INSERTION OF A CONTRA-BAIRE-1 (BAIRE-.5) FUNCTION BETWEEN TWO COMPARABLE REAL-VALUED FUNCTIONS

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Abstract. A necessary and sufficient condition in terms of lower cut sets are given for the insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that $F_\alpha$—kernel of sets are $F_\alpha$—sets.

1. Introduction

A generalized class of closed sets was considered by Maki in 1986 [16]. He investigated the sets that can be represented as union of closed sets and called them $V$—sets. Complements of $V$—sets, i.e., sets that are intersection of open sets are called $A$—sets [16].

Recall that a real-valued function $f$ defined on a topological space $X$ is called $A$—continuous [22] if the preimage of every open subset of $\mathbb{R}$ belongs to $A$, where $A$ is a collection of subsets of $X$. Most of the definitions of function used throughout this paper are consequences of the definition of $A$—continuity. However, for unknown concepts the reader may refer to [4, 10]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [5] introduced a new class of mappings called contra-continuity. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [1, 3, 7, 8, 9, 11, 12, 21].

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Results of Katětov [13, 14] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [2], are used in order to give a necessary and sufficient condition for the insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that \( F_\sigma - \) kernel of sets are \( F_\sigma - \) sets.

A real-valued function \( f \) defined on a topological space \( X \) is called contra-Baire-1 (Baire-.5) if the preimage of every open subset of \( \mathbb{R} \) is a \( G_\delta \) set in \( X \) [23].

If \( g \) and \( f \) are real-valued functions defined on a space \( X \), we write \( g \leq f \) (resp. \( g < f \)) in case \( g(x) \leq f(x) \) (resp. \( g(x) < f(x) \)) for all \( x \) in \( X \).

The following definitions are modifications of conditions considered in [15].

A property \( P \) defined relative to a real-valued function on a topological space is a \( B \) property provided that any constant function has property \( P \) and provided that the sum of a function with property \( P \) and any Baire-.5 function also has property \( P \). If \( P_1 \) and \( P_2 \) are \( B \) properties, the following terminology is used:

(i) A space \( X \) has the weak \( B - .5 \) insertion property for \( (P_1, P_2) \) if and only if for any functions \( g \) and \( f \) on \( X \) such that \( g \leq f, g \) has property \( P_1 \) and \( f \) has property \( P_2 \), then there exists a Baire-.5 function \( h \) such that \( g \leq h \leq f \).

(ii) A space \( X \) has the \( B - .5 \) insertion property for \( (P_1, P_2) \) if and only if for any functions \( g \) and \( f \) on \( X \) such that \( g < f, g \) has property \( P_1 \) and \( f \) has property \( P_2 \), then there exists a Baire-.5 function \( h \) such that \( g < h < f \).

In this paper, for a topological space that \( F_\sigma \) kernel of sets are \( F_\sigma \) sets, is given a sufficient condition for the weak \( B - .5 \) insertion property. Also for a space with the weak \( B - .5 \) insertion property, we give a necessary and sufficient condition for the space to have the \( B - .5 \) insertion property. Several insertion theorems are obtained as corollaries of these results. In addition, the insertion and weak insertion of a contra-continuous function between two comparable real-valued functions has also recently considered by the authors in [18, 19].

2. The Main Results

Before giving a sufficient condition for insertability of a Baire-.5 function, the necessary definitions and terminology are stated.

**Definition 2.1.** Let \( A \) be a subset of a topological space \( (X, \tau) \). We define the subsets \( A^A \) and \( A^V \) as follows:

\[
A^A = \cap \{O : O \supseteq A, O \in (X, \tau)\} \text{ and } A^V = \cup \{F : F \subseteq A, F^c \in (X, \tau)\}.
\]

In [6, 17, 20], \( A^A \) is called the kernel of \( A \).

We define the subsets \( G_\delta(A) \) and \( F_\sigma(A) \) as follows:

\[
G_\delta(A) = \cup \{O : O \subseteq A, O \text{ is } G_\delta \text{ set}\} \text{ and } F_\sigma(A) = \cap \{F : F \supseteq A, F \text{ is } F_\sigma \text{ set}\}.
\]

\( F_\sigma(A) \) is called the \( F_\sigma - \) kernel of \( A \). The following first two definitions are modifications of conditions considered in [13, 14].
DEFINITION 2.2. If \( \rho \) is a binary relation in a set \( S \) then \( \bar{\rho} \) is defined as follows: \( x \, \bar{\rho} \, y \) if and only if \( y \, \rho \, v \) implies \( x \, \rho \, v \) and \( u \, \rho \, x \) implies \( u \, \rho \, y \) for any \( u \) and \( v \) in \( S \).

DEFINITION 2.3. A binary relation \( \rho \) in the power set \( P(X) \) of a topological space \( X \) is called a \textit{strong binary relation} in \( P(X) \) in case \( \rho \) satisfies each of the following conditions:

1) If \( A_i \, \rho \, B_j \) for any \( i \in \{1, \ldots, m\} \) and for any \( j \in \{1, \ldots, n\} \), then there exists a set \( C \) in \( P(X) \) such that \( A_i \, \rho \, C \) and \( C \, \rho \, B_j \) for any \( i \in \{1, \ldots, m\} \) and any \( j \in \{1, \ldots, n\} \).

2) If \( A \subseteq B \), then \( A \, \rho \, B \).

3) If \( A \, \rho \, B \), then \( F_\rho(A) \subseteq B \) and \( A \subseteq G_\rho(B) \).

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks \[2\] as follows:

DEFINITION 2.4. If \( f \) is a real-valued function defined on a space \( X \) and if \( \{ x \in X : f(x) < \ell \} \subseteq A(f, \ell) \subseteq \{ x \in X : f(x) \leq \ell \} \) for a real number \( \ell \), then \( A(f, \ell) \) is a lower indefinite cut set in the domain of \( f \) at the level \( \ell \).

We now give the following main results:

**Theorem 2.1.** Let \( g \) and \( f \) be real-valued functions on the topological space \( X \), that \( F_\rho \)-kernel of sets in \( X \) are \( F_\rho \)-sets, with \( g \leq f \). If there exists a strong binary relation \( \rho \) on the power set \( X \) and if there exist lower indefinite cut sets \( A(f, t) \) and \( A(g, t) \) in the domain of \( f \) and \( g \) at the level \( t \) for each rational number \( t \) such that if \( t_1 < t_2 \) then \( A(f, t_1) \rho A(g, t_2) \), then there exists a Baire-.5 function \( h \) defined on \( X \) such that \( g \leq h \leq f \).

**Proof.** Let \( g \) and \( f \) be real-valued functions defined on the \( X \) such that \( g \leq f \). By hypothesis there exists a strong binary relation \( \rho \) on the power set of \( X \) and there exist lower indefinite cut sets \( A(f, t) \) and \( A(g, t) \) in the domain of \( f \) and \( g \) at the level \( t \) for each rational number \( t \) such that if \( t_1 < t_2 \) then \( A(f, t_1) \rho A(g, t_2) \).

Define functions \( F \) and \( G \) mapping the rational numbers \( \mathbb{Q} \) into the power set of \( X \) by \( F(t) = A(f, t) \) and \( G(t) = A(g, t) \). If \( t_1 \) and \( t_2 \) are any elements of \( \mathbb{Q} \) with \( t_1 < t_2 \), then \( F(t_1) \rho F(t_2) \), \( G(t_1) \rho G(t_2) \), and \( F(t_1) \rho G(t_2) \). By Lemmas 1 and 2 of \[14\] it follows that there exists a function \( H \) mapping \( \mathbb{Q} \) into the power set of \( X \) such that if \( t_1 \) and \( t_2 \) are any rational numbers with \( t_1 < t_2 \), then \( F(t_1) \rho H(t_2) \), \( H(t_1) \rho H(t_2) \), and \( F(t_1) \rho G(t_2) \).

For any \( x \) in \( X \), let \( h(x) = \inf \{ t \in \mathbb{Q} : x \in H(t) \} \).

We first verify that \( g \leq h \leq f \). If \( x \) is in \( H(t) \) then \( x \) is in \( G(t') \) for any \( t' > t \); since \( x \) in \( G(t') = A(g, t') \) implies that \( g(x) \leq t' \), it follows that \( g(x) \leq t \). Hence \( g \leq h \). If \( x \) is not in \( H(t) \), then \( x \) is not in \( F(t') \) for any \( t' < t \); since \( x \) is not in \( F(t') = A(f, t') \) implies that \( f(x) > t' \), it follows that \( f(x) \geq t \). Hence \( h \leq f \).

Also, for any rational numbers \( t_1 \) and \( t_2 \) with \( t_1 < t_2 \), we have \( h^{-1}(t_1, t_2) = G_\delta(H(t_2)) \setminus F_\rho(H(t_1)) \). Hence \( h^{-1}(t_1, t_2) \) is a \( G_\delta \)-set in \( X \), i.e., \( h \) is a Baire-.5 function on \( X \).

The above proof used the technique of Theorem 1 of \[13\].
Theorem 2.2. Let $P_1$ and $P_2$ be $B-\frac{1}{2}$-property and $X$ be a space that satisfies the weak $B-\frac{1}{2}$-insertion property for $(P_1, P_2)$. Also assume that $g$ and $f$ are functions on $X$ such that $g < f, g$ has property $P_1$ and $f$ has property $P_2$. The space $X$ has the $B-\frac{1}{2}$-insertion property for $(P_1, P_2)$ if and only if there exist lower cut sets $A(f - g, 3^{-n+1})$ and there exists a decreasing sequence $\{D_n\}$ of subsets of $X$ with empty intersection and such that for each $n$, $X \setminus D_n$ and $A(f - g, 3^{-n+1})$ are completely separated by Baire-$\frac{1}{2}$ functions.

Proof. Assume that $X$ has the weak $B-\frac{1}{2}$-insertion property for $(P_1, P_2)$. Let $g$ and $f$ be functions such that $g < f, g$ has property $P_1$ and $f$ has property $P_2$. By hypothesis there exist lower cut sets $A(f - g, 3^{-n+1})$ and there exists a sequence $(D_n)$ such that $\bigcap_{n=1}^{\infty} D_n = \emptyset$ and such that for each $n, X \setminus D_n$ and $A(f - g, 3^{-n+1})$ are completely separated by Baire-$\frac{1}{2}$ functions. Let $k_n$ be a Baire-$\frac{1}{2}$ function such that $k_n = 0$ on $A(f - g, 3^{-n+1})$ and $k_n = 1$ on $X \setminus D_n$. Let a function $k$ on $X$ be defined by

$$k(x) = 1/2 \sum_{n=1}^{\infty} 3^{-n} k_n(x) .$$

By the Cauchy condition and the $B-\frac{1}{2}$-properties, the function $k$ is a Baire-$\frac{1}{2}$ function. Since $\bigcap_{n=1}^{\infty} D_n = \emptyset$ and since $k_n = 1$ on $X \setminus D_n$, it follows that $0 < k$. Also $2k < f - g$: In order to see this, observe first that if $x$ is in $A(f - g, 3^{-n+1})$, then $k(x) \leq 1/4(3^{-n})$. If $x$ is any point in $X$, then $x \notin A(f - g, 1)$ or for some $n$,

$$x \in A(f - g, 3^{-n+1}) - A(f - g, 3^{-n});$$

in the former case $2k(x) < 1$, and in the latter $2k(x) \leq 1/2(3^{-n}) < f(x) - g(x)$. Thus if $f_1 = f - k$ and if $g_1 = g + k$, then $g < g_1 < f_1 < f$. Since $P_1$ and $P_2$ are $B-\frac{1}{2}$-properties, then $g_1$ has property $P_1$ and $f_1$ has property $P_2$. Since $X$ has the weak $B-\frac{1}{2}$-insertion property for $(P_1, P_2)$, then there exists a Baire-$\frac{1}{2}$ function such that $g_1 \leq h \leq f_1$. Thus $g < h < f$, it follows that $X$ satisfies the $B-\frac{1}{2}$-insertion property for $(P_1, P_2)$. (The technique of this proof is by Katětov [13]).

Conversely, let $g$ and $f$ be functions on $X$ such that $g$ has property $P_1, f$ has property $P_2$ and $g < f$. By hypothesis, there exists a Baire-$\frac{1}{2}$ function such that $g < h < f$. We follow an idea contained in Lane [15]. Since the constant function 0 has property $P_1$, since $f - h$ has property $P_2$, and since $X$ has the $B-\frac{1}{2}$-insertion property for $(P_1, P_2)$, then there exists a Baire-$\frac{1}{2}$ function $k$ such that $0 < k < f - h$. Let $A(f - g, 3^{-n+1})$ be any lower cut set for $f - g$ and let $D_n = \{x \in X : k(x) < 3^{-n+2}\}$. Since $k > 0$ it follows that $\bigcap_{n=1}^{\infty} D_n = \emptyset$. Since

$$A(f - g, 3^{-n+1}) \subseteq \{x \in X : (f - g)(x) \leq 3^{-n+1}\} \subseteq \{x \in X : k(x) \leq 3^{-n+1}\}$$

and since $\{x \in X : k(x) \leq 3^{-n+1}\}$ and $\{x \in X : k(x) \geq 3^{-n+2}\} = X \setminus D_n$ are completely separated by Baire-$\frac{1}{2}$ function $sup\{3^{-n+1}, inf\{k, 3^{-n+2}\}\}$, it follows that for each $n, A(f - g, 3^{-n+1})$ and $X \setminus D_n$ are completely separated by Baire-$\frac{1}{2}$ functions.

$\Box$
3. Applications

Definition 3.1. A real-valued function $f$ defined on a space $X$ is called contra-upper semi-Baire-.5 (resp. contra-lower semi-Baire-.5) if $f^{-1}(-\infty, t)$ (resp. $f^{-1}(t, +\infty)$) is a $G_\delta$-set for any real number $t$.

The abbreviations $usc, lsc, cusB.5$ and $clsB.5$ are used for upper semicontinuous, lower semicontinuous, contra-upper semi-Baire-.5, and contra-lower semi-Baire-.5, respectively.

Remark 3.1. ([13, 14]) A space $X$ has the weak $c-$insertion property for $(usc, lsc)$ if and only if $X$ is normal.

Before stating the consequences of Theorem 2.1, and Theorem 2.2 we suppose that $X$ is a topological space that $F_\sigma-$kernel of sets are $F_\sigma-$sets.

Corollary 3.1. For each pair of disjoint $F_\sigma-$sets $F_1, F_2$, there are two $G_\delta-$sets $G_1$ and $G_2$ such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ if and only if $X$ has the weak $B - .5-$insertion property for $(cusB - .5, clsB - .5)$.

Proof. Let $g$ and $f$ be real-valued functions defined on the $X$, such that $f$ is $lsB_1, g$ is $usB_1$, and $g \leq f$. If a binary relation $\rho$ is defined by $A \rho B$ in case $F_\sigma(A) \subseteq G_\delta(B)$, then by hypothesis $\rho$ is a strong binary relation in the power set of $X$. If $t_1$ and $t_2$ are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f,t_1) \subseteq \{x \in X : f(x) \leq t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g,t_2);$$

since $\{x \in X : f(x) \leq t_1\}$ is a $F_\sigma-$set and since $\{x \in X : g(x) < t_2\}$ is a $G_\delta-$set, it follows that $F_\sigma(A(f,t_1)) \subseteq G_\delta(A(g,t_2))$. Hence $t_1 < t_2$ implies that $A(f,t_1) \rho A(g,t_2)$. The proof follows from Theorem 2.1.

On the other hand, let $F_1$ and $F_2$ are disjoint $F_\sigma-$sets. Set $f = \chi_{F_1}$ and $g = \chi_{F_2}$, then $f$ is $clsB - .5$, $g$ is $cusB - .5$, and $g \leq f$. Thus there exists Baire-.5 function $h$ such that $g \leq h \leq f$. Set $G_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $G_2 = \{x \in X : h(x) > \frac{1}{2}\}$, then $G_1$ and $G_2$ are disjoint $G_\delta-$sets such that $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$. 

Remark 3.2. ([24]) A space $X$ has the weak $c-$insertion property for $(lsc, usc)$ if and only if $X$ is extremally disconnected.

Corollary 3.2. For every $G$ of $G_\delta-$set, $F_\sigma(G)$ is a $G_\delta-$set if and only if $X$ has the weak $B - .5-$insertion property for $(clsB - .5, cusB - .5)$.

Before giving the proof of this corollary, the necessary lemma is stated.

Lemma 3.1. The following conditions on the space $X$ are equivalent:

(i) For every $G$ of $G_\delta-$set we have $F_\sigma(G)$ is a $G_\delta-$set.

(ii) For each pair of disjoint $G_\delta-$sets as $G_1$ and $G_2$ we have $F_\sigma(G_1) \cap F_\sigma(G_2) = \emptyset$.

Proof. The proof of Lemma 3.1 is a direct consequence of the definition $F_\sigma-$kernel of sets. 

We now give the proof of corollary 3.2.

**Proof.** Let $g$ and $f$ be real-valued functions defined on the $X$, such that $f$ is $\text{cls}B-1/2$, $g$ is $\text{cus}B-1/2$, and $f \leq g$. If a binary relation $p$ is defined by $A \rho B$ in case $F_p(A) \subseteq G \subseteq F_p(B)$ for some $G_\delta$-set $g$ in $X$, then by hypothesis and Lemma 3.1, $\rho$ is a strong binary relation in the power set of $X$. If $t_1$ and $t_2$ are any elements of $Q$ with $t_1 < t_2$, then

$$A(g, t_1) = \{x \in X : g(x) < t_1\} \subseteq \{x \in X : f(x) \leq t_2\} = A(f, t_2);$$

since $\{x \in X : g(x) < t_1\}$ is a $G_\delta$-set and since $\{x \in X : f(x) \leq t_2\}$ is a $F_\sigma$-set, by hypothesis it follows that $A(g, t_1) \rho A(f, t_2)$. The proof follows from Theorem 2.1.

On the other hand, Let $G_1$ and $G_2$ are disjoint $G_\delta$-sets. Set $f = \chi_{G_2}$ and $g = \chi_{G_1}$, then $f$ is $\text{cls}B-1/2$, $g$ is $\text{cus}B-1/2$, and $f \leq g$.

Thus there exists Baire-$1/2$ function $h$ such that $f \leq h \leq g$. Set $F_1 = \{x \in X : h(x) \leq 1/2\}$ and $F_2 = \{x \in X : h(x) \geq 2/3\}$ then $F_1$ and $F_2$ are disjoint $F_\sigma$-sets such that $G_1 \subseteq F_1$ and $G_2 \subseteq F_2$. Hence $F_\sigma(F_1) \cap F_\sigma(F_2) = \emptyset$. □

Before stating the consequences of Theorem 2.2, we state and prove the necessary lemmas.

**Lemma 3.2.** The following conditions on the space $X$ are equivalent:

(i) Every two disjoint $F_\sigma$-sets of $X$ can be separated by $G_\delta$-sets of $X$.

(ii) If $F$ is a $F_\sigma$-set of $X$ which is contained in a $G_\delta$-set $G$, then there exists a $G_\delta$-set $H$ such that $F \subseteq H \subseteq F_\sigma(H) \subseteq G$.

**Proof.** (i) $\Rightarrow$ (ii) Suppose that $F \subseteq G$, where $F$ and $G$ are $F_\sigma$-set and $G_\delta$-set of $X$, respectively. Hence, $G^c$ is a $F_\delta$-set and $F \cap G^c = \emptyset$.

By (i) there exists two disjoint $G_\delta$-sets $G_1$, $G_2$ such that $F \subseteq G_1$ and $G^c \subseteq G_2$.

But

$$G^c \subseteq G_2 \Rightarrow G_2^c \subseteq G,$$

and

$$G_1 \cap G_2 = \emptyset \Rightarrow G_1 \subseteq G^c_2$$

hence

$$F \subseteq G_1 \subseteq G^c_2 \subseteq G$$

and since $G^c_2$ is a $F_\sigma$-set containing $G_1$ we conclude that $F_\sigma(G_1) \subseteq G^c_2$, i.e.,

$$F \subseteq G_1 \subseteq F_\sigma(G_1) \subseteq G.$$

By setting $H = G_1$, condition (ii) holds.

(ii) $\Rightarrow$ (i) Suppose that $F_1$, $F_2$ are two disjoint $F_\sigma$-sets of $X$.

This implies that $F_1 \subseteq F_1^c$ and $F_2$ is a $G_\delta$-set. Hence by (ii) there exists a $G_\delta$-set $H$ such that, $F_1 \subseteq H \subseteq F_\sigma(H) \subseteq F_2^c$.

But

$$H \subseteq F_\sigma(H) \Rightarrow H \cap (F_\sigma(H))^c = \emptyset$$

and

$$F_\sigma(H) \subseteq F_2^c \Rightarrow F_2 \subseteq (F_\sigma(H))^c.$$
Furthermore, \((F_\sigma(H))^c\) is a \(G_\delta\)-set of \(X\). Hence \(F_1 \subseteq H, F_2 \subseteq (F_\sigma(H))^c\) and 
\(H \cap (F_\sigma(H))^c = \emptyset\). This means that condition (i) holds.

**Lemma 3.3.** Suppose that \(X\) is the topological space such that we can separate every two disjoint \(F_\sigma\)-sets by \(G_\delta\)-sets. If \(F_1\) and \(F_2\) are two disjoint \(F_\sigma\)-sets of \(X\), then there exists a Baire-.5 function \(h : X \to [0,1]\) such that \(h(F_1) = \{0\}\) and \(h(F_2) = \{1\}\).

**Proof.** Suppose \(F_1\) and \(F_2\) are two disjoint \(F_\sigma\)-sets of \(X\). Since \(F_1 \cap F_2 = \emptyset\), hence \(F_1 \subseteq F_2^c\). In particular, since \(F_2^c\) is a \(G_\delta\)-set of \(X\) containing \(F_1\), by Lemma 3.2, there exists a \(G_\delta\)-set \(H_{1/2}\) such that,

\[F_1 \subseteq H_{1/2} \subseteq F_\sigma(H_{1/2}) \subseteq F_2^c.\]

Note that \(H_{1/2}\) is a \(G_\delta\)-set and contains \(F_1\), and \(F_2^c\) is a \(G_\delta\)-set and contains \(F_\sigma(H_{1/2})\). Hence, by Lemma 3.2, there exists \(G_\delta\)-sets \(H_{1/4}\) and \(H_{3/4}\) such that,

\[F_1 \subseteq H_{1/4} \subseteq F_\sigma(H_{1/4}) \subseteq H_{1/2} \subseteq F_\sigma(H_{1/2}) \subseteq H_{3/4} \subseteq F_\sigma(H_{3/4}) \subseteq F_2^c.\]

By continuing this method for every \(t \in D\), where \(D \subseteq [0,1]\) is the set of rational numbers that their denominators are exponents of 2, we obtain \(G_\delta\)-sets \(H_t\) with the property that if \(t_1, t_2 \in D\) and \(t_1 < t_2\), then \(H_{t_1} \subseteq H_{t_2}\). We define the function \(h\) on \(X\) by \(h(x) = \inf\{t : x \in H_t\}\) for \(x \notin F_2\) and \(h(x) = 1\) for \(x \in F_2\).

Note that for every \(x \in X\), \(0 < h(x) < 1\), i.e., \(h\) maps \(X\) into \([0,1]\). Also, we note that for any \(t \in D\), \(F_1 \subseteq H_t\); hence \(h(F_1) = \{0\}\). Furthermore, by definition, \(h(F_2) = \{1\}\). It remains only to prove that \(h\) is a Baire-5 function on \(X\). For every \(\alpha \in \mathbb{R}\), we have if \(\alpha \leq 0\) then \(\{x \in X : h(x) < \alpha\} = \emptyset\) and if \(0 < \alpha\) then \(\{x \in X : h(x) < \alpha\} = \cup\{H_t : t < \alpha\}\), hence, they are \(G_\delta\)-sets of \(X\). Similarly, if \(\alpha < 0\) then \(\{x \in X : h(x) > \alpha\} = X\) and if \(0 \leq \alpha\) then \(\{x \in X : h(x) > \alpha\} = \cup\{F_\sigma(H_t) : t > \alpha\}\) hence, every of them is a \(G_\delta\)-set. Consequently \(h\) is a Baire-5 function.

**Lemma 3.4.** Suppose that \(X\) is the topological space such that every two disjoint \(F_\sigma\)-sets can be separated by \(G_\delta\)-sets. The following conditions are equivalent:

(i) Every countable covering of \(G_\delta\)-sets of \(X\) has a refinement consisting of \(G_\delta\)-sets such that, for every \(x \in X\), there exists a \(G_\delta\)-set containing \(x\) such that it intersects only finitely many members of the refinement.

(ii) Corresponding to every decreasing sequence \(\{F_n\}\) of \(F_\sigma\)-sets with empty intersection there exists a decreasing sequence \(\{G_n\}\) of \(G_\delta\)-sets such that, \(\bigcap_{n=1}^{\infty} G_n = 0\) and for every \(n \in \mathbb{N}\), \(F_n \subseteq G_n\).

**Proof.** (i) \(\Rightarrow\) (ii). Suppose that \(\{F_n\}\) be a decreasing sequence of \(F_\sigma\)-sets with empty intersection. Then \(\{F_n^c : n \in \mathbb{N}\}\) is a countable covering of \(G_\delta\)-sets.

By hypothesis (i) and Lemma 3.2, this covering has a refinement \(\{V_n : n \in \mathbb{N}\}\) such that every \(V_n\) is a \(G_\delta\)-set and \(F_\sigma(V_n) \subseteq F_n^c\). By setting \(F_n = (F_\sigma(V_n))^c\), we obtain a decreasing sequence of \(G_\delta\)-sets with the required properties.

(ii) \(\Rightarrow\) (i). Now if \(\{H_n : n \in \mathbb{N}\}\) is a countable covering of \(G_\delta\)-sets, we set for \(n \in \mathbb{N}\), \(F_n = \bigcup_{i=1}^{n} H_i^c\). Then \(\{F_n\}\) is a decreasing sequence of \(F_\sigma\)-sets with empty intersection. By (ii) there exists a decreasing sequence \(\{G_n\}\) consisting of
$G_\delta$–sets such that, $\bigcap_{n=1}^{\infty} G_n = \emptyset$ and for every $n \in \mathbb{N}, F_n \subseteq G_n$. Now we define the subsets $W_n$ of $X$ in the following manner:

- $W_1$ is a $G_\delta$–set of $X$ such that $G_1 \subseteq W_1$ and $F_\sigma(W_1) \cap F_1 = \emptyset$.
- $W_2$ is a $G_\delta$–set of $X$ such that $F_\sigma(W_1) \cup G_2 \subseteq W_2$ and $F_\sigma(W_2) \cap F_2 = \emptyset$, and so on. (By Lemma 3.2, $W_n$ exists).

Then since $\{G_n^c : n \in \mathbb{N}\}$ is a covering for $X$, hence $\{W_n : n \in \mathbb{N}\}$ is a covering for $X$ consisting of $G_\delta$–sets. Moreover, we have

(i) $F_\sigma(W_n) \subseteq W_{n+1}$
(ii) $G_n^c \subseteq W_n$
(iii) $W_n \subseteq \bigcup_{i=1}^{n} F_i$.

Now suppose that $S_1 = W_1$ and for $n \geq 2$, we set $S_n = W_{n+1} \setminus F_\sigma(W_{n-1})$.

Then since $F_\sigma(W_{n-1}) \subseteq W_n$ and $S_n \supseteq W_{n+1} \setminus W_n$, it follows that $\{S_n : n \in \mathbb{N}\}$ consists of $G_\delta$–sets and covers $X$. Furthermore, $S_i \cap S_j \neq \emptyset$ if and only if $|i - j| \leq 1$.

Finally, consider the following sets:

- $S_1 \cap H_1$
- $S_1 \cap H_2$
- $S_2 \cap H_1$
- $S_2 \cap H_2$
- $S_3 \cap H_1$
- $S_3 \cap H_2$
- $S_3 \cap H_3$
- $S_3 \cap H_4$

and continue ad infinitum. These sets are $G_\delta$–sets, cover $X$ and refine $\{H_n : n \in \mathbb{N}\}$. In addition, $S_i \cap H_j$ can intersect at most the sets in its row, immediately above, or immediately below row.

Hence if $x \in X$ and $x \in S_n \cap H_m$, then $S_n \cap H_m$ is a $G_\delta$–set containing $x$ that intersects at most finitely many of sets $S_i \cap H_j$. Consequently, $\{S_i \cap H_j : i \in \mathbb{N}, j = 1, \ldots, i + 1\}$ refines $\{H_n : n \in \mathbb{N}\}$ such that its elements are $G_\delta$–sets, and for every point in $X$ we can find a $G_\delta$–set containing the point that intersects only finitely many elements of that refinement.

**Remark 3.3.** ([13, 14]) A space $X$ has the $c$–insertion property for $(usc, lsc)$ if and only if $X$ is normal and countably paracompact.

**Corollary 3.3.** $X$ has the $B - .5$–insertion property for $(cusB - .5, clsB - .5)$ if and only if every two disjoint $F_\sigma$–sets of $X$ can be separated by $G_\delta$–sets, and in addition, every countable covering of $G_\delta$–sets has a refinement that consists of $G_\delta$–sets such that, for every point of $X$ we can find a $G_\delta$–set containing that point such that, it intersects only a finite number of refining members.

**Proof.** Suppose that $F_1$ and $F_2$ are disjoint $F_\sigma$–sets. Since $F_1 \cap F_2 = \emptyset$, it follows that $F_2 \subseteq F_1^c$. We set $f(x) = 2$ for $x \in F_1^c$, $f(x) = \frac{1}{2}$ for $x \notin F_1^c$, and $g = \chi_{F_1}$.

Since $F_3$ is a $F_\sigma$–set, and $F_1^c$ is a $G_\delta$–set, therefore $g$ is $cusB - .5$,$ f$ is $clsB - .5$ and furthermore $g < f$. Hence by hypothesis there exists a Baire-.5 function $h$ such that, $g < h < f$. Now by setting $G_1 = \{x \in X : h(x) < 1\}$ and $G_2 = \{x \in X : h(x) > 1\}$. We can say that $G_1$ and $G_2$ are disjoint $G_\delta$–sets that contain $F_1$ and $F_2$, respectively. Now suppose that $\{F_n\}$ is a decreasing sequence of $F_\sigma$–sets with empty intersection. Set $F_0 = X$ and define for every $x \in F_n \setminus F_{n+1}$.

$$f(x) = \frac{1}{\#n}.$$ Since $\bigcap_{n=0}^{\infty} F_n = \emptyset$ and for every $x \in X$, there exists $n \in \mathbb{N}$, such that, $x \in F_n \setminus F_{n+1}, f$ is well defined. Furthermore, for every $r \in \mathbb{R}$, if $r \leq 0$
then \( \{ x \in X : f(x) > r \} = X \) is a \( G_δ \)-set and if \( r > 0 \) then by Archimedean property of \( \mathbb{R} \), we can find \( i \in \mathbb{N} \) such that \( \frac{1}{i+1} \leq r \). Now suppose that \( k \) is the least natural number such that \( \frac{1}{i+1} \leq r \). Hence \( \frac{1}{k} > r \) and consequently, \( \{ x \in X : f(x) > r \} = X \setminus F_k \) is a \( G_δ \)-set. Therefore, \( f \) is \( \text{cls}B - .5 \). By setting \( g = 0 \), we have \( g \) is \( \text{c}usB - .5 \) and \( g < f \). Hence by hypothesis there exists a Baire-.5 function \( h \) on \( X \) such that, \( g < h < f \).

By setting \( G_n = \{ x \in X : h(x) < \frac{1}{n+1} \} \), we have \( G_n \) is a \( G_δ \)-set. But for every \( x \in F_n \), we have \( f(x) \leq \frac{1}{n+1} \) and since \( g < h < f \) therefore \( 0 < h(x) < \frac{1}{n+1} \), i.e., \( x \in G_n \) therefore \( F_n \subseteq G_n \) and since \( h > 0 \) it follows that \( \bigcap_{n=1}^{\infty} G_n = \emptyset \). Hence by Lemma 3.4, the conditions holds.

On the other hand, since every two disjoint \( F_σ \)-sets can be separated by \( G_δ \)-sets, therefore by corollary 3.1, \( X \) has the weak \( B - .5 \)-insertion property for \( (\text{c}usB - .5, \text{cls}B - .5) \). Now suppose that \( f \) and \( g \) are real-valued functions on \( X \) with \( g < f \), such that, \( g \) is \( \text{c}usB - .5 \) and \( f \) is \( \text{cls}B - .5 \). For every \( n \in \mathbb{N} \), set

\[
A(f-g, 3^{-n+1}) = \{ x \in X : (f-g)(x) \leq 3^{-n+1} \}.
\]

Since \( g \) is \( \text{c}usB - .5 \), and \( f \) is \( \text{cls}B - .5 \), therefore \( f - g \) is \( \text{cls}B - .5 \). Hence \( A(f-g, 3^{-n+1}) \) is a \( F_σ \)-set of \( X \). Consequently, \( \{ A(f-g, 3^{-n+1}) \} \) is a decreasing sequence of \( F_σ \)-sets and furthermore since \( 0 < f - g \), it follows that \( \bigcap_{n=1}^{\infty} A(f-g, 3^{-n+1}) = \emptyset \).

Now by Lemma 3.4, there exists a decreasing sequence \( \{ D_n \} \) of \( G_δ \)-sets such that \( A(f-g, 3^{-n+1}) \subseteq D_n \) and \( \bigcap_{n=1}^{\infty} D_n = \emptyset \). But by Lemma 3.3, \( A(f-g, 3^{-n+1}) \) and \( X \setminus D_n \) of \( F_σ \)-sets can be completely separated by Baire-.5 functions. Hence by Theorem 2.2, there exists a Baire-.5 function \( h \) defined on \( X \) such that, \( g < h < f \), i.e., \( X \) has the \( B - .5 \)-insertion property for \( (\text{c}usB - .5, \text{cls}B - .5) \).

**Remark 3.4.** ([15]) A space \( X \) has the \( c \)-insertion property for \((\text{isc}, \text{usc})\) iff \( X \) is extremally disconnected and if for any decreasing sequence \( \{ G_n \} \) of open subsets of \( X \) with empty intersection there exists a decreasing sequence \( \{ F_n \} \) of closed subsets of \( X \) with empty intersection such that \( G_n \subseteq F_n \) for each \( n \).

**Corollary 3.4.** For every \( G \) of \( G_δ \)-set, \( F_σ(G) \) is a \( G_δ \)-set and in addition for every decreasing sequence \( \{ G_n \} \) of \( G_δ \)-sets with empty intersection, there exists a decreasing sequence \( \{ F_n \} \) of \( F_σ \)-sets with empty intersection such that for every \( n \in \mathbb{N}, G_n \subseteq F_n \) if and only if \( X \) has the \( B - .5 \)-insertion property for \((\text{c}usB - .5, \text{cls}B - .5) \).

**Proof.** Since for every \( G \) of \( G_δ \)-set, \( F_σ(G) \) is a \( G_δ \)-set, therefore by Corollary 3.2, \( X \) has the weak \( B - .5 \)-insertion property for \((\text{c}usB - .5, \text{cls}B - .5) \). Now suppose that \( f \) and \( g \) are real-valued functions defined on \( X \) with \( g < f \), \( g \) is \( \text{c}usB - .5 \), and \( f \) is \( \text{cls}B - .5 \). Set \( A(f-g, 3^{-n+1}) = \{ x \in X : (f-g)(x) < 3^{-n+1} \} \). Then since \( f - g \) is \( \text{c}usB - .5 \), hence \( A(f-g, 3^{-n+1}) \) is a decreasing sequence of \( G_δ \)-sets with empty intersection. By hypothesis, there exists a decreasing sequence \( \{ D_n \} \) of \( F_σ \)-sets with empty intersection such that, for every \( n \in \mathbb{N}, A(f-g, 3^{-n+1}) \subseteq D_n \). Hence \( X \setminus D_n \) and \( A(f-g, 3^{-n+1}) \) are two disjoint \( G_δ \)-sets and therefore by
Lemma 3.1, we have
\[ F_\sigma(A(f - g, 3^{-n+1})) \cap F_\sigma((X \setminus D_n)) = \emptyset \]
and therefore by Lemma 3.3, \( X \setminus D_n \) and \( A(f - g, 3^{-n+1}) \) are completely separable by Baire-.5 functions. Therefore by theorem 2.2, there exists a Baire-.5 function \( h \) on \( X \) such that, \( g < h < f \), i.e., \( X \) has the \( B - .5 \)-insertion property for \( (cls B - .5, cus B - .5) \).

On the other hand, suppose that \( G_1 \) and \( G_2 \) be two disjoint \( G_δ \)-sets. Since \( G_1 \cap G_2 = \emptyset \). We have \( G_2 \subseteq G_1^c \). We set \( f(x) = 2 \) for \( x \in G_1^c \), \( f(x) = \frac{1}{r} \) for \( x \notin G_1^c \) and \( g = \chi g_2 \).

Then since \( G_2 \) is a \( G_δ \)-set and \( G_1^c \) is a \( F_\sigma \)-set, we conclude that \( g \) is \( cls B - .5 \) and \( f \) is \( cus B - .5 \) and furthermore \( g < f \). By hypothesis, there exists a Baire-.5 function \( h \) on \( X \) such that, \( g < h < f \). Now we set \( F_1 = \{ x \in X : h(x) \leq \frac{1}{r} \} \) and \( F_2 = \{ x \in X : h(x) \geq 1 \} \). Then \( F_1 \) and \( F_2 \) are two disjoint \( F_\sigma \)-sets contain \( G_1 \) and \( G_2 \), respectively. Hence \( F_\sigma(G_1) \subseteq F_1 \) and \( F_\sigma(G_2) \subseteq F_2 \) and consequently \( F_\sigma(G_1) \cap F_\sigma(G_2) = \emptyset \). By Lemma 3.1, for every \( G \) of \( G_δ \)-set, the set \( F_\sigma(G) \) is a \( G_δ \)-set.

Now suppose that \( \{ G_n \} \) is a decreasing sequence of \( G_δ \)-sets with empty intersection.

We set \( G_0 = X \) and \( f(x) = \frac{1}{r + 1} \) for \( x \in G_n \setminus G_{n+1} \). Since \( \bigcap_{n=0}^{\infty} G_n = \emptyset \) and for every \( n \in \mathbb{N} \) there exists \( x \in G_n \setminus G_{n+1} \), \( f \) is well-defined. Furthermore, for every \( r \in \mathbb{R} \), if \( r \leq 0 \) then \( \{ x \in X : f(x) < r \} = \emptyset \) is a \( G_δ \)-set and if \( r > 0 \) then by Archimedean property of \( \mathbb{R} \), there exists \( i \in \mathbb{N} \) such that \( \frac{1}{i+1} \leq r \).

Suppose that \( k \) is the least natural number with this property. Hence \( \frac{1}{k} > r \). Now if \( \frac{1}{k+1} < r \) then \( \{ x \in X : f(x) < r \} = G_k \) is a \( G_δ \)-set and if \( \frac{1}{k+1} = r \) then \( \{ x \in X : f(x) < r \} = G_{k+1} \) is a \( G_δ \)-set. Hence \( f \) is a \( cus B - .5 \) on \( X \). By setting \( g = 0 \), we have conclude that \( g \) is \( cls B - .5 \) on \( X \) and in addition \( g < f \). By hypothesis there exists a Baire-.5 function \( h \) on \( X \) such that, \( g < h < f \).

Set \( F_n = \{ x \in X : h(x) \leq \frac{1}{n+1} \} \). This set is a \( F_\sigma \)-set. But for every \( x \in G_n \), we have \( f(x) \leq \frac{1}{n+1} \) and since \( g < h < f \) thus \( h(x) < \frac{1}{n+1} \), this means that \( x \in F_n \) and consequently \( G_n \subseteq F_n \).

By definition of \( F_n \), \( \{ F_n \} \) is a decreasing sequence of \( F_\sigma \)-sets and since \( h > 0 \), \( \bigcap_{n=1}^{\infty} F_n = \emptyset \). Thus the conditions holds.

\[ \square \]

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References


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