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DOMINATION WEAK INTEGRITY IN GRAPHS

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Abstract. The domination weak integrity of a graph G is defined as $\min\{|S|+m_e(G-S) : S \subseteq V(G)\}$ and S is a dominating set of G where $m_e(G-S)$ represents the order of the largest component in $G-S$ and denoted by $DI_w(G)$. Since the vertex cover of G is a domination weak integrity - set, the existence of a domination weak integrity set in any graph is guaranteed. In this paper, the domination weak integrity of some graphs is obtained and the relations between domination weak integrity and other parameters are also determined.

1. Introduction

All the graphs considered here are finite, non-trivial, undirected and connected without loops or multiple edges. The symbols $\triangle(G)$, $\delta(G)$, αG, $\kappa(G)$, $\lambda(G)$ and $\beta(G)$ denote the maximum degree, the minimum degree, the vertex cover number, the connectivity, the edge-connectivity and the independence number of G respectively. Further, a cut-set is any set of vertices whose removal leaves a disconnected graph. For a vertex $v \in V(G)$, the open neighbourhood of v in G, denoted by $N(v)$ is the set of all vertices that are adjacent to v. $\lceil x \rceil$ denote the smallest integer number that greater than or equals to x with $\lfloor x \rfloor$to the greatest integer number that smaller than or equals to x. The complement $\overline{G}$ of a graph G has $V(G)$ as its vertex set, two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G[S]$.

The concept of network vulnerability is motivated by the design and analysis of networks under a hostile environment. Several graph theoretic models under various assumptions have been proposed for the study and assessment of network vulnerability. Vertex integrity, introduced by Barefoot et al. [**3**] is one of these models that has received wide attention and also studied two measures of network

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vulnerability, the integrity and the edge integrity of a graph. Bagga et al. [**1**] introduced a similar measure called pure-edge integrity. The integrity $I(G)$ of a graph G is defined as $I(G) = min\{|S| + m(G - S) : S \subseteq V(G)\}$, where $m(G - S)$ denotes the order of a maximum component of $G - S$. Moreover the pure edge-integrity $I_p(G)$ of a graph G is defined as $I_p(G) = min\{|S| + m_e(G - S) : S \subseteq E(G)\}$, where $m_e(G - S)$ denotes the number of edges in a largest component of $G - s$. Also the weak integrity was introduced by Kirlangic [**7**] and is defined as $I_w(G) = min\{|S| + m_e(G - S) : S \subseteq V(G)\}$.

Definition 1.1. A subset S of $V(G)$ is called dominating set if for every $v \in VS$, there exist a $u \in S$ such that v is adjacent to u. The minimum cardinality of a minimal dominating set in G is called the domination number of G denoted as $\gamma(G)$ and the corresponding minimal dominating set is called a γ-set of G.

The theory of domination plays vital role in determining decision making bodies of minimum strength or weakness of a network when certain part of it is paralysed. In the case of disruption of a network, the damage will be more when vital node are under siege. This motivated the study of domination integrity when the sets of nodes disturbed are dominating sets.

R. Sundareswaran and V. Swaminathan [**8**] have introduced the concept of domination integrity of a graph as a new measure of vulnerability which is defined as follows.

Definition 1.2. The domination integrity of a connected graph G denoted by $DI(G)$ and defined as

$$DIG) = min\{|X| + m(GX) : X \text{ is a dominating set }\}$$

where $m(GX)$ is the order of a maximum component of GX.

They also have investigated domination integrity of some standard graphs. In the same paper they have investigated domination integrity of Binomial trees and Complete k-ary trees.

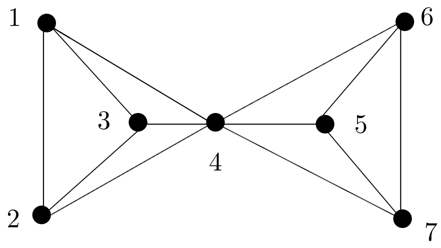
2. Main Results

Definition 2.1. The domination weak integrity of a graph G is defined as $min\{|S|+m_e(G-S) : S \subseteq V(G)\}$ and S is a dominating set of G where $m_e(G-S)$ represents the order of the largest component in $G - S$ and denoted by $DI_w(G)$. A subset S of $V(G)$ is a DI_w -set if $DI_w(G) = min\{|S| + m_e(G - S) : S \subseteq V(G)\}$ and S is a dominating set of G.

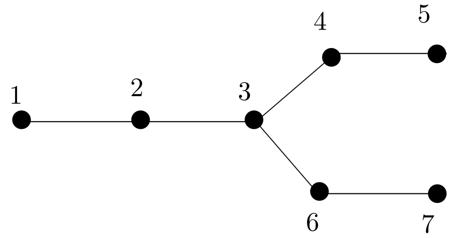
It is easy to observe that both the domination integrity and the domination weak integrity are nonincreasing with respect to subgraph inclusion. That is, if H is a subgraph of G, then $DI(H) \leqslant DI(G)$ and $DI_w(H) \leqslant DI_w(G)$ if G is a nontrivial connected graph of order n. It is also not difficult to see that the domination integrity never exceeds the domination weak integrity, so that if G is a nontrivial connected graph of order p, then $2 \leqslant DI(G) \leqslant DI_w(G)$.

Example 2.1.

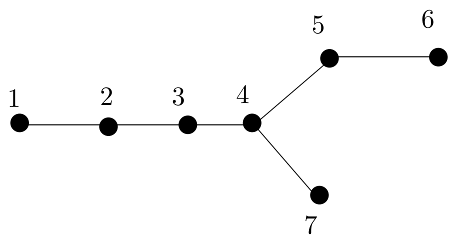
G_1 :



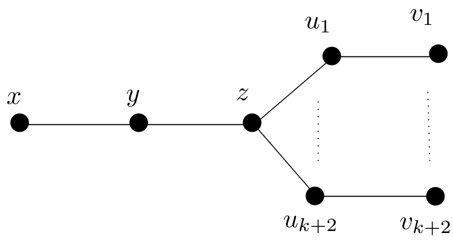
G_2 :



G_3 :



G :



Graph	I_w	DI_w
G_1	4	4
G_2	2	3
G_3	3	3

OBSERVATION 2.1. *For any graph G, let H be a subgraph of G. Then $DI_w(H)$ need not bee less than or equal to $DI_w(G)$.*

For example, $DI_w(K_{1,n}) = 2$ and $DI_w(K_{1,n} - \{u\} = |V(K_{1,n}) - 1|$ where u is the centre of the star $K_{1,n}$. Even if H is a spanning subgraph of G, $DI_w(H)$ may be greater than $DI_w(G)$.

OBSERVATION 2.2. *For any graph G and $v \in V(G), DI_w(G-v) \geqslant DI_w(G)-1$.*

PROOF. Let S be an DI_w set of $G-v$. Then S is a dominating set of $G-v$ and $DI_w(G-v) = |S| + m_e((G-v)-S)$. Therefore, $DI_w(G) \leqslant |T| + m_e(G-T) = |S| + m_e((G-v)-S) = DI_w(G-v)+1$. $DI_w(G-v) \geqslant DI(G)-1$. □

OBSERVATION 2.3. *For any graph G and $e \in E(G)$, $DI_w(G-e) \geqslant DI_w(G)-1$.*

PROOF. Let S be an DI_w set of $G-e$. Then S is a dominating set of $G-e$ and $DI_w(G-e) = |S| + m((G-e)-S)$. Let $e = uv$. Let $T = S \cup \{u\}$ or $T = S \cup \{v\}$. If u or v belongs to S, then $T = S$. If both u and v do not belong to S, then $|T| = |S|+1$. T is a dominating set of G and $m_e(G-T) = m_e((G-e)-S)$. Therefore $DI_w(G) \leqslant |T| + m_e(G-T) \leqslant |S| + m_e((G-e)-S)+1 = DI(G-e)+1$. Hence $DI_w(G-e) \geqslant DI_w(G)-1$. □

OBSERVATION 2.4. *If $e \in E(G)$, $e = uv$ and if there exists a DI_w-set S of G containing u or v,then $DI_w(G-e) \geqslant DI_w(G)$.*

OBSERVATION 2.5. *$DI_w(G-v) = |S| + m_e((G-v)-S)$. Let $T = S \cup \{v\}$. Then T is a dominating set of G and $m_e(G-T) = m_e((G-v)-S)$. Therefore $DI_w(G) \leqslant |T| + m_e(G-T) = |S| + m_e((G-v)-S) = DI_w(G-v)+1$. $DI_w(G-v) \geqslant DI(G)-1$.*

OBSERVATION 2.6. *Let G be a spanning sub graph of H and let $DI_w(G) = DI_w(H)$. Then, clearly DI_w-set of G is a DI_w-set of H.*

PROOF. Let S be a DI_w-set of G. Then $DI_w(G) \leqslant |S| + m_e(H-S) \leqslant |S| + m_e(G-S)$. Since $DI_w(G) = DI_w(H)$, we have $DI_w(H) = |S| + m_e(H-S)$. But S is a dominating set of H. Therefore, S is a DI_w-set of H. □

PROPOSITION 2.1. *The following*

(1) $DI_w(K_{(a_1,a_2,\cdots,a_k)}) = p-r$ *where* $p = \sum_i a_i$ *and* $r = \max_i a_i$.
(2) $DI_w(K_{(a,b)}) = min\{a,b\}$.
(3) $DI_w(K_{(1,n)}) = 2$.
(4) $DI_w(C_n) = \begin{cases} 3 & if \;\; n = 3,4 \\ \lceil \frac{n}{3} \rceil & if \;\; n \geqslant 5 \end{cases}$
(5) $DI_w(P_n) = \begin{cases} \lfloor \frac{n}{2} \rfloor & if \;\; n = 2,3,4,5,6,7 \\ \lceil \frac{n}{3} \rceil & if \;\; n \geqslant 8 \end{cases}$

are valid.

PROOF. 1. In [**3**], $I(K_{(a_1,a_2,\cdots,a_r)} = p-r+1$. Therefore $DI(K_{(a_1,a_2,\cdots,a_r)} \geqslant p-r+1$. Let S be the largest partite set of $K_{(a_1,a_2,\cdots,a_r)}$. Let $T = V-S$. Then T is a dominating set and $|T| + m_e(G-T) = p-r$. Therefore $DI(K_{(a_1,a_2,\cdots,a_r)} \leqslant p-r+1$. Hence the result.

2. and 3. follows from (1).

4. Let $V(C_n) = \{u_1, u_2, \cdots, u_n\}$. It has been proved that $\gamma(C_n) = \lfloor \frac{n}{3} \rfloor$. It is easy to verify that $DI_w(C_n) = 3$ if $n = 3,4$. Let $n \geqslant 5$. For any minimum dominating set D of C_n, $m_e(C_n - D) = 1$. Therefore, $DI_w(C_n) \leqslant \gamma(C_n) + 2 =$

$\lceil \frac{n}{3} + 1$. If S is any dominating set other than a minimum dominating set of C_n, then $|S| + m_e(C_n - S) \geqslant \lceil \frac{n}{3} \rceil$.

5. Let $V(P_n) = \{u_1, u_2, \cdots, u_n\}$. It has been proved that $\gamma(P_n) = \lfloor \frac{n}{3} \rfloor$. It is easy to verify that $DI_w(P_n) = \lfloor \frac{n}{2} + 1$ if $n = 2, 3, 4, 5, 6, 7$. Let $n \geqslant 8$. For any minimum dominating set D of P_n, $m_e(P_n - D) = 1$. Therefore, $DI_w(P_n) \leqslant \gamma(P_n) + 2 = \lceil \frac{n}{3} + 1$. If S is any dominating set other than a minimum dominating set of P_n, then $|S| + m_e(P_n - S) \geqslant \lceil \frac{n}{3} \rceil + 1$. □

PROPOSITION 2.2. (1) *Given any positive integer k, there exists a graph G and a spanning subgraph H of G such that $DI_w(H) - DI_w(G) = k$.*

(2) *Given any positive integer k, there exists a graph G and a spanning subgraph H of G such that $DI_w(G) - DI_w(H) = k$.*

PROOF. 1. Consider $G = K_{3k+3,3}$. G contains a spanning path H of length $3k + 5$. Then $DI_w(G) = 4, DI_w(H) = \lceil \frac{3k+5}{3} \rceil + 2 = k + 4$.

2. Let k be even. Consider $G = K_{\frac{3(k+2)}{2}}$. Let H be the spanning cycle $C_{\frac{3(k+3)}{2}-1}$ of G. Then $DI_w(G) = \frac{3(k+3)}{2} - 1$ and $DI_w(H) = \left\lceil \frac{\frac{3(k+3)}{2} - 1}{3} \right\rceil + 2 = \frac{k+3}{2} + 2$. □

REMARK 2.1. There exists a connected graph G in which $DI_w(G) = DI_w(H)$ for some connected spanning subgraph of G. For: let $G = C_n, n \geqslant 8$. Let H be the spanning path of G. Then $DI_w(G) = DI_w(H) = \lceil \frac{n}{3} \rceil + 2$.

OBSERVATION 2.7.

(1) $DI_w(G) \geqslant DI(G) - 1$.
(2) $DI_w(G) \geqslant \alpha(G)$.

PROOF. 1. Let S be a DI_w-set of G. Then $DI_w(G) \leqslant |S| + m_e(G - S)$ and $m(G - S) \leqslant m_e(G - S) + 1$. Finally

$$DI_w(G) = |S| + m_e(G - S) \geqslant |S| + m(G - S) - 1 = DI(G) - 1$$

. □

OBSERVATION 2.8. $DI_w(G) = n - 1$ *if and only if* $G \cong K_n$.

PROOF. Let $G \cong K_n$. Then we through. Assume $DI_w(G) = n - 1$. Suppose G is not complete, let u and v two non adjacent vertices of G. Since G is connected, $V(G) - \{u, v\}$ is a dominating set of G.
$m(V(G) - \{u, v\}) = 0$.
$DI_w(G) \leqslant |V(G) - \{u, v\}| + 0 = n - 2 + 0 = n - 2$. □

OBSERVATION 2.9. *Let S be a DI_w-set of G. Then $m_e(G - S) \leqslant DI_w(G - S)$.*

PROOF. Let T be a DI_w-set of $G - S$.

$$\begin{aligned} |S| + m_e(G - S) =& DI_w(G) \leqslant m_e(G - (S \cup T)) + |S \cup T| \\ =& |S| + |T| + m_e(G - S - T) \\ =& |S| + DI_w(G - S). \end{aligned}$$

Hence $m_e(G-S) \leqslant DI_w(G-S)$. $\square$

OBSERVATION 2.10. *For any graph G, $DI_w(G) \geqslant \delta(G)$.*

PROOF. Let S be DI_w set of G such that $DI_w(G) = |S|+m_e(G-S)$. Therefore $m_e(G-S) \geqslant \delta(G-S) \geqslant \delta(G) = |S|$.

$$\begin{aligned} DI_W(G) =& |S| + m_e(G-S) \\ \geqslant& |S| + \delta(G) - |S| \\ =& \delta(G). \end{aligned}$$

$\square$

LEMMA 2.1. *Let G be a graph. If G is non-complete, then every DI_w-set of G is a cut-set of G and hence has cardinality at least $\kappa(G)$.*

PROPOSITION 2.3. *For any graph G, $1 \leqslant DI_w(G) \leqslant p-1$. The lower bound attains for $K_{1,p-1}$ and the upper bound attains for a complete graph $K_p, p \geqslant 2$.*

THEOREM 2.1. *For any tree T, $DI_w(G) \geqslant \alpha(T)$.*

PROOF. Let S be a DI_w-set of T and S^* be a minimum covering set of T. Then

$$\begin{aligned} DI_w(T) =& |S| + m_e(T-S) \\ \geqslant& |S^*| + m_e(T-S^*) \\ \geqslant& |S^*| \\ =& \alpha(T) \end{aligned}$$

$\square$

THEOREM 2.2. *Let G be a connected graph of order $p > 1$. Then $DI_w(G) = 1$ if and only if $\alpha(G) = 1$.*

PROOF. Let S be a DI_w-set of G. Since $DI_w(G) = |S| + m_e(G-S) = 1$ and $m_e(G-S) \geqslant 0$, it follows that $|S| = 1$ and $m_e(G-S) = 0$. Thus $|S| = \alpha(G) = 1$. Conversely, consider $\alpha(G) = 1$. Then $G \cong K_{1,p-1}$. Thus $DI_w(G) = 1$. $\square$

THEOREM 2.3. *For any connected graph G, $DI_w(G) = \kappa(G)$ if and only if $\kappa(G) = \alpha(G)$.*

PROOF. Suppose that $DI_w(G) = \kappa(G)$. Let S be a DI_w-set of a graph G such that $DI_w(G) = |S| + m_e(G-S)$. If G is complete, then by proposition (), $DI_w(G) = \kappa(G)$. Thus we may assume that G is non-complete. Since $DI_w(G) = |S| + m_e(G-S) = \kappa(G)$, it follows by Lemma (), that $|S| \geqslant \kappa(G)$. Thus $\kappa(G) + m_e(G-S) \leqslant \kappa(G)$. Therefore, $m_e(G-S) \leqslant 0$. Since $m_e(G-S)$
$geq0$, we have $m_e(G-S) = 0$ and $m(G-S) = 1$. So we have $m(G-S) = 1$. S is a cover set and we have $|S| = \alpha(G)$.

Conversely, let S be a hub set of a graph G. Then we have $|S| \leqslant \kappa(G)$. By lemma (), $G \cong K_p$. Therefore $DI_w(K_p) = p-1$. $\square$

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