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SOME NEW CLASSES OF MAPPINGS BETWEEN RELATIONAL SYSTEMS

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ABSTRACT. In this paper we describe some types of mappings between relational systems such as $1-$, $2-$ and $3-$ homomorphism between the relational systems. The analogous claims with the quasi-order relation obtained by some other authors are shown as the applications of the obtained results in this article.

1. Introduction

There are many possibilities of how to define homomorphisms between ordered sets. For information on such homomorphisms, the reader can see the following articles [**1, 2, 3, 4, 8**] in the literature. For notations and terminologies on posets not given in this paper, we rely on [**1, 2, 6, 7**]. It turns out that these results can be extended to arbitrary relational systems. This is the aim of this paper.

By a relational system we think that an ordered pair (X, R) consists of a set X and a binary relation R on X. In this paper, we introduce the notion of a 1-, 2- and 3-homomorphism between two relational systems. Some characterisations of such homomorphisms between relational systems are given.

2. Preliminaries

Let X and Y be arbitrary sets and $\mathfrak{J}(X, Y)$ be the family of all mappings from X to Y. For a relation $\varrho \subseteq Y \times Y$ and for a mapping $\varphi \in \mathfrak{J}(X, Y)$ we define relation $\varphi^{-1}(\varrho)$ in the following way

$$(x, y) \in \varphi^{-1}(\varrho) \Longleftrightarrow (\varphi(x), \varphi(y)) \in \varrho.$$

DEFINITION 2.1. Let (X, α) and (Y, β) be relational systems.

The mapping $\varphi \in \mathfrak{J}(X, Y)$ is called isotone with respect to relations α and β if $\alpha \subseteq \varphi^{-1}(\beta)$ holds.

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The mapping φ is reverse isotone with respect to relations α and β if $\varphi^{-1}(\beta) \subseteq \alpha$ holds.

Now we state some characterizations of the notion of homomorphism (= isotone mapping = relation-preserving mappings). We determine $Ker\varphi = \varphi^{-1} \circ \varphi$.

LEMMA 2.1 ([**1**], Lemma 3.4). *Let (A, α) and (B, β) be relational systems and $\varphi : X \longrightarrow Y$. Then the following are equivalent:*
(i) *φ is a isotone mapping from X to Y.*
(ii) *$\varphi^{-1}(\beta) \supseteq Ker\varphi \circ \alpha$.*
(iii) *$\varphi^{-1}(\beta) \supseteq \alpha \circ Ker\varphi$.*
(iv) *$\varphi^{-1}(\beta) \supseteq Ker\varphi \circ \alpha \circ Ker\varphi$.*

DEFINITION 2.2 ([**5**]). A reverse isotone mapping $\varphi : X \longrightarrow Y$ is called a reverse isotone strong mapping of X to Y if holds

$$\varphi^{-1}(\beta) \subseteq \alpha \subseteq Ker\varphi \circ \varphi^{-1}(\beta) \circ Ker\varphi;$$

A isotone mapping $\varphi : X \longrightarrow Y$ is called a isotone strong mapping of X to Y if holds

$$\alpha \subseteq \varphi^{-1}(\beta) \subseteq Ker\varphi \circ \alpha \circ Ker\varphi.$$

DEFINITION 2.3 ([**2**]). A mapping φ from relational system (X, α) into a relational system (Y, β) is called:

u-super strong if holds

$$\varphi^{-1}(\beta) \subseteq Ker\varphi \circ \alpha.$$

l-super strong if holds

$$\varphi^{-1}(\beta) \subseteq \alpha \circ Ker\varphi.$$

DEFINITION 2.4. Let (X, R) be a relational system and $\emptyset \neq A \subseteq X$. By $U_R(A)$ we denote the set $\{t \in X : (\exists a \in A)((a, t) \in R)\}$.

If $A = \{a\}$, then we write $U_R(a) = aR$ which is a left class of R generated by the element a. So, $U_R(A) = \bigcup_{a \in A} aR$.

Some others authors (see, for example [**1**]) determine an upper set $U_R(A)$ generated by a set A in the following way: $U_R(A) = \{x \in X : (\forall a \in A)((a, x) \in R)\}$. In that case $U_R(A) = \bigcap_{a \in A} aR$ holds.

3. The Main Results

Although some of the claims of homomorphism between relational systems are well known, here we them state and proving again for the consistency of the exposing material and in order to enable the reader to have a deeper insight into the proposed classification. Of course, for every such claim, it was pointed out where it was taken from.

THEOREM 3.1. *Let $\varphi : (X, \alpha) \longrightarrow (Y, \beta)$ be a mapping between two relational systems. Then the following are equivalent:*
(1) *φ is an isotone mapping;*
(2) $(\forall x \in X)(\varphi(U_\alpha(x)) \subseteq U_\beta(\varphi(x)))$;

(3) $(\forall x \in X)(U_\alpha((\varphi^{-1} \circ \varphi)(x)) \subseteq \varphi^{-1}(U_\beta(\varphi(x))))$;
(4) $(\forall x \in X)((\varphi^{-1} \circ \varphi)(U_\alpha((\varphi^{-1} \circ \varphi)(x))) \subseteq \varphi^{-1}(U_\beta(\varphi(x))))$.

PROOF. (1) $\Longrightarrow$ (2). ([**7**], Lemma 2.2) Suppose that φ is an isotone mapping and let $x \in X$ be an arbitrary element. Suppose $y \in \varphi(U_\alpha(x))$. Then there exists an element $x' \in U_\alpha(x)$, which means $(x, x') \in \alpha$, such that $y = \varphi(x')$. Since φ is isotone, we have $(\varphi(x), \varphi(x')) \in \beta$, which implies $y = \varphi(x') \in U_\beta(\varphi(x))$. Thus $\varphi(U_\alpha(x)) \subseteq U_\beta(\varphi(x))$.

(2) $\Longrightarrow$ (3) Let $x \in X$ be an arbitrary element and suppose that $\varphi(U_\alpha(x) \subseteq U_\beta(\varphi(x))$. Let $t \in U_\alpha(((\varphi^{-1} \circ \varphi)(x))$ be an arbitrary element. Then there exists $x' \in (\varphi^{-1} \circ \varphi)(x)$, i.e. $\varphi(x') = \varphi(x)$ such that $(x', t) \in \alpha$. This means $t \in U_\alpha(x')$. Thus $\varphi(t) \in \varphi(U_\alpha(x')) \subseteq U_\beta(\varphi(x))$. Hence $t \in \varphi^{-1}(U_\beta(\varphi(x)))$. Therefore, $U_\alpha((\varphi^{-1} \circ \varphi)(x)) \subseteq \varphi^{-1}(U_\beta(\varphi(x))$.

(3) $\Longrightarrow$ (1) Suppose that (3) is valid and let $x, x' \in X$ be elements such that $(x, x') \in \alpha$. Since $x \in \varphi^{-1}(\varphi(x))$ we have $x' \in U_\alpha(x) \subseteq U_\alpha((\varphi^{-1} \circ \varphi)(x)) \subseteq \varphi^{-1}(U_\beta(\varphi(x)))$. Hence $\varphi(x') \in U_\beta(\varphi(x))$ and $(\varphi(x), \varphi(x')) \in \beta$. Consequently $(x, x') \in \alpha \Longrightarrow (\varphi(x), \varphi(x')) \in \beta$. Therefore, φ is an isotone mapping.

(1) $\Longleftrightarrow$ (4). Let $x \in X$ be an arbitrary element. Suppose that the mapping φ is an isotone mapping and $t \in (\varphi^{-1} \circ \varphi)(U_\alpha((\varphi^{-1} \circ \varphi)(x))$ which means $\varphi(t) \in \varphi(U_\alpha((\varphi^{-1} \circ \varphi)(x))$. Then there exists an element $x' \in U_\alpha((\varphi^{-1} \circ \varphi)(x))$ such that $\varphi(t) = \varphi(x')$. Thus, there exists an element $x'' \in (\varphi^{-1} \circ \varphi)(x)$, i.e. $\varphi(x'') = \varphi(x)$, such that $x'', x') \in \alpha$. Since $(\varphi(x''), \varphi(x')) \in \beta$ because φ is an isotone mapping with respect to relations α and β, we have $\varphi(x') \in U_\beta(\varphi(x'')) = U_\beta(\varphi(x))$ and cosequently $x' \in \varphi^{-1}(U_\beta(\varphi(x)))$. So, we have $(\varphi^{-1} \circ \varphi)(U_\alpha((\varphi^{-1} \circ \varphi)(x))) \subseteq \varphi^{-1}(U_\beta(\varphi(x)))$.

Opposite. Let $x, x' \in X$ be elements such that $(x, x') \in \alpha$. Since $x \in (\varphi^{-1} \circ \varphi)(x)$ we have $x' \in U_\alpha((\varphi^{-1} \circ \varphi)(x)) \subseteq (\varphi^{-1} \circ \varphi)U_\alpha((\varphi^{-1} \circ \varphi)(x)) \subseteq \varphi^{-1}(U_\beta(\varphi(x)))$ by hypothesis. This implies $\varphi(x') \in U_\beta(x))$ and $(\varphi(x), \varphi(x')) \in \beta$. Finally, the mapping φ is an isotone mapping. □

COROLLARY 3.1 ([**3**], Lemma 2; [**2**], Lemma 1). *Let $\varphi : (X, \leqslant_X) \longrightarrow (Y, \leqslant_Y)$ be a mapping between two ordered sets under quasi-orders. Then the following are equivalent:*

(1') *φ is an isotone mapping;*
(2') $(\forall x \in X)(\varphi(U_{\leqslant_X}(x)) \subseteq U_{\leqslant_X}(\varphi(x)))$;
(3') $(\forall x \in X)(U_{\leqslant_X}((\varphi^{-1} \circ \varphi)(x)) \subseteq \varphi^{-1}(U_{\leqslant_Y}(\varphi(x))))$.
(4') $(\forall x \in X)((\varphi^{-1} \circ \varphi)(U_{\leqslant_X}((\varphi^{-1} \circ \varphi)(x))) \subseteq \varphi^{-1}(U_{\leqslant_Y}(\varphi(x))))$.

Let us note the following proposition

PROPOSITION 3.1. *Let $\varphi : (X, \alpha) \longrightarrow (Y, \beta)$ be a surjective mapping between two relational systems. Then the following are equivalent:*

(5) *φ is a reverse isotone mapping;*
(6) $(\forall x \in X)(U_\beta(\varphi(x)) \subseteq \varphi(U_\alpha(x)))$.

PROOF. (5) $\Longrightarrow$ (6) ([**7**], Lemma 2.1). Suppose that φ is a reverse isotone surjective mapping and let x be an arbitrary element of X. Suppose $y \in U_\beta(\varphi(x))$.

Then $(\varphi(x), y) \in \beta$. Since φ is a surjective mapping, there exists an element $t \in X$ such that $y = \varphi(t)$ and $(\varphi(x), \varphi(t)) \in \beta$. Hence $(x, t) \in \alpha$ because φ is a reverse isotone mapping. Thus $t \in U_\alpha(x)$ and $y = \varphi(t) \in \varphi(U_\alpha(x))$. Consequently, $U_\beta(\varphi(x)) \subseteq \varphi(U_\alpha(x))$.

(6) $\Longrightarrow$ (5). Opposite, suppose that the condition (6) is valid. Let $x, x' \in X$ be elements such that $(\varphi(x), \varphi(x')) \in \beta$. Thus $\varphi(x') \in U_\beta(\varphi(x)) \subseteq \varphi(U_\alpha(x))$. Hence $x' \in U_\alpha(x)$ and therefore $(x, x') \in \alpha$. So, the mapping φ is a reverse isotone mapping. □

Remark 3.1. (a) Let z be an arbitrary element and suppose that $U_\beta(\varphi(z)) \subseteq \varphi(U_\alpha(z))$. Suppose $t \in \varphi^{-1}(U_\beta(\varphi(x)))$. Then $\varphi(t) \in U_\beta(\varphi(x)) \subseteq \varphi(U_\alpha(x))$. Thus $t \in U_\alpha(x)$, which means $(x, t) \in \alpha$. Since $x \in (\varphi^{-1} \circ \varphi)(x)$, we have $t \in U_\alpha((\varphi^{-1} \circ \varphi)(x))$. Finally,

$$\varphi^{-1}(U_\beta(\varphi(x))) \subseteq U_\alpha((\varphi^{-1} \circ \varphi)(x)).$$

(b) Also, let us remember the Theorem 3.10 in [**6**]: Let (X, α) and (Y, β) be relational systems and let $\varphi : X \longrightarrow Y$ be a surjective reverse isotone mapping. Then

$$\varphi^{-1} \circ \beta \subseteq \alpha \circ \varphi^{-1}.$$

(c) Let φ be a reverse isotone surjective mapping between two relational systems. Suppose $t \in \varphi^{-1}(U_\beta(\varphi(x))$. This means $\varphi(t) \in U_\beta(\varphi(x))$ and $(\varphi(x), \varphi(t)) \in \beta$. Thus $(x, t) \in \alpha$ because the mapping φ is a reverse isotone mapping. Since $x \in (\varphi^{-1} \circ \varphi)(x)$, we have $t \in U_\alpha((\varphi^{-1} \circ \varphi)(x)) \subseteq (\varphi^{-1} \circ \varphi)(U_\alpha((\varphi^{-1} \circ \varphi)(x)))$. Finally, we have

$$\varphi^{-1}(U_\beta(\varphi(x))) \subseteq (\varphi^{-1} \circ \varphi)(U_\alpha((\varphi^{-1} \circ \varphi)(x))).$$

In the following theorem we give a connection between these three conditions.

Theorem 3.2. *Let $\varphi : (X, \alpha) \longrightarrow (Y, \beta)$ be a surjective mapping between two relational systems. Then the following are equivalent:*

(7) $(\forall x \in X)(\varphi^{-1}(U_\beta(\varphi(x))) \subseteq U_\alpha((\varphi^{-1} \circ \varphi)(x)))$;

(8) $\varphi^{-1}(\beta) \subseteq \alpha \circ (\varphi^{-1} \circ \varphi)$;

(9) $\varphi^{-1} \circ \beta \subseteq \alpha \circ \varphi^{-1}$;

(10) $(\forall x \in X)(\varphi^{-1}(U_\beta(\varphi(x))) \subseteq (\varphi^{-1} \circ \varphi)(U_\alpha((\varphi^{-1} \circ \varphi)(x))))$.

Proof. (7) $\Longrightarrow$ (8). Suppose that $\varphi^{-1}(U_\beta(\varphi(x))) \subseteq U_\alpha((\varphi^{-1} \circ \varphi)(x)$ holds for any $x \in X$. Let $x, x' \in X$ be arbitrary elements such that $(\varphi(x), \varphi(x')) \in \beta$. It is means $(x, x') \in \varphi^{-1}(\beta)$. Then $\varphi(x') \in U_\beta(\varphi(x))$ and $x' \in \varphi^{-1}(U_\beta(\varphi(x)))$. Thus $x' \in U_\alpha((\varphi^{-1} \circ \varphi)(x))$. Hence there exists an element $x'' \in X$ such that $x'' \in (\varphi^{-1} \circ \varphi)(x)$ and $(x'', x') \in \alpha$. Thus $\varphi^{-1}(\beta) \subseteq \alpha \circ Ker\varphi$.

(9) $\Longleftrightarrow$ (8) ([**6**], Theorem 3.7). Suppose that the condition (9) holds. Let $x, x' \in X$ be arbitrary elements such that $(x, x') \in \varphi^{-1}(\beta)$. Then $(\varphi(x), \varphi(x')) \in \beta$. Since $(\varphi(x'), x') \in \varphi^{-1}$ we have $(\varphi(x), x') \in \varphi^{-1} \circ \beta \subseteq \alpha \circ \varphi^{-1}$. Thus there exists an element $x'' \in X$ such that $(\varphi(x), x'') \in \varphi^{-1}$ and $(x'', x') \in \alpha$. This means $(x, x'') \in \varphi^{-1} \circ \varphi$ and $(x'', x') \in \alpha$. So, $(x, x') \in \alpha \circ (\varphi^{-1} \circ \varphi)$. Therefore $\varphi^{-1}(\beta) \subseteq \alpha \circ (\varphi^{-1} \circ \varphi)$.

Opposite, suppose that the condition (8) holds and let $(y, x') \in \varphi^{-1} \circ \beta$. Then there exists an element $y' \in X$ such that $(y, y') \in \beta$ and $(y', x') \in \varphi^{-1}$. Since φ is a surjective mapping, there exists an element $x \in X$ such that $(x, y) \in \varphi$. So $(x, x') \in \varphi^{-1}(\beta) \subseteq \alpha \circ Ker\varphi$. Hence, there exists an element $x'' \in X$ such that $(x, x'') \in \varphi^{-1} \circ \varphi$ and $(x'', x') \in \alpha$. Now, we have $(y, x'') \in \varphi$ and $(x'', x') \in \alpha$. Finally, $(y, x') \in \alpha \circ \varphi^{-1}$.

(9) $\Longrightarrow$ (7). Let the condition (8) is valid and let $t \in \varphi^{-1}(U_\beta(\varphi(x)))$. Then $\varphi(t) \in U_\beta(\varphi(x))$ and $(\varphi(x), \varphi(t)) \in \beta$. Since $(\varphi(t), t) \in \varphi^{-1}$ we have $(\varphi(x), t) \in \varphi^{-1} \circ \beta \subseteq \alpha \circ \varphi^{-1}$. By determination of the composition, there exists an element $x' \in X$ such that $(\varphi(x), x') \in \varphi^{-1}$ and $(x', t) \in \alpha$. So, $x' \in (\varphi^{-1} \circ \varphi)(x)$ and $(x', t) \in \alpha$. Therefore, $t \in U_\alpha((\varphi^{-1} \circ \varphi)(x))$.

(7) $\Longrightarrow$ (10). Obviously.

(10) $\Longrightarrow$ (8). Suppose that the condition (10) holds. Let $x, x' \in X$ be elements such that $(x, x') \in \varphi^{-1}(\beta)$, i.e. $(\varphi(x), \varphi(x')) \in \beta$. Thus $\varphi(x') \in U_\beta(\varphi(x))$ and $x' \in \varphi^{-1}(U_\beta(\varphi(x)) \subseteq (\varphi^{-1} \circ \varphi)(U_\alpha((\varphi^{-1} \circ \varphi)(x)))$. Hence $\varphi(x') \in \varphi(U_\alpha((\varphi^{-1} \circ \varphi)(x)))$. This means $x' \in U_\alpha((\varphi^{-1} \circ \varphi)(x))$. So, there exists an element $x'' \in (\varphi^{-1} \circ \varphi)(x)$ such that $(x'', x') \in \alpha$. Finally, we have $(x, x') \in \alpha \circ Ker\varphi$. Therefore, $\varphi^{-1}(\beta) \subseteq \alpha \circ (\varphi^{-1} \circ \varphi)$. □

Corollary 3.2. *Let $\varphi : (X, \leqslant_X) \longrightarrow (Y, \leqslant_Y)$ be a surjective mapping between two ordered sets under quasi-orders. Then the following are equivalent:*

(7') $(\forall x \in X)(\varphi^{-1}(U_{\leqslant_Y}(\varphi(x))) \subseteq U_{\leqslant_X}((\varphi^{-1} \circ \varphi)(x)))$.

(8') $\varphi^{-1}(\leqslant_Y) \subseteq \leqslant_X \circ(\varphi^{-1} \circ \varphi)$

(9') $\varphi^{-1}\circ \leqslant_Y \subseteq \leqslant_X \circ\varphi^{-1}$

(10') $(\forall x \in X)(\varphi^{-1}(U_{\leqslant_Y}(\varphi(x))) \subseteq (\varphi^{-1} \circ \varphi)(U_{\leqslant_X}((\varphi^{-1} \circ \varphi)(x))))$.

Following the ideas of Definition 1 in [**3**], Definition 1 in [**2**] and relying on our statements (3), (4), (7) and (10) in this article, we introduce the following notions.

Definition 3.1. Let $X, \alpha)$ and (Y, β) be relational systems. A mapping $\varphi : X \longrightarrow Y$ is called

- a 1-homomorphism if it satisfies the condition

$$(\forall x \in X)(\varphi^{-1}(U_\beta(\varphi(x))) = (\varphi^{-1} \circ \varphi)(U_\alpha((\varphi^{-1} \circ \varphi)(x))));$$

- a 2-homomorphism if it satisfies the condition

$$(\forall x \in X)(\varphi^{-1}(U_\beta(\varphi(x))) = \varphi^{-1}(\varphi(U_\alpha(x))));$$

- a 3-homomorphism if it satisfies the condition

$$(\forall x \in X)(\varphi^{-1}(U_\beta(\varphi(x))) = U_\alpha((\varphi^{-1} \circ \varphi)(x))).$$

In the following theorems we give description on $1-$, $2-$ and $3-$homomorphism between two relational systems.

Theorem 3.3. *Let (X, α) and (Y, β) be relational systems and $\varphi : X \longrightarrow Y$ be a mapping. Then the mapping φ is a 1-homomorphism if and only if φ is an isotone strong mapping.*

Proof. ($\Longrightarrow$) Suppose that φ is a 1-mapping. Then φ is an isotone mapping by condition (4) of Theorem 1. Let $x, x' \in X$ such that $(x, x') \in \varphi^{-1}(\beta)$, i.e. $(\varphi(x), \varphi(x')) \in \beta$. Thus $\varphi(x') \in U_\beta(\varphi(x))$ and $x' \in \varphi^{-1}(U_\beta(\varphi(x)))$. By hypothesis, we have $x' \in (\varphi^{-1} \circ \varphi)(U_\beta((\varphi^{-1} \circ \varphi)(x)))$ and $\varphi(x') \in \varphi(U_\beta(\varphi^{-1} \circ \varphi)(x)))$. Hence, there exists an element $x'' \in U_\beta((\varphi^{-1} \circ \varphi)(x))$ such that $\varphi(x') = \varphi(x'')$. Further on, there exists an element $x''' \in (\varphi^{-1} \circ \varphi)(x)$ such that $(x''', x'') \in \alpha$. Finally, we have $(x, x''') \in Ker\varphi$, $(x''', x'') \in \alpha$ and $(x'', x') \in Ker\varphi$. Therefore, $(x, x') \in Ker\varphi \circ \alpha \circ Ker\varphi$. So, the mapping φ is an isotone strong mapping.

($\Longleftarrow$) Suppose that φ is an isotone strong mapping. Since φ is isotone, the inclusion (4) holds. Let $t \in \varphi^{-1}(U_\beta(\varphi(x))$ be an arbitrary element. Then $\varphi(t) \in U_\beta(\varphi(x))$ and $(\varphi(x), \varphi(t)) \in \beta$. Thus $(x, t) \in \varphi^{-1}(\beta) \subseteq Ker\varphi \circ \alpha \circ Ker\varphi$. So, there exist elements $x;, x'' \in X$ such that $x' \in (\varphi^{-1} \circ \varphi)(x)$, $(x', x'') \in \alpha$ and $t \in (\varphi^{-1} \circ \varphi)(x'')$. Finally, we have $t \in (\varphi^{-1} \circ \varphi)(U_\alpha(\varphi^{-1} \circ \varphi)((x)))$. □

Corollary 3.3 ([**2**], Proposition 3). *Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be ordered sets under quasi-orders and $\varphi : X \longrightarrow Y$ be a mapping. Then the mapping φ is a 1-homomorphism if and only if φ is an isotone strong mapping.*

Theorem 3.4. *Let (X, α) and (Y, β) be relational systems and $\varphi : X \longrightarrow Y$ be a mapping. Then the mapping φ is a 2-homomorphism if and only if φ is an isotone u-super strong mapping.*

Proof. ($\Longrightarrow$) Suppose that $\varphi : X \longrightarrow Y$ is a 2-homomorphism, i.e. suppose that $\varphi^{-1}(U_\beta(\varphi(x))) = \varphi^{-1}(\varphi(U_\alpha(x)))$ holds. Let $(x, x') \in \varphi^{-1}(\beta)$ be arbitrary element. Then $(\varphi(x), \varphi(x')) \in \beta$ and $\varphi(x') \in U_\beta(\varphi(x))$. Thus $x' \in \varphi^{-1}(U_\beta(\varphi(x))) = \varphi^{-1}(\varphi(U_\alpha(x)))$. So, there exists an element $y \in Y$ such that $x' = \varphi^{-1}(y)$ and $y \in \varphi(U_\alpha(x))$. Further on, there exists an element $x'' \in U_\alpha(x))$ such that $y = \varphi(x'')$ and $(x, x'') \in \alpha$. Thus $(x', x'') \in Ker\varphi$ and $(x, x') \in Ker\varphi \circ \alpha$.

Suppose $x, x' \in X$ are arbitrary elements such that $(x, x') \in \alpha$. Then $x' \in U_\alpha(x)$, which implies $\varphi(x') \in \varphi(U_\alpha(x))$ and $x' \in (\varphi^{-1} \circ \varphi)(x) \subseteq \varphi^{-1}(\varphi(U_\alpha(x))) = \varphi^{-1}(U_\beta(\varphi(x))$. Hence, $\varphi(x') \in U_\beta(\varphi(x))$, which means $(\varphi(x), \varphi(x')) \in \beta$. So, the mapping φ is an isotone mapping.

Therefore, φ is an u-super strong homomorphism.

($\Longleftarrow$) Suppose that $\varphi : X \longrightarrow Y$ is a 2$-$super strong homomorphism. Let $x' \in \varphi^{-1}(U_\beta(\varphi(x)))$, which implies $\varphi(x') \in U_\beta(\varphi(x))$. Thus $(\varphi(x), \varphi(x')) \in \beta$ and $(x, x') \in \varphi^{-1}(\beta) \subseteq Ker\varphi \circ \alpha$. So, there exists an element $x'' \in X$ such that $(x, x'') \in \alpha$ and $\varphi(x'') = \varphi(x')$. Hence $x'' \in U_\alpha(x)$, which implies $\varphi 9x'') \in \varphi(U_\alpha(x))$. Finally $x' \in \varphi^{-1}(\varphi(U_\alpha(x)))$. The converse inclusion follows from the condition (2) in Theorem 1. So, we have $\varphi^{-1}(U_\beta(\varphi(x)) = \varphi^{-1}(\varphi(U_\alpha(x)))$. □

Corollary 3.4 ([**2**], Proposition 4). *Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be ordered sets under quasi-orders and $\varphi : X \longrightarrow Y$ be a mapping. Then the mapping φ is a 2-homomorphism if and only if φ is an isotone u-super strong mapping.*

Theorem 3.5. *Let (X, α) and (Y, β) be relational systems and $\varphi : X \longrightarrow Y$ be a mapping. Then the mapping φ is a 3-homomorphism if and only if φ is an isotone l-super strong mapping.*

PROOF. ($\Longrightarrow$) Let φ be a $3-$homomorphism, i.e. suppose that

$$\varphi^{-1}(U_\beta(\varphi(x))) = U_\alpha((\varphi^{-1} \circ \varphi)(x))$$

holds.

Suppose $(x, x') \in \alpha$ for $x, x' \in X$. Since $x \in (\varphi^{-1} \circ \varphi)(x)$, we have $x' \in U_\alpha(x) \subseteq U_\alpha((\varphi^{-1} \circ \varphi)(x)) = \varphi^{-1}(U_\beta(\varphi(x))$. Hence $\varphi(x') \in U_\beta(\varphi(x))$ and $(\varphi(x), \varphi(x')) \in \beta$. So, the mapping φ is an isotone mapping.

Let $(x, x') \in \varphi^{-1}(\beta)$ be arbitrary element. This means $(\varphi(x), \varphi(x')) \in \beta$. Thus $\varphi(x') \in U_\beta(\varphi(x))$ and $x' \in \varphi^{-1}(U_\beta(\varphi(x))) = U_\alpha((\varphi^{-1} \circ \varphi)(x))$. Hence, there exists an element $x'' \in (\varphi^{-1} \circ \varphi)(x)$ such that $(x'', x') \in \alpha$. Therefore, $(x, x') \in \alpha \circ Ker\varphi$. So, the φ is a $l-$super strong homomorphism.

($\Longleftarrow$) Suppose that $\varphi : X \longrightarrow Y$ is an $l-$super strong homomorphism between two relational systems. Since φ is a homomorphism, then, by condition (3) in Theorem 1, the inclusion $U_\alpha((\varphi^{-1} \circ \varphi)(x)) \subseteq \varphi^{-1}(U_\beta(\varphi(x)))$ holds. Opposite, let $x' \in \varphi^{-1}(U_\beta(\varphi(x)))$, i.e. let $\varphi(x) \in U_\beta(\varphi(x))$ holds. Then $(\varphi(x), \varphi(x')) \in \beta$ and $(x, x') \in \varphi^{-1}(\beta) \subseteq \alpha \circ Ker\varphi$. So, there exists an element $x'' \in X$ such that $x'' \in (\varphi^{-1} \circ \varphi)(x)$ and $(x'', x') \in \alpha$. Thus $x' \in U_\alpha((\varphi^{-1} \circ \varphi(x))$. Therefore, the mapping φ is an isotone $3-$homomorphism. □

COROLLARY 3.5 ([**2**], Proposition 5). *Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be ordered sets under quasi-orders and $\varphi : X \longrightarrow Y$ be a mapping. Then the mapping φ is a 3-homomorphism if and only if φ is an isotone l-super strong mapping.*

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