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INSERTION OF A CONTRA-BAIRE-1 (BAIRE-.5) FUNCTION BETWEEN TWO COMPARABLE REAL-VALUED FUNCTIONS

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ABSTRACT. A necessary and sufficient condition in terms of lower cut sets are given for the insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that $F_\sigma-$kernel of sets are $F_\sigma-$sets.

1. Introduction

A generalized class of closed sets was considered by Maki in 1986 [**16**]. He investigated the sets that can be represented as union of closed sets and called them $V-$sets. Complements of $V-$sets, i.e., sets that are intersection of open sets are called $\Lambda-$sets [**16**].

Recall that a real-valued function f defined on a topological space X is called $A-$continuous [**22**] if the preimage of every open subset of $\mathbb{R}$ belongs to A, where A is a collection of subsets of X. Most of the definitions of function used throughout this paper are consequences of the definition of $A-$continuity. However, for unknown concepts the reader may refer to [**4, 10**]. In the recent literature many topologists had focused their research in the direction of investigating different types of generalized continuity.

J. Dontchev in [**5**] introduced a new class of mappings called contra-continuity. A good number of researchers have also initiated different types of contra-continuous like mappings in the papers [**1, 3, 7, 8, 9, 11, 12, 21**].

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Results of Katětov [**13, 14**] concerning binary relations and the concept of an indefinite lower cut set for a real-valued function, which is due to Brooks [**2**], are used in order to give a necessary and sufficient condition for the insertion of a Baire-.5 function between two comparable real-valued functions on the topological spaces that $F_\sigma-$kernel of sets are $F_\sigma-$sets.

A real-valued function f defined on a topological space X is called *contra-Baire-1 (Baire-.5)* if the preimage of every open subset of $\mathbb{R}$ is a $G_\delta-$set in X [**23**].

If g and f are real-valued functions defined on a space X, we write $g \leqslant f$ (resp. $g < f$) in case $g(x) \leqslant f(x)$ (resp. $g(x) < f(x)$) for all x in X.

The following definitions are modifications of conditions considered in [**15**].

A property P defined relative to a real-valued function on a topological space is a $B-.5-$*property* provided that any constant function has property P and provided that the sum of a function with property P and any Baire-.5 function also has property P. If P_1 and P_2 are $B-.5-$properties, the following terminology is used:

(i) A space X has the *weak* $B-.5-$*insertion property for* (P_1, P_2) if and only if for any functions g and f on X such that $g \leqslant f, g$ has property P_1 and f has property P_2, then there exists a Baire-.5 function h such that $g \leqslant h \leqslant f$.

(ii) A space X has the $B-.5-$*insertion property for* (P_1, P_2) if and only if for any functions g and f on X such that $g < f, g$ has property P_1 and f has property P_2, then there exists a Baire-.5 function h such that $g < h < f$.

In this paper, for a topological space that $F_\sigma-$kernel of sets are $F_\sigma-$sets, is given a sufficient condition for the weak $B-.5-$insertion property. Also for a space with the weak $B-.5-$insertion property, we give a necessary and sufficient condition for the space to have the $B-.5-$insertion property. Several insertion theorems are obtained as corollaries of these results. In addition, the insertion and weak insertion of a contra-continuous function between two comparable real-valued functions has also recently considered by the authors in [**18, 19**].

2. The Main Results

Before giving a sufficient condition for insertability of a Baire-.5 function, the necessary definitions and terminology are stated.

Definition 2.1. Let A be a subset of a topological space (X, τ). We define the subsets A^Λ and A^V as follows:

$$A^\Lambda = \cap\{O : O \supseteq A, O \in (X,\tau)\} \text{ and } A^V = \cup\{F : F \subseteq A, F^c \in (X,\tau)\}.$$

In [**6, 17, 20**], A^Λ is called the *kernel* of A.

We define the subsets $G_\delta(A)$ and $F_\sigma(A)$ as follows:

$$G_\delta(A) = \cup\{O : O \subseteq A, O \,\text{is}\, G_\delta - set\} \text{ and } F_\sigma(A) = \cap\{F : F \supseteq A, F \,\text{is}\, F_\sigma - set\}.$$

$F_\sigma(A)$ is called the $F_\sigma - kernel$ of A. The following first two definitions are modifications of conditions considered in [**13, 14**].

DEFINITION 2.2. If ρ is a binary relation in a set S then $\bar{\rho}$ is defined as follows: $x\ \bar{\rho}\ y$ if and only if $y\ \rho\ \nu$ implies $x\ \rho\ \nu$ and $u\ \rho\ x$ implies $u\ \rho\ y$ for any u and v in S.

DEFINITION 2.3. A binary relation ρ in the power set $P(X)$ of a topological space X is called a *strong binary relation* in $P(X)$ in case ρ satisfies each of the following conditions:

1) If $A_i\ \rho\ B_j$ for any $i \in \{1,\ldots,m\}$ and for any $j \in \{1,\ldots,n\}$, then there exists a set C in $P(X)$ such that $A_i\ \rho\ C$ and $C\ \rho\ B_j$ for any $i \in \{1,\ldots,m\}$ and any $j \in \{1,\ldots,n\}$.

2) If $A \subseteq B$, then $A\ \bar{\rho}\ B$.

3) If $A\ \rho\ B$, then $F_\sigma(A) \subseteq B$ and $A \subseteq G_\delta(B)$.

The concept of a lower indefinite cut set for a real-valued function was defined by Brooks [**2**] as follows:

DEFINITION 2.4. If f is a real-valued function defined on a space X and if $\{x \in X : f(x) < \ell\} \subseteq A(f,\ell) \subseteq \{x \in X : f(x) \leqslant \ell\}$ for a real number ℓ, then $A(f,\ell)$ is a *lower indefinite cut set* in the domain of f at the level ℓ.

We now give the following main results:

THEOREM 2.1. *Let g and f be real-valued functions on the topological space X, that $F_\sigma-$kernel of sets in X are $F_\sigma-$ sets , with $g \leqslant f$. If there exists a strong binary relation ρ on the power set of X and if there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\ \rho\ A(g,t_2)$, then there exists a Baire-.5 function h defined on X such that $g \leqslant h \leqslant f$.*

PROOF. Let g and f be real-valued functions defined on the X such that $g \leqslant f$. By hypothesis there exists a strong binary relation ρ on the power set of X and there exist lower indefinite cut sets $A(f,t)$ and $A(g,t)$ in the domain of f and g at the level t for each rational number t such that if $t_1 < t_2$ then $A(f,t_1)\ \rho\ A(g,t_2)$.

Define functions F and G mapping the rational numbers $\mathbb{Q}$ into the power set of X by $F(t) = A(f,t)$ and $G(t) = A(g,t)$. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then $F(t_1)\ \bar{\rho}\ F(t_2), G(t_1)\ \bar{\rho}\ G(t_2)$, and $F(t_1)\ \rho\ G(t_2)$. By Lemmas 1 and 2 of [14] it follows that there exists a function H mapping $\mathbb{Q}$ into the power set of X such that if t_1 and t_2 are any rational numbers with $t_1 < t_2$, then $F(t_1)\ \rho\ H(t_2), H(t_1)\ \rho\ H(t_2)$ and $H(t_1)\ \rho\ G(t_2)$.

For any x in X, let $h(x) = \inf\{t \in \mathbb{Q} : x \in H(t)\}$.

We first verify that $g \leqslant h \leqslant f$: If x is in $H(t)$ then x is in $G(t')$ for any $t' > t$; since x in $G(t') = A(g,t')$ implies that $g(x) \leqslant t'$, it follows that $g(x) \leqslant t$. Hence $g \leqslant h$. If x is not in $H(t)$, then x is not in $F(t')$ for any $t' < t$; since x is not in $F(t') = A(f,t')$ implies that $f(x) > t'$, it follows that $f(x) \geqslant t$. Hence $h \leqslant f$.

Also, for any rational numbers t_1 and t_2 with $t_1 < t_2$, we have $h^{-1}(t_1,t_2) = G_\delta(H(t_2)) \smallsetminus F_\sigma(H(t_1))$. Hence $h^{-1}(t_1,t_2)$ is a $G_\delta-$set in X, i.e., h is a Baire-.5 function on X. □

The above proof used the technique of Theorem 1 of [**13**].

Theorem 2.2. *Let P_1 and P_2 be $B-.5-$property and X be a space that satisfies the weak $B-.5-$insertion property for (P_1, P_2). Also assume that g and f are functions on X such that $g < f, g$ has property P_1 and f has property P_2. The space X has the $B-.5-$insertion property for (P_1, P_2) if and only if there exist lower cut sets $A(f-g, 3^{-n+1})$ and there exists a decreasing sequence $\{D_n\}$ of subsets of X with empty intersection and such that for each $n, X \setminus D_n$ and $A(f-g, 3^{-n+1})$ are completely separated by Baire-.5 functions.*

Proof. Assume that X has the weak $B-.5-$insertion property for (P_1, P_2). Let g and f be functions such that $g < f, g$ has property P_1 and f has property P_2. By hypothesis there exist lower cut sets $A(f-g, 3^{-n+1})$ and there exists a sequence (D_n) such that $\bigcap_{n=1}^{\infty} D_n = \emptyset$ and such that for each $n, X \setminus D_n$ and $A(f-g, 3^{-n+1})$ are completely separated by Baire-.5 functions. Let k_n be a Baire-.5 function such that $k_n = 0$ on $A(f-g, 3^{-n+1})$ and $k_n = 1$ on $X \setminus D_n$. Let a function k on X be defined by

$$k(x) = 1/2 \sum_{n=1}^{\infty} 3^{-n} k_n(x).$$

By the Cauchy condition and the $B-.5-$properties, the function k is a Baire-.5 function. Since $\bigcap_{n=1}^{\infty} D_n = \emptyset$ and since $k_n = 1$ on $X \setminus D_n$, it follows that $0 < k$. Also $2k < f - g$: In order to see this, observe first that if x is in $A(f-g, 3^{-n+1})$, then $k(x) \leqslant 1/4(3^{-n})$. If x is any point in X, then $x \notin A(f-g, 1)$ or for some n,

$$x \in A(f-g, 3^{-n+1}) - A(f-g, 3^{-n});$$

in the former case $2k(x) < 1$, and in the latter $2k(x) \leqslant 1/2(3^{-n}) < f(x) - g(x)$. Thus if $f_1 = f - k$ and if $g_1 = g + k$, then $g < g_1 < f_1 < f$. Since P_1 and P_2 are $B-.5-$properties, then g_1 has property P_1 and f_1 has property P_2. Since X has the weak $B-.5-$insertion property for (P_1, P_2), then there exists a Baire-.5 function such that $g_1 \leqslant h \leqslant f_1$. Thus $g < h < f$, it follows that X satisfies the $B-.5-$insertion property for (P_1, P_2). (The technique of this proof is by Katětov [**13**]).

Conversely, let g and f be functions on X such that g has property P_1, f has property P_2 and $g < f$. By hypothesis, there exists a Baire-.5 function such that $g < h < f$. We follow an idea contained in Lane [**15**]. Since the constant function 0 has property P_1, since $f - h$ has property P_2, and since X has the $B-.5-$insertion property for (P_1, P_2), then there exists a Baire-.5 function k such that $0 < k < f - h$. Let $A(f-g, 3^{-n+1})$ be any lower cut set for $f - g$ and let $D_n = \{x \in X : k(x) < 3^{-n+2}\}$. Since $k > 0$ it follows that $\bigcap_{n=1}^{\infty} D_n =$???. Since

$$A(f-g, 3^{-n+1}) \subseteq \{x \in X : (f-g)(x) \leqslant 3^{-n+1}\} \subseteq \{x \in X : k(x) \leqslant 3^{-n+1}\}$$

and since $\{x \in X : k(x) \leqslant 3^{-n+1}\}$ and $\{x \in X : k(x) \geqslant 3^{-n+2}\} = X \setminus D_n$ are completely separated by Baire-.5 function $\sup\{3^{-n+1}, \inf\{k, 3^{-n+2}\}\}$, it follows that for each $n, A(f-g, 3^{-n+1})$ and $X \setminus D_n$ are completely separated by Baire-.5 functions. □

3. Applications

Definition 3.1. A real-valued function f defined on a space X is called *contra-upper semi-Baire-.5* (resp. *contra-lower semi-Baire-.5*) if $f^{-1}(-\infty, t)$ (resp. $f^{-1}(t, +\infty)$) is a $G_\delta-$set for any real number t.

The abbreviations $usc, lsc, cusB_.5$ and $clsB_.5$ are used for upper semicontinuous, lower semicontinuous, contra-upper semi-Baire-.5, and contra-lower semi-Baire-.5, respectively.

Remark 3.1. ([**13, 14**]). A space X has the weak $c-$insertion property for (usc, lsc) if and only if X is normal.

Before stating the consequences of Theorem 2.1, and Theorem 2.2 we suppose that X is a topological space that $F_\sigma-$kernel of sets are $F_\sigma-$sets.

Corollary 3.1. *For each pair of disjoint $F_\sigma-$sets F_1, F_2, there are two $G_\delta-$sets G_1 and G_2 such that $F_1 \subseteq G_1$, $F_2 \subseteq G_2$ and $G_1 \cap G_2 = \emptyset$ if and only if X has the weak $B-.5-$insertion property for $(cusB-.5, clsB-.5)$.*

Proof. Let g and f be real-valued functions defined on the X, such that f is lsB_1, g is usB_1, and $g \leqslant f$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $F_\sigma(A) \subseteq G_\delta(B)$, then by hypothesis ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(f, t_1) \subseteq \{x \in X : f(x) \leqslant t_1\} \subseteq \{x \in X : g(x) < t_2\} \subseteq A(g, t_2);$$

since $\{x \in X : f(x) \leqslant t_1\}$ is a $F_\sigma-$set and since $\{x \in X : g(x) < t_2\}$ is a $G_\delta-$set, it follows that $F_\sigma(A(f, t_1)) \subseteq G_\delta(A(g, t_2))$. Hence $t_1 < t_2$ implies that $A(f, t_1)\ \rho\ A(g, t_2)$. The proof follows from Theorem 2. 1.

On the other hand, let F_1 and F_2 are disjoint $F_\sigma-$sets. Set $f = \chi_{F_1^c}$ and $g = \chi_{F_2}$, then f is $clsB-.5, g$ is $cusB-.5$, and $g \leqslant f$. Thus there exists Baire-.5 function h such that $g \leqslant h \leqslant f$. Set $G_1 = \{x \in X : h(x) < \frac{1}{2}\}$ and $G_2 = \{x \in X : h(x) > \frac{1}{2}\}$, then G_1 and G_2 are disjoint $G_\delta-$sets such that $F_1 \subseteq G_1$ and $F_2 \subseteq G_2$. □

Remark 3.2. ([**24**]) A space X has the weak $c-$insertion property for (lsc, usc) if and only if X is extremally disconnected.

Corollary 3.2. *For every G of $G_\delta-$set, $F_\sigma(G)$ is a $G_\delta-$set if and only if X has the weak $B-.5-$insertion property for $(clsB-.5, cusB-.5)$.*

Before giving the proof of this corollary, the necessary lemma is stated.

Lemma 3.1. *The following conditions on the space X are equivalent:*
(i) *For every G of $G_\delta-$set we have $F_\sigma(G)$ is a $G_\delta-$set.*
(ii) *For each pair of disjoint $G_\delta-$sets as G_1 and G_2 we have $F_\sigma(G_1) \cap F_\sigma(G_2) = \emptyset$.*

Proof. The proof of Lemma 3.1 is a direct consequence of the definition $F_\sigma-$kernel of sets. □

We now give the proof of corollary 3.2.

PROOF. Let g and f be real-valued functions defined on the X, such that f is $clsB-.5, g$ is $cusB-.5$, and $f \leqslant g$.If a binary relation ρ is defined by $A\ \rho\ B$ in case $F_\sigma(A) \subseteq G \subseteq F_\sigma(G) \subseteq G_\delta(B)$ for some $G_\delta-$set g in X, then by hypothesis and Lemma 3.1 ρ is a strong binary relation in the power set of X. If t_1 and t_2 are any elements of $\mathbb{Q}$ with $t_1 < t_2$, then

$$A(g,t_1) = \{x \in X : g(x) < t_1\} \subseteq \{x \in X : f(x) \leqslant t_2\} = A(f,t_2);$$

since $\{x \in X : g(x) < t_1\}$ is a $G_\delta-$set and since $\{x \in X : f(x) \leqslant t_2\}$ is a $F_\sigma-$set, by hypothesis it follows that $A(g,t_1)\ \rho\ A(f,t_2)$. The proof follows from Theorem 2.1.

On the other hand, Let G_1 and G_2 are disjoint $G_\delta-$sets. Set $f = \chi_{G_2}$ and $g = \chi_{G_1^c}$, then f is $clsB-.5, g$ is $cusB-.5$, and $f \leqslant g$.

Thus there exists Baire-.5 function h such that $f \leqslant h \leqslant g$. Set $F_1 = \{x \in X : h(x) \leqslant \frac{1}{3}\}$ and $F_2 = \{x \in X : h(x) \geqslant 2/3\}$ then F_1 and F_2 are disjoint $F_\sigma-$sets such that $G_1 \subseteq F_1$ and $G_2 \subseteq F_2$. Hence $F_\sigma(F_1) \cap F_\sigma(F_2) = \emptyset$. □

Before stating the consequences of Theorem 2.2, we state and prove the necessary lemmas.

LEMMA 3.2. *The following conditions on the space X are equivalent:*

(i) *Every two disjoint $F_\sigma-$sets of X can be separated by $G_\delta-$sets of X.*

(ii) *If F is a $F_\sigma-$set of X which is contained in a $G_\delta-$set G, then there exists a $G_\delta-$set H such that $F \subseteq H \subseteq F_\sigma(H) \subseteq G$.*

PROOF. (i) ⇒ (ii) Suppose that $F \subseteq G$, where F and G are $F_\sigma-$set and $G_\delta-$set of X, respectively. Hence, G^c is a $F_\sigma-$set and $F \cap G^c = \emptyset$.

By (i) there exists two disjoint $G_\delta-$sets G_1, G_2 such that $F \subseteq G_1$ and $G^c \subseteq G_2$. But

$$G^c \subseteq G_2 \Rightarrow G_2^c \subseteq G,$$

and

$$G_1 \cap G_2 = \emptyset \Rightarrow G_1 \subseteq G_2^c$$

hence

$$F \subseteq G_1 \subseteq G_2^c \subseteq G$$

and since G_2^c is a $F_\sigma-$set containing G_1 we conclude that $F_\sigma(G_1) \subseteq G_2^c$, i.e.,

$$F \subseteq G_1 \subseteq F_\sigma(G_1) \subseteq G.$$

By setting $H = G_1$, condition (ii) holds.

(ii) ⇒ (i) Suppose that F_1, F_2 are two disjoint $F_\sigma-$sets of X.

This implies that $F_1 \subseteq F_2^c$ and F_2^c is a $G_\delta-$set. Hence by (ii) there exists a $G_\delta-$set H such that, $F_1 \subseteq H \subseteq F_\sigma(H) \subseteq F_2^c$.
But

$$H \subseteq F_\sigma(H) \Rightarrow H \cap (F_\sigma(H))^c = \emptyset$$

and

$$F_\sigma(H) \subseteq F_2^c \Rightarrow F_2 \subseteq (F_\sigma(H))^c.$$

Furthermore, $(F_\sigma(H))^c$ is a $G_\delta-$set of X. Hence $F_1 \subseteq H, F_2 \subseteq (F_\sigma(H))^c$ and $H \cap (F_\sigma(H))^c = \emptyset$. This means that condition (i) holds. □

Lemma 3.3. *Suppose that X is the topological space such that we can separate every two disjoint $F_\sigma-$sets by $G_\delta-$sets. If F_1 and F_2 are two disjoint $F_\sigma-$sets of X, then there exists a Baire-.5 function $h : X \to [0,1]$ such that $h(F_1) = \{0\}$ and $h(F_2) = \{1\}$.*

Proof. Suppose F_1 and F_2 are two disjoint $F_\sigma-$sets of X. Since $F_1 \cap F_2 = \emptyset$, hence $F_1 \subseteq F_2^c$. In particular, since F_2^c is a $G_\delta-$set of X containing F_1, by Lemma 3.2, there exists a $G_\delta-$set $H_{1/2}$ such that,

$$F_1 \subseteq H_{1/2} \subseteq F_\sigma(H_{1/2}) \subseteq F_2^c.$$

Note that $H_{1/2}$ is a $G_\delta-$set and contains F_1, and F_2^c is a $G_\delta-$set and contains $F_\sigma(H_{1/2})$. Hence, by Lemma 3.2, there exists $G_\delta-$sets $H_{1/4}$ and $H_{3/4}$ such that,

$$F_1 \subseteq H_{1/4} \subseteq F_\sigma(H_{1/4}) \subseteq H_{1/2} \subseteq F_\sigma(H_{1/2}) \subseteq H_{3/4} \subseteq F_\sigma(H_{3/4}) \subseteq F_2^c.$$

By continuing this method for every $t \in D$, where $D \subseteq [0,1]$ is the set of rational numbers that their denominators are exponents of 2, we obtain $G_\delta-$sets H_t with the property that if $t_1, t_2 \in D$ and $t_1 < t_2$, then $H_{t_1} \subseteq H_{t_2}$. We define the function h on X by $h(x) = \inf\{t : x \in H_t\}$ for $x \notin F_2$ and $h(x) = 1$ for $x \in F_2$.

Note that for every $x \in X, 0 \leqslant h(x) \leqslant 1$, i.e., h maps X into [0,1]. Also, we note that for any $t \in D, F_1 \subseteq H_t$; hence $h(F_1) = \{0\}$. Furthermore, by definition, $h(F_2) = \{1\}$. It remains only to prove that h is a Baire-.5 function on X. For every $\alpha \in \mathbb{R}$, we have if $\alpha \leqslant 0$ then $\{x \in X : h(x) < \alpha\} = \emptyset$ and if $0 < \alpha$ then $\{x \in X : h(x) < \alpha\} = \cup\{H_t : t < \alpha\}$, hence, they are $G_\delta-$sets of X. Similarly, if $\alpha < 0$ then $\{x \in X : h(x) > \alpha\} = X$ and if $0 \leqslant \alpha$ then $\{x \in X : h(x) > \alpha\} = \cup\{(F_\sigma(H_t))^c : t > \alpha\}$ hence, every of them is a $G_\delta-$set. Consequently h is a Baire-.5 function. □

Lemma 3.4. *Suppose that X is the topological space such that every two disjoint $F_\sigma-$sets can be separated by $G_\delta-$sets. The following conditions are equivalent:*

(i) *Every countable convering of $G_\delta-$sets of X has a refinement consisting of $G_\delta-$sets such that, for every $x \in X$, there exists a $G_\delta-$set containing x such that it intersects only finitely many members of the refinement.*

(ii) *Corresponding to every decreasing sequence $\{F_n\}$ of $F_\sigma-$sets with empty intersection there exists a decreasing sequence $\{G_n\}$ of $G_\delta-$sets such that, $\bigcap_{n=1}^{\infty} G_n = \emptyset$ and for every $n \in \mathbb{N}, F_n \subseteq G_n$.*

Proof. (i) ⇒ (ii). suppose that $\{F_n\}$ be a decreasing sequence of $F_\sigma-$sets with empty intersection. Then $\{F_n^c : n \in \mathbb{N}\}$ is a countable covering of $G_\delta-$sets. By hypothesis (i) and Lemma 3.2, this covering has a refinement $\{V_n : n \in \mathbb{N}\}$ such that every V_n is a $G_\delta-$set and $F_\sigma(V_n) \subseteq F_n^c$. By setting $F_n = (F_\sigma(V_n))^c$, we obtain a decreasing sequence of $G_\delta-$sets with the required properties.

(ii) ⇒ (i). Now if $\{H_n : n \in \mathbb{N}\}$ is a countable covering of $G_\delta-$sets, we set for $n \in \mathbb{N}, F_n = (\bigcup_{i=1}^{n} H_i)^c$. Then $\{F_n\}$ is a decreasing sequence of $F_\sigma-$sets with empty intersection. By (ii) there exists a decreasing sequence $\{G_n\}$ consisting of

$G_\delta-$sets such that, $\bigcap_{n=1}^{\infty} G_n = \emptyset$ and for every $n \in \mathbb{N}, F_n \subseteq G_n$. Now we define the subsets W_n of X in the following manner:

W_1 is a $G_\delta-$set of X such that $G_1^c \subseteq W_1$ and $F_\sigma(W_1) \cap F_1 = \emptyset$.

W_2 is a $G_\delta-$set of X such that $F_\sigma(W_1) \cup G_2^c \subseteq W_2$ and $F_\sigma(W_2) \cap F_2 = \emptyset$, and so on. (By Lemma 3.2, W_n exists).

Then since $\{G_n^c : n \in \mathbb{N}\}$ is a covering for X, hence $\{W_n : n \in \mathbb{N}\}$ is a covering for X consisting of $G_\delta-$sets. Moreover, we have

(i) $F_\sigma(W_n) \subseteq W_{n+1}$

(ii) $G_n^c \subseteq W_n$

(iii) $W_n \subseteq \bigcup_{i=1}^{n} H_i$.

Now suppose that $S_1 = W_1$ and for $n \geqslant 2$, we set $S_n = W_{n+1} \setminus F_\sigma(W_{n-1})$.

Then since $F_\sigma(W_{n-1}) \subseteq W_n$ and $S_n \supseteq W_{n+1} \setminus W_n$, it follows that $\{S_n : n \in \mathbb{N}\}$ consists of $G_\delta-$sets and covers X. Furthermore, $S_i \cap S_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$. Finally, consider the following sets:

$S_1 \cap H_1, \quad S_1 \cap H_2$

$S_2 \cap H_1, \quad S_2 \cap H_2, \quad S_2 \cap H_3$

$S_3 \cap H_1, \quad S_3 \cap H_2, \quad S_3 \cap H_3, \quad S_3 \cap H_4$

and continue ad infinitum. These sets are $G_\delta-$sets, cover X and refine $\{H_n : n \in \mathbb{N}\}$. In addition, $S_i \cap H_j$ can intersect at most the sets in its row, immediately above, or immediately below row.

Hence if $x \in X$ and $x \in S_n \cap H_m$, then $S_n \cap H_m$ is a $G_\delta-$set containing x that intersects at most finitely many of sets $S_i \cap H_j$. Consequently, $\{S_i \cap H_j : i \in \mathbb{N}, j = 1, \ldots, i+1\}$ refines $\{H_n : n \in \mathbb{N}\}$ such that its elements are $G_\delta-$sets, and for every point in X we can find a $G_\delta-$set containing the point that intersects only finitely many elements of that refinement. □

Remark 3.3. ([**13, 14**]) A space X has the $c-$insertion property for (usc, lsc) if and only if X is normal and countably paracompact.

Corollary 3.3. *X has the $B-.5-$insertion property for $(cusB-.5, clsB-.5)$ if and only if every two disjoint $F_\sigma-$sets of X can be separated by $G_\delta-$sets, and in addition, every countable covering of $G_\delta-$sets has a refinement that consists of $G_\delta-$sets such that, for every point of X we can find a $G_\delta-$set containing that point such that, it intersects only a finite number of refining members.*

Proof. Suppose that F_1 and F_2 are disjoint $F_\sigma-$sets . Since $F_1 \cap F_2 = \emptyset$, it follows that $F_2 \subseteq F_1^c$. We set $f(x) = 2$ for $x \in F_1^c$, $f(x) = \frac{1}{2}$ for $x \notin F_1^c$, and $g = \chi_{F_2}$.

Since F_2 is a $F_\sigma-$set, and F_1^c is a $G_\delta-$set, therefore g is $cusB-.5$,f is $clsB-.5$ and furthermore $g < f$. Hence by hypothesis there exists a Baire-.5 function h such that, $g < h < f$. Now by setting $G_1 = \{x \in X : h(x) < 1\}$ and $G_2 = \{x \in X : h(x) > 1\}$. We can say that G_1 and G_2 are disjoint $G_\delta-$sets that contain F_1 and F_2, respectively. Now suppose that $\{F_n\}$ is a decreasing sequence of $F_\sigma-$sets with empty intersection. Set $F_0 = X$ and define for every $x \in F_n \setminus F_{n+1}$, $f(x) = \frac{1}{n+1}$. Since $\bigcap_{n=0}^{\infty} F_n = \emptyset$ and for every $x \in X$, there exists $n \in \mathbb{N}$, such that, $x \in F_n \setminus F_{n+1}, f$ is well defined. Furthermore, for every $r \in \mathbb{R}$, if $r \leqslant 0$

then $\{x \in X : f(x) > r\} = X$ is a $G_\delta-$set and if $r > 0$ then by Archimedean property of $\mathbb{R}$, we can find $i \in \mathbb{N}$ such that $\frac{1}{i+1} \leqslant r$. Now suppose that k is the least natural number such that $\frac{1}{k+1} \leqslant r$. Hence $\frac{1}{k} > r$ and consequently, $\{x \in X : f(x) > r\} = X \smallsetminus F_k$ is a $G_\delta-$set. Therefore, f is $clsB-.5$. By setting $g = 0$, we have g is $cusB-.5$ and $g < f$. Hence by hypothesis there exists a Baire-.5 function h on X such that, $g < h < f$.

By setting $G_n = \{x \in X : h(x) < \frac{1}{n+1}\}$, we have G_n is a $G_\delta-$set. But for every $x \in F_n$, we have $f(x) \leqslant \frac{1}{n+1}$ and since $g < h < f$ therefore $0 < h(x) < \frac{1}{n+1}$, i.e., $x \in G_n$ therefore $F_n \subseteq G_n$ and since $h > 0$ it follows that $\bigcap_{n=1}^{\infty} G_n = \emptyset$. Hence by Lemma 3.4, the conditions holds.

On the other hand, since every two disjoint $F_\sigma-$sets can be separated by $G_\delta-$sets, therefore by corollary 3.1, X has the weak $B-.5-$insertion property for $(cusB-.5, clsB-.5)$. Now suppose that f and g are real-valued functions on X with $g < f$, such that, g is $cusB-.5$ and f is $clsB-.5$. For every $n \in \mathbb{N}$, set

$$A(f-g, 3^{-n+1}) = \{x \in X : (f-g)(x) \leqslant 3^{-n+1}\}.$$

Since g is $cusB-.5$, and f is $clsB-.5$, therefore $f-g$ is $clsB-.5$. Hence $A(f-g, 3^{-n+1})$ is a $F_\sigma-$set of X. Consequently, $\{A(f-g, 3^{-n+1})\}$ is a decreasing sequence of $F_\sigma-$sets and furthermore since $0 < f-g$, it follows that $\bigcap_{n=1}^{\infty} A(f-g, 3^{-n+1}) = \emptyset$. Now by Lemma 3.4, there exists a decreasing sequence $\{D_n\}$ of $G_\delta-$sets such that $A(f-g, 3^{-n+1}) \subseteq D_n$ and $\bigcap_{n=1}^{\infty} D_n = \emptyset$. But by Lemma 3.3, $A(f-g, 3^{-n+1})$ and $X \smallsetminus D_n$ of $F_\sigma-$sets can be completely separated by Baire-.5 functions. Hence by Theorem 2.2, there exists a Baire-.5 function h defined on X such that, $g < h < f$, i.e., X has the $B-.5-$insertion property for $(cusB-.5, clsB-.5)$. □

Remark 3.4. ([**15**]) A space X has the $c-$insertion property for (lsc, usc) iff X is extremally disconnected and if for any decreasing sequence $\{G_n\}$ of open subsets of X with empty intersection there exists a decreasing sequence $\{F_n\}$ of closed subsets of X with empty intersection such that $G_n \subseteq F_n$ for each n.

Corollary 3.4. *For every G of $G_\delta-$set, $F_\sigma(G)$ is a $G_\delta-$set and in addition for every decreasing sequence $\{G_n\}$ of $G_\delta-$sets with empty intersection, there exists a decreasing sequence $\{F_n\}$ of $F_\sigma-$sets with empty intersection such that for every $n \in \mathbb{N}, G_n \subseteq F_n$ if and only if X has the $B-.5-$insertion property for $(clsB-.5, cusB-.5)$.*

Proof. Since for every G of $G_\delta-$set, $F_\sigma(G)$ is a $G_\delta-$set, therefore by Corollary 3.2, X has the weak $B-.5-$insertion property for $(clsB-.5, cusB-.5)$. Now suppose that f and g are real-valued functions defined on X with $g < f, g$ is $clsB-.5$, and f is $cusB-.5$. Set $A(f-g, 3^{-n+1}) = \{x \in X : (f-g)(x) < 3^{-n+1}\}$. Then since $f-g$ is $cusB-.5$, hence $\{A(f-g, 3^{-n+1})\}$ is a decreasing sequence of $G_\delta-$sets with empty intersection. By hypothesis, there exists a decreasing sequence $\{D_n\}$ of $F_\sigma-$sets with empty intersection such that, for every $n \in \mathbb{N}, A(f-g, 3^{-n+1}) \subseteq D_n$. Hence $X \smallsetminus D_n$ and $A(f-g, 3^{-n+1})$ are two disjoint $G_\delta-$sets and therefore by

Lemma 3.1, we have

$$F_\sigma(A(f-g,3^{-n+1}))\cap F_\sigma((X\setminus D_n))=\emptyset$$

and therefore by Lemma 3.3, $X\setminus D_n$ and $A(f-g,3^{-n+1})$ are completely separable by Baire-.5 functions. Therefore by theorem 2.2, there exists a Baire-.5 function h on X such that, $g<h<f$, i.e., X has the $B-.5-$insertion property for $(clsB-.5, cusB-.5)$.

On the other hand, suppose that G_1 and G_2 be two disjoint $G_\delta-$sets. Since $G_1\cap G_2=\emptyset$. We have $G_2\subseteq G_1^c$. We set $f(x)=2$ for $x\in G_1^c$, $f(x)=\frac{1}{2}$ for $x\notin G_1^c$ and $g=\chi_{G_2}$.

Then since G_2 is a $G_\delta-$set and G_1^c is a $F_\sigma-$set, we conclude that g is $clsB-.5$ and f is $cusB-.5$ and furthermore $g<f$. By hypothesis, there exists a Baire-.5 function h on X such that, $g<h<f$. Now we set $F_1=\{x\in X: h(x)\leqslant \frac{3}{4}\}$ and $F_2=\{x\in X: h(x)\geqslant 1\}$. Then F_1 and F_2 are two disjoint $F_\sigma-$sets contain G_1 and G_2, respectively. Hence $F_\sigma(G_1)\subseteq F_1$ and $F_\sigma(G_2)\subseteq F_2$ and consequently $F_\sigma(G_1)\cap F_\sigma(G_2)=\emptyset$. By Lemma 3.1, for every G of $G_\delta-$set, the set $F_\sigma(G)$ is a $G_\delta-$set.

Now suppose that $\{G_n\}$ is a decreasing sequence of $G_\delta-$sets with empty intersection.

We set $G_0=X$ and $f(x)=\frac{1}{n+1}$ for $x\in G_n\setminus G_{n+1}$. Since $\bigcap_{n=0}^{\infty}G_n=\emptyset$ and for every $n\in\mathbb{N}$ there exists $x\in G_n\setminus G_{n+1}, f$ is well-defined. Furthermore, for every $r\in\mathbb{R}$, if $r\leqslant 0$ then $\{x\in X: f(x)<r\}=\emptyset$ is a $G_\delta-$set and if $r>0$ then by Archimedean property of $\mathbb{R}$, there exists $i\in\mathbb{N}$ such that $\frac{1}{i+1}\leqslant r$. Suppose that k is the least natural number with this property. Hence $\frac{1}{k}>r$. Now if $\frac{1}{k+1}<r$ then $\{x\in X: f(x)<r\}=G_k$ is a $G_\delta-$set and if $\frac{1}{k+1}=r$ then $\{x\in X: f(x)<r\}=G_{k+1}$ is a $G_\delta-$set. Hence f is a $cusB-.5$ on X. By setting $g=0$, we have conclude that g is $clsB-.5$ on X and in addition $g<f$. By hypothesis there exists a Baire-.5 function h on X suvh that, $g<h<f$.

Set $F_n=\{x\in X: h(x)\leqslant\frac{1}{n+1}\}$. This set is a $F_\sigma-$set. But for every $x\in G_n$, we have $f(x)\leqslant\frac{1}{n+1}$ and since $g<h<f$ thus $h(x)<\frac{1}{n+1}$, this means that $x\in F_n$ and consequently $G_n\subseteq F_n$.

By definition of $F_n,\{F_n\}$ is a decreasing sequence of $F_\sigma-$sets and since $h>0,\bigcap_{n=1}^{\infty}F_n=\emptyset$. Thus the conditions holds. □

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References

[1] A. Al-Omari and M.S. Md Noorani. Some properties of contra-b-continuous and almost contra-b-continuous functions. *European J. Pure. Appl. Math.*, **2**(2)(2009), 213–230.

[2] F. Brooks. Indefinite cut sets for real functions. *Amer. Math. Monthly*, **78**(9)(1971), 1007–1010.

[3] M. Caldas and S. Jafari. Some properties of contra-$\beta-$continuous functions. *Mem. Fac. Sci. Kochi. Univ.*, **22**(2001), 19–28.

[4] J. Dontchev. The characterization of some peculiar topological space via $\mathcal{A}$ and $\mathcal{B}$-sets. *Acta Math. Hungar.*, **69**(1-2)(1995), 67–71.
[5] J. Dontchev. Contra-continuous functions and strongly S-closed space. *Intrnat. J. Math. Math. Sci.*, **19**(2)(1996), 303–310.
[6] J. Dontchev, and H. Maki. On sg-closed sets and semi$-\lambda-$closed sets. *Questions Answers Gen. Topology*, **15**(2)(1997), 259–266.
[7] E. Ekici. On contra-continuity. *Annales Univ. Sci. Bodapest*, **47**(2004), 127–137.
[8] E. Ekici. New forms of contra-continuity. *Carpathian J. Math.*, **24**(1)(2008), 37–45.
[9] A. I. El-Magbrabi. Some properties of contra-continuous mappings. *Int. J. General Topol.*, **3**(1-2)(2010), 55–64.
[10] M. Ganster and I. Reilly. A decomposition of continuity. *Acta Math. Hungar.*, **56**(3-4)(1990), 299–301.
[11] S. Jafari and T. Noiri. Contra-α-continuous function between topological spaces. *Iranian Int. J. Sci.*, **2**(2)(2001), 153–167.
[12] S. Jafari and T. Noiri. On contra-precontinuous functions. *Bull. Malaysian Math. Sc. Soc.*, **25**(2002), 115–128.
[13] M. Katětov. On real-valued functions in topological spaces. *Fund. Math.*, **38**(1951), 85–91.
[14] M. Katětov. Correction to, "On real-valued functions in topological spaces". *Fund. Math.*, **40**(1953), 203–205.
[15] E. Lane. Insertion of a continuous function. *Pacific J. Math.*, **66**(1)(1976), 181–190.
[16] H. Maki. Generalized $\Lambda-$sets and the associated closure operator. The special Issue in commemoration of Prof. Kazuada IKEDA's Retirement, (1986), 139–146.
[17] S. N. Maheshwari and R. Prasad. On $R_{Os}-$spaces. *Portugal. Math.*, **34**(4)(1975), 213–217.
[18] M. Mirmiran and B. Naderi. Insertion of a contra-continuous function between two comparable contra-$\alpha-$continuous (contra-$C-$continuous) functions. *Facta Universitatis (Nis) Ser. Math.*, **34**(1)(2019), 13–22.
[19] M. Mirmiran. Weak insertion of a contra-continuous function between two comparable contra- precontinuous (contra-*semi*$-$continuous) functions. *Mathematica Montisnigri*, **41**(2018), 16–20.
[20] M. Mršević. On pairwise R_0 and pairwise R_1 bitopological spaces. *Bull. Math. Soc. Sci. Math. R. S. Roumanie*, **30**(78)(2)(1986), 141–145.
[21] A. A. Nasef. Some properties of contra-γ-continuous functions. *Chaos, Solitons and Fractals*, **24**(2)(2005), 471–477.
[22] M. Przemski. A decomposition of continuity and $\alpha-$continuity. *Acta Math. Hungar.*, **61**(1-2)(1993), 93–98.
[23] H. Rosen. Darboux Baire-.5 functions. *Proc. Amer. Math. Soc.*, **110**(1)(1990), 285–286.
[24] M. H. Stone. Boundedness properties in function-lattices. *Canad. J. Math.*, **1**(2)(1949), 176–186.

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