that such smallest element a in M is denoted by x^M.

COROLLARY 2.1. *Let M be a maximal set in an AL L and $x \in L$ be M-amicable. Then x^M is the unique element of M such that $x^M \wedge x = x$ and $x \wedge x^M = x^M$.*

Note that if M is a maximal set in an AL L, then we denote the set of all M-amicable elements of L by $A_M(L)$.

DEFINITION 2.10. A maximal set M in an AL is said to be amicable if $A_M(L) = L$.

THEOREM 2.5. *Let M be a maximal set in an AL L and $x, y \in L$ be M-amicable such that $x \sim y$. Then $x^M = y^M$ if and only if $x = y$.*

3. Initial segments in ALs

In this section, we introduce the concept of an initial segment in an AL L and prove some basic properties of an initial segment. Also, we prove that the set of all initial segments of an AL L with 0 is a complete lattice. We describe the smallest initial segment generated by a nonempty subset of an AL L with 0. First, we begin with the following.

LEMMA 3.1. *Let L be an AL with 0 and I be a nonempty subset of L. Then the following are equivalent to each other.*

(1) *If $x \in I$ and $a \in L$ such that $a \leqslant x$, then $a \in I$.*
(2) *If $a \in L$, then $a \wedge x \in I$ for all $x \in I$.*

PROOF. (1) $\Longrightarrow$ (2):-Assume (1). Let $x \in$ and $a \in L$. Then we have $a \wedge x \leqslant x$. Therefore by (1), we get $a \wedge x \in I$.
(2) $\Longrightarrow$ (1):- Assume (2). Let $x \in I$ and $a \in L$ such that $a \leqslant x$. Then we have $a = a \wedge x$. It follows by (2), $a \in I$. □

DEFINITION 3.1. A nonempty subset I of an AL L is called an initial segment in L if it satisfies any one of the above two condition.

Now, we prove the following.

Theorem 3.1. *Let I be a nonempty subset of an AL L such that $x \wedge a \in I$ for all $x \in I$ and for all $a \in L$. Then I is an initial segment.*

Proof. Let $x \in I$ and $a \in L$ such that $a \leqslant x$. Now, consider $a = a \wedge x = x \wedge a \in I$. Thus $a \in I$. Therefore I is an initial segment. □

But, the converse of the above theorem is not true. For, consider the discrete AL L with 0 with at least two elements. Choose $x \neq y \in L - \{0\}$ and put $I = \{0, x\}$. Then clearly I is an initial segment of L. But, $x \wedge y = y \notin I$. It is clear that if L is an AL with 0, then every initial segment I of L contains the zero element 0. In the following, we prove that the intersection (union) of any class of initial segments of an AL L is again an initial segment.

Theorem 3.2. *Let L be an AL with 0. Then the intersection(union) of any class of an initial segments of L is also an initial segment of L.*

Proof. Let $\mathcal{I} = \{I_\alpha \ / \alpha \in \Delta\}$ be any class of initial segments of L. Then clearly $L \in \mathcal{I}$ and hence $\mathcal{I} \neq \emptyset$. Also, since $0 \in I_\alpha$ for all α in $\Delta, 0 \in \bigcap\limits_{\alpha \in \Delta} I_\alpha (\bigcup\limits_{\alpha \in \Delta} I_\alpha)$ and hence $\bigcap\limits_{\alpha \in \Delta} I_\alpha (\bigcup\limits_{\alpha \in \Delta} I_\alpha) \neq \emptyset$. Clearly, $\bigcap\limits_{\alpha \in \Delta} I_\alpha (\bigcup\limits_{\alpha \in \Delta} I_\alpha)$ is an initial segment of L. □

In view of Theorem 3.2, we have the following.

Theorem 3.3. *The set $In(L)$, of all initial segments of an AL L with 0 is a complete lattice with respect to set inclusion.*

Next, we prove the smallest initial segment generated by a nonempty subset of an AL L.

Theorem 3.4. *Let L be an AL and $X (\neq \emptyset) \subseteq L$. Then $X^{\downarrow} = \{a \in L \mid a \leqslant x \ for \ some \ x \in X\}$ is the smallest initial segment containing X.*

Proof. Let $X (\neq \emptyset) \subseteq L$ and $X^{\downarrow} = \{a \in L \mid a \leqslant x \ for \ some \ x \in X\}$. Then since $X \neq \emptyset, X^{\downarrow} \neq \emptyset$. Let $a \in X^{\downarrow}$ and $t \in L$ such that $t \leqslant a$. Then we have $a \leqslant x$ for some $x \in X$. Hence we get $t \leqslant a \leqslant x$. Therefore $t \leqslant x, x \in X$. Thus $t \in X^{\downarrow}$. Therefore $X^{\downarrow}$ is an initial segment in L. Clearly $X \subseteq X^{\downarrow}$. Suppose H is an initial segment of L such that $X \subseteq H$. Now, let $a \in X^{\downarrow}$. Then $a \leqslant x$ for some $x \in X$. It follows that, $a \leqslant x$ and $x \in H$. Therefore $a \in H$ and hence $X^{\downarrow} \subseteq H$. Therefore $X^{\downarrow}$ is the smallest initial segment containing X. □

Definition 3.2. Let L be an AL and $X (\neq \emptyset) \subseteq L$. Then the initial segment $X^{\downarrow}$ is called the initial segment generated by X.

Corollary 3.1. *Let L be an AL and $x \in L$. Then $x^{\downarrow} = \{a \in L | a \leqslant x\}$ is an initial segment and is called the principal initial segment generated by x.*

Corollary 3.2. *Let L be an AL and $X (\neq \emptyset) \subseteq L$. Then $\bigcup\limits_{x \in X} x^{\downarrow} = X^{\downarrow}$.*

PROOF. Let $a \in \bigcup_{x \in X} x^{\downarrow}$. Then $a \in x^{\downarrow}$ for some $x \in X$. This implies $a \leqslant x$. Hence $a \in X^{\downarrow}$. Therefore $\bigcup_{x \in X} x^{\downarrow} \subseteq X^{\downarrow}$. Conversely, suppose $a \in x^{\downarrow}$. Then $a \leqslant x$ for some $x \in X$. It follows that, $a \in x^{\downarrow}$ and hence $a \in \bigcup_{x \in X} x^{\downarrow}$. Thus $X^{\downarrow} \subseteq \bigcup_{x \in X} x^{\downarrow}$. Therefore $X^{\downarrow} = \bigcup_{x \in X} x^{\downarrow}$. □

4. Ideals in ALs

In this section, we introduce the concept of an ideal in an AL L and describe the smallest ideal containing a given nonempty subset of L. We prove the set $\mathcal{I}(L)$ of all ideals of an AL L form a lattice and the set $P\mathcal{I}(L)$ of all principal ideals of L is a sublattice of the lattice $\mathcal{I}(L)$. Also, we establish a set of identities that the lattice $\mathcal{I}(L)$ is a complete lattice. Now, we begin this section with the following.

DEFINITION 4.1. Let L be an AL. Then a nonempty subset I of L is said to be an ideal of L if it satisfies the following:

(1) If $x, y \in I$, then there exists $d \in I$ such that $d \wedge x = x$ and $d \wedge y = y$.
(2) If $x \in I$ *and* $a \in L$, *then* $x \wedge a \in I$.

LEMMA 4.1. *Let L be an AL and I be an ideal of L. Then the following are equivalent:*

(1) *$x, y \in I$, implies $x \vee y \in I$*
(2) *$x, y \in I$, implies there exists $d \in I$ such that $d \wedge x = x$ and $d \wedge y = y$.*

PROOF. (1) $\Longrightarrow$ (2) Assume (1). Suppose $x, y \in I$. Then by (1), $x \vee y \in I$ and $(x \vee y) \wedge x = x,\ (x \vee y) \wedge y = y$.
(2) $\Longrightarrow$ (1):- Assume (2). Let $x, y \in L$. Then there exists $d \in I$ such that $d \wedge x = x$ and $d \wedge y = y$. It follows that, $d \vee x = d$ and $d \vee y = d$. Consider, $d \vee (x \vee y) = (d \vee x) \vee y = d \vee y = d$ and hence $x \vee y = d \wedge (x \vee y) \in I$. □

COROLLARY 4.1. *Let L be an AL and I be an ideal of L. Then I is a sub AL of L.*

But, converse of the above corollary is not true. For, consider the following example.

EXAMPLE 4.1. Let $L = \{a, b, c\}$. Define a binary operation $\vee$ *and* $\wedge$ on L as below:

$\vee$	a	b	c
a	a	b	c
b	b	b	b
c	c	c	c

and

$\wedge$	a	b	c
a	a	a	a
b	a	b	c
c	a	b	c

Then clearly, $(L, \vee, \wedge)$ is an AL. Put $I = \{a, b\}$. Clearly, I is sub AL of L. But, since $b \wedge c = c \notin I$, I is not an ideal.

COROLLARY 4.2. *Let L be an AL and I be an ideal of L. If $x \in I$ and $a \in L$, then $a \wedge x \in I$.*

PROOF. Suppose $x \in I$ and $a \in L$. Then $x \wedge a \in I$. Now, consider $a \wedge x = a \wedge (x \wedge x) = (a \wedge x) \wedge x = (x \wedge a) \wedge x \in I$. Therefore $a \wedge x \in I$. □

In the following, we describe the ideal generated by a given nonempty subset S of an AL L and prove that this is the smallest ideal of L containing S.

THEOREM 4.1. *Let S be a nonempty subset of an AL L. Then*

$$(S] = \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S,\ x \in L \text{ and } n \text{ is a positive integer}\}$$

is the smallest ideal of L containing S.

PROOF. Suppose S is a nonempty subset of L. We shall prove that

$$(S] = \{(\bigvee_{i=1}^{n} s_i) \wedge x \mid s_i \in S,\ x \in L \text{ and } n \text{ is a positive integer}\}$$

is the smallest ideal of L containing S. Clearly, $S \subseteq (S]$ and hence $(S] \neq \emptyset$. Let $(\bigvee_{i=1}^{n} s_i) \wedge x,\ (\bigvee_{j=1}^{m} t_j) \wedge y \in (S]$. Put $d = (\bigvee_{i=1}^{n} s_i) \vee (\bigvee_{j=1}^{m} t_j)$. Then clearly $d \in (S]$ and $d \wedge ((\bigvee_{i=1}^{n} s_i) \wedge x) = (d \wedge (\bigvee_{i=1}^{n} s_i)) \wedge x = (\bigvee_{i=1}^{n} s_i) \wedge x$. Similarly, we can prove that, $d \wedge ((\bigvee_{j=1}^{m} t_j) \wedge y) = (\bigvee_{j=1}^{m} t_j) \wedge y$. Also, if $(\bigvee_{i=1}^{n} s_i) \wedge x \in (S]$ and $p \in L$, then $((\bigvee_{i=1}^{n} s_i) \wedge x) \wedge p = (\bigvee_{i=1}^{n} s_i) \wedge (x \wedge p) \in (S]$. Therefore $(S]$ is an ideal of L containing S. Suppose H is an ideal of L such that $S \subseteq H$. Now, let $(\bigvee_{i=1}^{n} s_i) \wedge x \in (S]$ where $s_i \in S$ for all i and $x \in L$. Then $s_i \in H$ for all i and $x \in L$ and hence $\bigvee_{i=1}^{n} s_i \in H$. Now, since H is an ideal, $(\bigvee_{i=1}^{n} s_i) \wedge x \in H$. It follows that, $(S] \subseteq H$. Therefore $(S]$ is the smallest ideal containing S. □

If $S = \{a\}$, then we write $(a]$ instead of $(\{a\}]$. With this notion we have the following.

COROLLARY 4.3. *Let L be an AL and $a \in L$. Then $(a] = \{a \wedge x \mid x \in L\}$ is an ideal of L and is called principal ideal generated by a.*

COROLLARY 4.4. *Let L be an AL and $a, b \in L$. Then $a \in (b]$ if and only if $a = b \wedge a$.*

PROOF. Suppose $a \in (b]$. Then $a = b \wedge t$ for some $t \in L$. Now, $b \wedge a = b \wedge (b \wedge t) = b \wedge t = a$. Therefore $a = b \wedge a$. Converse follows by the definition of $(b]$. □

COROLLARY 4.5. *Let I be an ideal of an AL L and $a, b \in L$. Then $a \wedge b \in I$ if and only if $b \wedge a \in I$.*

PROOF. Suppose $a \wedge b \in I$. Then $(a \wedge b) \wedge a \in I$. Now, consider $b \wedge a = b \wedge (a \wedge a) = (b \wedge a) \wedge a = (a \wedge b) \wedge a \in I$. Hence $b \wedge a \in I$. Similarly, we can prove that if $b \wedge a \in I$, then $a \wedge b \in I$. □

COROLLARY 4.6. *Let I be an ideal of an AL L and $a, b \in L$. Then $a \vee b \in I$ if and only if $b \vee a \in I$.*

PROOF. Suppose $a \vee b \in I$. Then $(a \vee b) \wedge (b \vee a) \in I$. It follows that, $b \vee a = (b \vee a) \wedge (b \vee a) \in I$. Conversely, suppose $b \vee a \in I$. Then $(b \vee a) \wedge (a \vee b) \in I$. Hence $a \vee b = (a \vee b) \wedge (a \vee b) \in I$. $\square$

COROLLARY 4.7. *Let L be an AL and $a, b \in L$. Then $(a \wedge b] = (b \wedge a]$.*

Recall that, for any $a, b \in L$ with $a \leqslant b$, we have $a \wedge b = b \wedge a$. It follows that, every ideal of an AL L is an initial segment.

REMARK 4.1. One may suspect that an initial segment of an AL L which is closed under the operation $\vee$ is an ideal. Though, it is true when L is a lattice, it is not true in a general AL. For, consider a discrete AL L with at least two elements. Now, choose $x \in L$. Then clearly, $\{x\}$ is an initial segment and is closed under the operation $\vee$. But $\{x\}$ is not an ideal; since for any $a(\neq x) \in L,\ x \wedge a = a \notin \{x\}$.

THEOREM 4.2. *Let L be an AL. Then the set $\mathcal{I}(L)$ of all ideals of L form a lattice under set inclusion in which the glb and lub for any $I, J \in \mathcal{I}(L)$ are respectively*

$$I \wedge J = I \cap J$$

and

$$I \vee J = \{x \in L |\ (a \vee b) \wedge x = x\ for\ some\ a \in I\ and\ b \in J\}.$$

PROOF. Clearly, $\mathcal{I}(L)$ is non empty. Also, clearly $\mathcal{I}(L)$ is a poset with respect to set inclusion. Now, let $I, J \in \mathcal{I}(L)$. Then clearly $I \cap J$ is the glb of I and J in $\mathcal{I}(L)$. Now, we shall prove $I \vee J = \{x \in L |\ (a \vee b) \wedge x = x\ for\ some\ a \in I\ and\ b \in J\}$ is the lub of I and J in $\mathcal{I}(L)$. Since I and J are nonempty, it follows that, $I \vee J$ is nonempty. Let $x, y \in I \vee J$. Then $(a_1 \vee b_1) \wedge x = x$ and $(a_2 \vee b_2) \wedge y = y$ for some $a_1, a_2 \in I$ and $b_1, b_2 \in J$. It follows that, there exists $d_1 \in I$ and $d_2 \in J$ such that $d_1 \wedge a_1 = a_1,\ d_1 \wedge a_2 = a_2$ and $d_2 \wedge b_1 = b_1,\ d_2 \wedge b_2 = b_2$. Hence $d_1 \vee a_1 = d_1,\ d_1 \vee a_2 = d_1$ and $d_2 \vee b_1 = d_2,\ d_2 \vee b_2 = d_2$. Now, we have $d_1 \in I,\ d_2 \in J$ and hence $d_1 \vee d_2 \in I \vee J$. Consider, $(d_1 \vee d_2) \wedge x = ((d_1 \vee a_1) \vee (d_2 \vee b_1)) \wedge x = ((d_1 \vee d_2) \vee (a_1 \vee b_1)) \wedge x = ((d_1 \vee d_2) \vee (a_1 \vee b_1)) \wedge ((a_1 \vee b_1) \wedge x) = (((d_1 \vee d_2) \vee (a_1 \vee b_1)) \wedge (a_1 \vee b_1)) \wedge x = (a_1 \vee b_1) \wedge x = x$. Similarly, we can prove that $(d_1 \vee d_2) \wedge y = y$. Again, let $x \in I \vee J$ and $p \in L$. Then $(a \vee b) \wedge x = x$ for some $a \in I$ and $b \in J$. Now, $(a \vee b) \wedge (x \wedge p) = ((a \vee b) \wedge x) \wedge p = x \wedge p$. Hence $x \wedge p \in I \vee J$. Therefore $I \vee J$ is an ideal of L. Let $x \in I$ and $y \in J$. Then $x = (x \vee y) \wedge x$ and $y = (x \vee y) \wedge y$. Hence $x, y \in I \vee J$. So that $I, J \subseteq I \vee J$. Therefore $I \vee J$ is an upper bound of I and J in $\mathcal{I}(L)$. Suppose that $H \in \mathcal{I}(L)$ such that H is an upper bound of I and J. Then $I \subseteq H$ and $J \subseteq H$. Now, let $x \in I \vee J$. Then $(a \vee b) \wedge x = x$ for some $a \in I$ and $b \in J$. Hence $a, b \in H$ since $I, J \subseteq H$. Therefore by Corollary 4.1, we get $a \vee b \in H$. It follows that, $x = (a \vee b) \wedge x \in H$. Therefore $I \vee J \subseteq H$. Thus $I \vee J$ is the lub of I and J in $\mathcal{I}(L)$. Therefore $\mathcal{I}(L)$ is a lattice. $\square$

LEMMA 4.2. *For any a, b in an AL L, $b \in (a]$ if and only if $(b] \subseteq (a]$.*

PROOF. Suppose $b \in (a]$. Then $b = a \wedge b$. Now, let $t \in (b]$. Then $t = b \wedge t = (a \wedge b) \wedge t = a \wedge (b \wedge t) \in (a]$. Thus $(b] \subseteq (a]$. Conversely, suppose $(b] \subseteq (a]$. Since $b \in (b]$, $b \in (a]$. Therefore $b \in (a]$. $\square$

LEMMA 4.3. *For any a, b in an AL L, $(a] \subseteq (b]$ whenever $a \leqslant b$.*

PROOF. Suppose $a \leqslant b$. Then $a = a \wedge b$. Now, let $t \in (a]$. Then $t = a \wedge t = (a \wedge b) \wedge t = (b \wedge a) \wedge t = b \wedge (a \wedge t) \in (b]$. Therefore $(a] \subseteq (b]$. □

LEMMA 4.4. *For any a, b in an AL L, we have the following.*

(1) $(a] \vee (b] = (a \vee b] = (b \vee a]$
(2) $(a] \cap (b] = (a \wedge b] = (b \wedge a]$

PROOF. Let $x \in (a] \vee (b]$. Then $x = (s \vee t) \wedge x$ for some $s \in (a]$ and $t \in (b]$. It follows that, $s = a \wedge s$ and $t = b \wedge t$. Hence $a = a \vee s,\ b = b \vee t$. Now, $a \vee b = (a \vee s) \vee (b \vee t)$. It follows that, $(a \vee b) \wedge x = ((a \vee s) \vee (b \vee t)) \wedge x = ((a \vee b) \vee (s \vee t)) \wedge x = ((a \vee b) \vee (s \vee t)) \wedge ((s \vee t) \wedge x) = ((a \vee b) \vee (s \vee t)) \wedge ((s \vee t)) \wedge x = (s \vee t) \wedge x = x$. Therefore $x \in (a \vee b]$. Hence $(a] \vee (b] \subseteq (a \vee b]$.

Conversely, suppose $x \in (a \vee b]$. Then $(a \vee b) \wedge x = x$. It follows that, $x \in (a] \vee (b]$; since $a \in (a],\ b \in (b]$. Therefore$(a \vee b] \subseteq (a] \vee (b]$. Hence $(a] \vee (b] = (a \vee b]$. Again, since $(a \vee b) \wedge t = (b \vee a) \wedge t$ for all $t \in L$, $(a \vee b] = (b \vee a]$.

Let $t \in (a] \cap (b]$. Then $t \in (a]$ *and* $t \in (b]$. This implies $t = a \wedge t$ *and* $t = b \wedge t$. Hence $t = t \wedge t = a \wedge t \wedge b \wedge t = a \wedge b \wedge t \in (a \wedge b]$. Therefore $(a] \cap (b] \subseteq (a \wedge b]$. Conversely, suppose $t \in (a \wedge b]$. Then $t = (a \wedge b) \wedge t = a \wedge (b \wedge t) \in (a]$ and also $t = (a \wedge b) \wedge t = (b \wedge a) \wedge t = b \wedge (a \wedge t) \in (b]$. Hence $t \in (a] \cap (b]$. It follows that, $(a \wedge b] \subseteq (a] \cap (b]$. Therefore $(a] \cap (b] = (a \wedge b] = (b \wedge a]$. □

Now, we have the following theorem whose proof follows from Lemma 4.4.

THEOREM 4.3. *Let L be an AL. Then the set $P\mathcal{I}(L)$, of all principal ideals of L is a sub lattice of the lattice $\mathcal{I}(L)$ of all ideals of L.*

Let us recall that, an element m of L is said to be maximal (minimal) if and only if $m \wedge x = x (x \wedge m = m)$ if and only if $m \vee x = m (x \vee m = x)$ for all $x \in L$. Therefore it can be easily seen that every ideal of an AL L contains all minimal elements in L. In the following, we prove that the set of all minimal elements in L is an ideal.

LEMMA 4.5. *Let L be an AL with a minimal element. Then the set of all minimal elements of L form an ideal of L.*

PROOF. Suppose L has a minimal element. Put $I = \{m/\ m$ is a minimal element in $L\}$. Then clearly I is nonempty. Let $x, y \in I$. Then x and y are minimal elements in L and hence $x \wedge x = x$ and $x \wedge y = y$. Now, put $d = x$. Then $d \in I$ and $d \wedge x = x,\ d \wedge y = y$. Again, let $x \in I$ *and* $t \in L$. Then for any $a \in L$, $a \wedge (x \wedge t) = (a \wedge x) \wedge t = x \wedge t$. Thus $x \wedge t$ is a minimal element of L. Hence $x \wedge t \in I$. Therefore I is an ideal in L. □

In view of Theorem 4.2, it can be easily seen that the intersection of a finite family of ideals is again an ideal in an AL L, but, the intersection of an arbitrary family of ideals need not be an ideal in general. In the following, we establish a set of identities that intersection of any family of ideals is again an ideal.

THEOREM 4.4. *Let L be an AL. Then the following conditions are equivalent in L:*

(1) *The intersection of any family of ideals is nonempty.*
(2) *The intersections of any family of ideals is again an ideal.*
(3) *The lattice $\mathcal{I}(L)$ has least element.*
(4) *The lattice $\mathcal{I}(L)$ is complete.*
(5) *The class $P\mathcal{I}(L)$ has least element.*
(6) *L has a minimal element.*

PROOF. (1) $\Longrightarrow$ (2): Suppose $\{I_\alpha\}_{\alpha\in\Delta}$ be a family of ideals in L and suppose $I = \bigcap\limits_{\alpha\in\Delta} I_\alpha$ is nonempty. Let $x, y \in I$. Then $x, y \in I_\alpha$ for all $\alpha \in \Delta$. Since each I_α is an ideal, $x \vee y \in I_\alpha$ for all $\alpha \in \Delta$. Therefore $x \vee y \in \bigcap\limits_{\alpha\in\Delta} I_\alpha$ such that $(x \vee y) \wedge x = x$ and $(x \vee y) \wedge y = y$. It follows that, I is an ideal of L.

Proof of (2) $\Longrightarrow$ (3) and (3) $\Longrightarrow$ (4) is clear.

(4) $\Longrightarrow$ (5): Suppose $\mathcal{I}(L)$ is complete. Since $P\mathcal{I}(L)$ is a nonempty subset of $\mathcal{I}(L)$, $P\mathcal{I}(L)$ has a greatest lower bound say I. Clearly, I is a principal ideal, since every element in I generates I. Thus $P\mathcal{I}(L)$ has least element.

(5) $\Longrightarrow$ (6): Suppose $P\mathcal{I}(L)$ has least element say $(a]$. Now, we shall prove that a is a minimal element in L. Suppose $x \in L$ such that $x \leqslant a$. Then we have $(x] \subseteq (a]$. It follows that, $(x] = (a]$. Since $a \in (a] = (x]$, $a = x \wedge a = x$. Therefore a is minimal.

(6) $\Longrightarrow$ (1): Suppose L has a minimal element. Since every ideal in L contains a minimal element, the intersections of any family of ideals is nonempty. ☐

5. The Lattice $P\mathcal{I}(L)$

In this section, we introduce the concept of prime ideals in an AL L and prove that a proper ideal P of L is prime if and only if for any ideal I and J of L, $I \cap J \subseteq P$ implies either $I \subseteq P$ or $J \subseteq P$. We have proved in the previous section that the class $P\mathcal{I}(L)$ of all principal ideals of an AL L form a lattice. In this section, we prove that the lattice $\mathcal{I}(L)$ of all ideals of an AL L is isomorphic with the lattice of all ideals of the lattice $P\mathcal{I}(L)$ of L and also we prove this correspondence gives a one-to-one correspondence between the prime ideals of L and those of $P\mathcal{I}(L)$. Also, we prove that the set $A_M(L)$, of all M-amicable elements of L is an ideal of L. Finally, we prove that an amicable set M of L (as lattice) and the lattice $P\mathcal{I}(L)$ are isomorphic.

First, we begin this section with the following definition.

DEFINITION 5.1. Let L be an AL. Then a proper ideal P of L is said to be prime if for any $x, y \in L$, $x \wedge y \in P$, then either $x \in P$ or $y \in P$.

Now, we derive a necessary and sufficient condition for a proper ideal P to become a prime ideal in an AL L.

THEOREM 5.1. *Let L be an AL. Then a proper ideal P of L is prime if and only if for any ideals I and J of L, $I \cap J \subseteq P$ implies either $I \subseteq P$ or $J \subseteq P$.*

PROOF. Suppose P is a prime ideal and suppose I, J are ideals of L such that $I \cap J \subseteq P$. Suppose $I \nsubseteq P$. Then there exists $x \in I$ such that $x \notin P$. Let $y \in J$. Then $y \wedge x \in J$ and hence $x \wedge y \in J$. Also, $x \wedge y \in I$. Hence $x \wedge y \in I \cap J \subseteq P$. Since P is prime ideal and $x \notin P$, $y \in P$. Thus $J \subseteq P$. Conversely, assume the condition. We shall prove that P is a prime ideal. Let $x, y \in L$ such that $x \wedge y \in P$. Then $(x] \cap (y] = (x \wedge y] \subseteq P$. Therefore either $(x] \subseteq P$ or $(y] \subseteq P$. It follows that, $x \in P$ or $y \in P$. Thus P is prime. □

In the following, we prove an isomorphism between the lattices $\mathcal{I}(L)$ and $I(P\mathcal{I}(L))$ and also derive a one-to-one correspondence between the prime ideals of L and those of $P\mathcal{I}(L)$. First, we need the following:

LEMMA 5.1. Let L be an AL. Then we have the following:

(1) For any ideal I of L, $I^e = \{(a] |\ a \in I\}$ is an ideal of $P\mathcal{I}(L)$. Moreover, I is prime if and only if I^e is prime.
(2) For any ideal K of the lattice $P\mathcal{I}(L)$, $K^c = \{a \in L |\ (a] \in K\}$ is an ideal of L. Further, K is prime if and only if K^c is prime.
(3) For any ideals I_1 and I_2 of L, $I_1 \subseteq I_2$ if and only if ${I_1}^e \subseteq {I_2}^e$.
(4) For any ideals K_1 and K_2 of $P\mathcal{I}(L)$, $K_1 \subseteq K_2$ if and only if ${K_1}^c \subseteq {K_2}^c$.
(5) $I^{ec} = I$ for all ideals I of L.
(6) $K^{ce} = K$ for all ideals K of $P\mathcal{I}(L)$.

PROOF. (1) Suppose I is an ideal of L. Now, we shall prove that I^e is an ideal of $P\mathcal{I}(L)$. Since I is nonempty, I^e is nonempty. Let $(a], (b] \in I^e$. Then $a, b \in I$ and hence $a \vee b \in I$. It follows that, $(a] \vee (b] = (a \vee b] \in I^e$. Again, we have $a \wedge b \in I$. Hence $(a] \cap (b] = (a \wedge b] \in I^e$. Thus I^e is an ideal of the lattice $P\mathcal{I}(L)$. Suppose I is a prime ideal of L. We shall prove that I^e is a prime ideal of $P\mathcal{I}(L)$. Let $(a], (b] \in P\mathcal{I}(L)$ such that $(a] \cap (b] \in I^e$. Then $(a \wedge b] \in I^e$. Therefore $(a \wedge b] = (t]$ for some $t \in I$. Since $a \wedge b \in (a \wedge b] = (t]$, $a \wedge b = t \wedge (a \wedge b) \in I$. Therefore $a \wedge b \in I$. Since I is prime, either $a \in I$ or $b \in I$. It follows that, $(a] \in I^e$ or $(b] \in I^e$. Thus I^e is a prime ideal of $P\mathcal{I}(L)$. Conversely, suppose I^e is a prime ideal of $P\mathcal{I}(L)$. Let $a, b \in L$ such that $a \wedge b \in I$. Then $(a] \cap (b] = (a \wedge b] \in I^e$. Therefore $(a] \in I^e$ or $(b] \in I^e$. Hence $a \in I$ or $b \in I$. Therefore I is prime.

(2) Suppose K is an ideal of $P\mathcal{I}(L)$. We shall prove that K^c is an ideal of L. Since K is nonempty, K^c is nonempty. Let $a,\ b \in K^c$. Then $(a],\ (b] \in K$. Hence $(a \vee b] = (a] \vee (b] \in K$. Therefore $a \vee b \in K^c$, $(a \vee b) \wedge a = a$ and $(a \vee b) \wedge b = b$. Let $a \in K^c$ and $t \in L$. Then $(a] \in K$ and $(t] \in P\mathcal{I}(L)$. Therefore $(a \wedge t] = (a] \cap (t] \in K$. Hence $a \wedge t \in K^c$. Thus K^c is an ideal of L. Now, suppose K is a prime ideal of $P\mathcal{I}(L)$. We shall prove that K^c is a prime ideal of L. Let $a, b \in L$ such that $a \wedge b \in K^c$. Then $(a] \cap (b] = (a \wedge b] \in K$. Therefore either $(a] \in K$ or $(b] \in K$, since K is prime. This implies $a \in K^c$ or $b \in K^c$. Hence K^c is a prime ideal of L. Conversely, suppose K^c is a prime ideal of L. Now, let $(a], (b] \in P\mathcal{I}(L)$ such that $(a] \cap (b] \in K$. Then $(a \wedge b] \in K$. It follows, $a \wedge b \in K^c$. Since K^c is prime, either $a \in K^c$ or $b \in K^c$. Therefore $(a] \in K$ or $(b] \in K$. Hence K is a prime ideal of $P\mathcal{I}(L)$.

(3) item Suppose I_1 and I_2 are an ideals of L such that $I_1 \subseteq I_2$. We shall prove that $I_1^e \subseteq I_2^e$. Let $(a] \in {I_1}^e$. Then $a \in I_1$ and hence $a \in I_2$. Therefore $(a] \in {I_2}^e$. Thus ${I_1}^e \subseteq {I_2}^e$. Conversely, suppose ${I_1}^e \subseteq {I_2}^e$. Let $a \in I_1$. Then $(a] \in {I_1}^e \subseteq {I_2}^e$. Therefore $(a] = (t]$ for some $t \in I_2$. Hence $a = t \wedge a \in I_2$. Thus $I_1 \subseteq I_2$.

(4) Suppose K_1 and K_2 are ideals of $P\mathcal{I}(L)$ such that $K_1 \subseteq K_2$. We shall prove that $K_1^c \subseteq K_2^c$. Let $a \in {K_1}^c$. Then $(a] \in K_1$. Hence $(a] \in K_2$. Therefore $(a] = (t]$ for some $t \in K_2$. It follows that $a = t \wedge a \in {K_2}^c$. Thus ${K_1}^c \subseteq {K_2}^c$. Conversely, suppose ${K_1}^c \subseteq {K_2}^c$. Let $(a] \in K_1$. Then $a \in {K_1}^c \subseteq {K_2}^c$. Therefore $a \in {K_2}^c$. Hence $(a] \in K_2$. Therefore $K_1 \subseteq K_2$.

(5) Suppose $a \in I^{ec}$. Then $(a] \in I^e$. Therefore $(a] = (t]$ for some $t \in I$. Hence $a = t \wedge a \in I$. Thus $I^{ec} \subseteq I$. Clearly $I \subseteq I^{ec}$. Thus $I = I^{ec}$.

(6) Suppose $(a] \in K^{ce}$. Then $a \in K^c$. This implies $(a] \in K$. Therefore $K^{ce} \subseteq K$. Conversely, suppose $(a] \in K$. Then $a \in K^c$. It follows that, $a \in K^{ce}$. Hence $K \subseteq K^{ce}$. Thus $K^{ce} = K$ □

LEMMA 5.2. Let I and J be an ideals of an AL L. Then $(I \cap J)^e = I^e \cap J^e$ and $(I \vee J)^e = I^e \vee J^e$.

PROOF. Suppose I and J are ideals of L. Then $I \cap J \subseteq I, J$ and hence $(I \cap J)^e \subseteq I^e, J^e$. Therefore $(I \cap J)^e \subseteq I^e \cap J^e$. Conversely, suppose $(a] \in I^e \cap J^e$. Then $(a] \in I^e$ and $(a] \in J^e$. Hence $(a] = (t]$ for some $t \in I$ and $(a] = (s]$ for some $s \in J$. Therefore $a \in (a] = (t]$ and hence $a = t \wedge a$. Similarly, we get $a = s \wedge a$. It follows that, $a \in I \cap J$. Thus $(a] \in (I \cap J)^e$. Therefore $I^e \cap J^e \subseteq (I \cap J)^e$. Thus $(I \cap J)^e = I^e \cap J^e$.

We have $I, J \subseteq I \vee J$. It follows that, $I^e, J^e \subseteq (I \vee J)^e$. Hence $I^e \vee J^e \subseteq (I \vee J)^e$. Conversely, let $(a] \in (I \vee J)^e$. Then $a \in I \vee J$. Hence $a = (x \vee y) \wedge a$ for some $x \in I$ and $y \in J$. Hence $(a] = ((x \vee y) \wedge a] = (x \vee y] \cap (a] = ((x] \vee (y]) \cap (a]$. Therefore $(a] \subseteq (x]) \vee (y]$, $(x] \in I^e$ *and* $(y] \in J^e$. Thus $(a] \in I^e \vee J^e$. Therefore $(I \vee J)^e \subseteq I^e \vee J^e$ and hence $(I \vee J)^e = I^e \vee J^e$ □

Now, we have the following theorem whose proof follows by Lemmas 5.1 and 5.2.

THEOREM 5.2. *Let L be an AL. Then the mapping $I \mapsto I^e$ is an isomorphism of the ideal lattice $\mathcal{I}(L)$ onto the ideal lattice $I(P\mathcal{I}(L))$. Moreover, this correspondence gives one-to-one correspondence between the prime ideals of L and those of $P\mathcal{I}(L)$.*

Recall that a maximal set M in an AL L is a maximal compatible set. Also, recall that an element $x \in L$ is said to be $M-$amicable if $a \wedge x = x$ for some $a \in M$. It follows that, if $x \in L$ is M-amicable, then there exists a unique element x^M in M with the property $x^M \wedge x = x$ and $x \wedge x^M = x^M$. Now, we prove the following.

THEOREM 5.3. *Let M be a maximal set in an AL L. Then the set $A_M(L)$, of all M-amicable elements of L is an ideal of L.*

PROOF. Suppose M is a maximal set in L. Then clearly $A_M(L)$ is non empty, since every element in M is M-amicable. Let $x, y \in A_M(L)$. Then there exists $x^M, y^M \in M$ such that $x^M \wedge x = x$ *and* $y^M \wedge y = y$. Since $x^M, y^M \in M$ and M is compatible set, $x^M \vee y^M,\ x^M \wedge y^M \in M$. Now, since $(x^M \vee y^M) \vee (x \vee y) = ((x^M \vee y^M) \vee (x \vee y)) \wedge ((x^M \vee y^M) \vee (x \vee y)) = (x^M \vee y^M) \wedge ((x^M \vee y^M) \vee (x \vee y)) = x^M \vee y^M$, by Theorem 2.1, we get $(x^M \vee y^M) \wedge (x \vee y) = x \vee y$. Hence $x \vee y \in A_M(L)$. Therefore $(x \vee y) \wedge x = x$ and $(x \vee y) \wedge y = y$. Now, if $a \in A_M(L)$ and $t \in L$, then there exists $m \in M$ such that $m \wedge a = a$. Consider, $a \wedge t = (m \wedge a) \wedge t = m \wedge (a \wedge t)$. Thus $a \wedge t \in A_M(L)$. Therefore $A_M(L)$ is an ideal of L. $\square$

Corollary 5.1. *If M is a maximal set in an AL L, then $A_M(L)$ is a sub AL of L.*

Finally, we prove that every amicable set in an AL L is isomorphic with the lattice $\mathcal{I}(L)$. For this, first we need the following.

Lemma 5.3. *Let M be an amicable set in an AL L and $x \in L$ be M-amicable. Then $(x] = (x^M]$.*

Proof. Suppose $x \in L$ is M-amicable. Then there exists a unique element $x^M \in M$ such that $x^M \wedge x = x$ and $x \wedge x^M = x^M$. It follows that, $(x] = (x^M]$. $\square$

Lemma 5.4. *Let M be an amicable set in an AL L. Then for any $x, y \in L$, the following are equivalent:*

(1) $(x] = (y]$
(2) $(x^M] = (y^M]$
(3) $x^M = y^M$.

Proof. Suppose M is an amicable set in L. Then $A_M(L) = L$.

(1) $\implies$ (2):- Assume (1). Let $x, y \in L = A_M(L)$ such that $(x] = (y]$. Then by Lemma 5.3, we have $(x] = (x^M]$ and $(y] = (y^M]$. Hence $(x^M] = (y^M]$.

(2) $\implies$ (3):-Assume (2). Since $x^M \in (x^M] = (y^M]$, we get $x^M = y^M \wedge x^M = x^M \wedge y^M$. Hence $x^M \leqslant y^M$. Similarly, we get $y^M \leqslant x^M$. Therefore $x^M = y^M$.

(3) $\implies$ (1):-Assume (3). We need to show that $(x] = (y]$. Let $t \in (x]$. Then $t = x \wedge t = (x^M \wedge x) \wedge t = (x \wedge x^M) \wedge t = x^M \wedge t = y^M \wedge t = (y \wedge y^M) \wedge t = (y^M \wedge y) \wedge t = y \wedge t \in (y]$. Hence $(x] \subseteq (y]$. Similarly, we can prove that $(y] \subseteq (x]$. Therefore $(x] = (y]$. $\square$

Now, we have the following corollary whose proof follows by Lemma 5.4 and Theorem 2.5.

Corollary 5.2. *Let M be a maximal set in an AL L. Then for any $x, y \in M$, the following are equivalent:*

(1) $x = y$
(2) $(x] = (y]$.

Recall that, if M is a maximal set in L and $a \in M$, then for any $x \in L$, $x \wedge a \in M$. Therefore we prove the following theorem.

Theorem 5.4. *Let M be an amicable set in an AL L.Then the mapping $x \mapsto (x]$ is an isomorphism of a lattice M onto the lattice $P\mathcal{I}(L)$.*

Proof. Let M be an amicable set in L. Define, $f : M \to P\mathcal{I}(L)$ by $f(x) = (x]$ for all $x \in M$. Then by the Corollary 5.2, f is both well defined and one-one. Let $(x] \in P\mathcal{I}(L)$. Then $x \in L = A_M(L)$. Therefore there exists $a \in M$ such that $a \wedge x = x$. Since $x \wedge a \in M$, we get $f(x \wedge a) = (x \wedge a] = (a \wedge x] = (x]$. Hence f is onto. Now, it remains to show that f is a homomorphism. Suppose $x, y \in M$. Then $f(x \wedge y) = (x \wedge y] = (x] \cap (y] = f(x) \cap f(y)$ and $f(x \vee y) = (x \vee y] = (x] \vee (y] = f(x) \vee f(y)$. Therefore f is a homomorphism and hence is an isomorphism. $\square$

References

[1] K. A. Baker and A. W. Hales. From a lattice to its ideal lattice. *Algebra Universalis*, **4**(1)(1974), 250–258.

[2] R. Freese. Ideal lattices of lattices. *Pacific J. Math.*, **57**(1)(1975), 125–133.

[3] G. N. Rao and T. A. Habtamu. Almost Lattices. *J. Int. Math. Virtual Inst.*, **9**(1)(2019), 155-171.

[4] G. N. Rao, T. A. Habtamu and G. B. Terefe. Amicable sets in almost lattices. *Bull. Int. Math. Virtual Inst.*, **9**(2)(2019), 319–331.

[5] G. N. Rao and G. B. Terefe. Ideals in almost semilattice. *Int. J. Math. Archive*, **7**(5)(2016), 60–70.

[6] U. M. Swamy and G. C. Rao. Almost distributive lattice. *J. Aust. Math. Soc.*, **31**(1)(1981), 77–91.

[7] G. Szasz. *Introduction to Lattice Theory* 3rd edition. Academic Press, New York and London, 1963.

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