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PERIODIC SOLUTIONS FOR SOME DIFFERENTIAL NONLINEAR SYSTEMS WITH SEVERAL DELAYS

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Abstract. By means of continuation theorem of coincidence degree theory and Krasnoselskii-Burton's fixed point theorem we study some differential nonlinear systems of several delays with a deviating argument having the form

$$\begin{cases} \dfrac{dx\left(t\right)}{dt} = \beta\left|x\left(t-\tau\left(t\right)\right)\right|^{\alpha} x\left(t\right) + f\left(t, u\left(t-\sigma\left(t\right)\right)\right) + p\left(t\right), \\ \\ \dfrac{du\left(t\right)}{dt} = a\left(t\right) g\left(u\left(t\right)\right) + G\left(t, x\left(t-\tau\left(t\right)\right), u\left(t-\sigma\left(t\right)\right)\right), \end{cases}$$

where α and β are two parameters with $0 < \alpha < 1$. We give sufficient conditions on β, α, f, g and G to offer, what we hope, an existence criteria of periodic solutions of above system. Some new results on the existence of periodic solutions are obtained. We end by giving an example to illustrate our claim.

1. Introduction

Ordinary and partial differential equations have played for long important roles in the history of theoretical population dynamics, and they will, with no doubt, continue to serve as indispensable tools in future investigations. However, they are generally the first approximations of the considered real systems. More realistic models should include some of the past states of these systems. That is, real problems or system should be modeled by differential equations with time delays. Indeed, the use of delay differential equations (DDEs) in the modeling of population dynamics is currently very active, largely due to the recent rapid progress achieved

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in the understanding of the dynamics of several important classes of delay differential equations and systems. In this paper, we are interested in the analysis of qualitative theory of periodic solutions of delay differential equations. In this work we have been motivated by the papers [**3**]-[**4**], [**6**]-[**15**], [**18**]-[**26**] and the references therein.

The main tool employed in this study is based on some mixed techniques of the Mawhin coincidence degree and the Krasnoselskii's fixed point theorem. For details on Mawhin techniques, we refer the reader to Gaines and Mawhin [**5**]. Here, we obtain various sufficient conditions for the existence of periodic solutions for the problem (1.1)–(1.2) below, by employing two available operators and by applying coincidence degree theorem and fixed point theorem.

We consider the nonlinear system of several delays equations

$$\frac{dx(t)}{dt} = \beta \left| x(t-\tau(t)) \right|^{\alpha} x(t) + f(t, u(t-\sigma(t))) + p(t), \tag{1.1}$$

$$\frac{du(t)}{dt} = a(t) g(u(t)) + G(t, x(t-\tau(t)), u(t-\sigma(t))), \tag{1.2}$$

where, $G \in C(\mathbb{R}^3, \mathbb{R})$, $f \in C(\mathbb{R}, \mathbb{R})$ and $a, p \in C(\mathbb{R}, \mathbb{R})$. All of the above functions are supposed to be continuous, T–periodic with $T > 0$ is a constant and $0 < \beta$ and $0 < \alpha < 1$ are two parameters.

2. Preliminaries

For $T > 0$, let C_T be the set of all continuous scalar functions x, periodic in t of period T. Let us begin with some known notions and notations used in the theory of coincidence degree theorem which are taken from [**17, 5, 16**] and which we will apply here. We seek conditions under which there exists a T–periodic function x which can be solution of (1.1) for all functions $u \in C_T$. Otherwise speaking, our result here of existence of T–periodic solutions of equation (1.1) doesn't depend on the choice of $y \in C_T$. For that end some preparations and notations are needed. Clearly, $(C_T, \|\cdot\|)$ is a Banach space when endowed with the supremum

$$\|x\| = \sup_{t\in\mathbb{R}} |x(t)| = \sup_{t\in[0,T]} |x(t)|.$$

The method we use, for proving existence, in this paper involves the applications of the continuous theorem of coincidence degree (see Gaines and Mawhin [**5**]). This theorem needs some introduction.

Let X and Z be two Banach spaces. Consider the operator equation

$$Lx = \lambda N(x, \lambda), \ \lambda \in (0, 1)$$

where $L : X \cap DomL \to Z$ is a linear operator and λ is a parameter. Let P and Q denote two projectors such that

$$P : X \cap DomL \to \ker L \text{ and } Q : Z \to Z/ImL.$$

A linear mapping $L : X \cap DomL \to Z$ with $\ker L = L^{-1}(0)$ and $ImL = L(DomL)$, will be called a Fredholm mapping if the following two conditions hold;

(i) $\ker L$ has a finite dimension;

(ii) ImL is closed and has a finite codimension.

Recall also that the codimension of ImL is the dimension of Z/ImL, i.e., the dimension of the cokernel $co\ker L$ of L. When L is a Fredholm mapping, its index is the integer $Ind(L) = \dim\ker L - co\dim ImL$. We shall say that a mapping N is L–compact on Ω if the mapping $QN : \bar{\Omega} \to Z$ is continuous, $QN\left(\bar{\Omega}\right)$ is bounded, and $K_P(I-Q)N : \bar{\Omega} \to X$ is compact, i.e., it is K_P is continuous and $K_P(I-Q)N\left(\bar{\Omega}\right)$ is relatively compact, where $K_P : ImL \to DomL \cap \ker P$ is the inverse of the restriction L_P of L to $DomL \cap \ker P$, so that $LK_P = I$ and $K_P L = I - P$.

Now, we state the continuous theorem of coincidence degree (Gaines, Mawhin [**5**]) which enables us to prove the existence of periodic solutions to (1.1). For its proof we refer the reader to [**5**].

Lemma 2.1. *Let X and Z be two Banach spaces and L a Fredholm mapping of index zero. Assume that $N : \bar{\Omega} \times [0,1] \to Z$ is L–compact on $\bar{\Omega} \times [0,1]$ with Ω open bounded in X. Furthermore, we assume that*

(a) *for each $\lambda \in (0,1)$, $x \in \partial\Omega \cap DomL$*

$$Lx \neq \lambda N(x,\lambda),$$

(b) *for each $x \in \partial\Omega \cap \ker L$,*

$$QNx \neq 0$$

and

$$\deg\{QNx, \Omega \cap \ker L, 0\} \neq 0.$$

Then the equation $Lx = N(x,1)$ has at least one solution in $\bar{\Omega}$.

One captivating theorem which constitutes a basis for our main result, is a reformulated version of Krasnoselskii's fixed point theorem due to Burton. Burton has noticed that the theorem of Krasnoselskii can be more interesting in application in existence and stability in differential equations with certain changes (see [**2**], Theorem 3 and [**3**]).

Definition 2.1 (Large Contraction). Let $(\mathcal{M}, d)$ be a metric space and consider $\mathcal{B} : \mathcal{M} \to \mathcal{M}$. Then, $\mathcal{B}$ is said to be a large contraction if given $\phi, \varphi \in \mathcal{M}$ with $\phi \neq \varphi$ then $d(\mathcal{B}\phi, \mathcal{B}\varphi) < d(\phi,\varphi)$ and if for all $\varepsilon > 0$, there exists a $\delta < 1$ such that

$$[\phi, \varphi \in \mathcal{M},\ d(\phi,\varphi) \geqslant \varepsilon] \Longrightarrow d(\mathcal{B}\phi, \mathcal{B}\varphi) \leqslant \delta d(\phi,\varphi).$$

Theorem 2.1. *Let $\mathcal{M}$ be a closed bounded convex nonempty subset of a Banach space $(X, \|\cdot\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathcal{M}$ into $\mathcal{M}$ such that*

(i) *$x, y \in \mathcal{M}$, implies $\mathcal{A}x + \mathcal{B}y \in \mathcal{M}$,*

(ii) *$\mathcal{A}$ is compact and continuous,*

(iii) *$\mathcal{B}$ is a large contraction mapping.*

Then there exists $z \in \mathcal{M}$ with $z = \mathcal{A}z + \mathcal{B}z$.

The following lemma is crucial to our results. It's prove is due to Adivar, Islam and Raffoul (see [**1**]).

Now, define the mapping H by $H(x) := x - g(x)$ where the function $g : \mathbb{R} \to \mathbb{R}$ satisfies the following assumptions,

(**H1**) g is continuous on $[-l, l]$ and differentiable on $(-l, l)$,
(**H2**) g is strictly increasing on $[-l, l]$,
(**H3**) $\sup_{x\in(-l,l)} g'(x) \leqslant 1$.

LEMMA 2.2. *Let $g : \mathbb{R} \to \mathbb{R}$ be a function satisfying (**H1**)–(**H3**). Then,*
(i) *the mapping H is a large contraction on the set C_T^l,*
(ii) *for $x, y \in C_T^l$ there exists $0 < \eta < 1$ such that $\|H(x) - H(y)\| \leqslant \eta \|x - y\|$.*

3. Existence of periodic solutions

In order to obtain the existence of a positive periodic solution of (1.1). we first make some preparations and begin with the following lemma.

LEMMA 3.1. *Suppose that $z(t)$ and $\omega(t)$ are continuous and nonnegative functions on $[0, T]$. Let $c_1 > 0$, $c_2 > 0$, and $\gamma > 1$ be constants. If*

$$z(t) \leqslant c_1 + c_2 \int_0^t \omega(s) z^\gamma(s) ds, \tag{3.1}$$

and $c_1^{1-\gamma} > (\gamma - 1) c_2 \int_0^t \omega(s) ds$, then

$$z(t) \leqslant \left[c_1^{1-\gamma} - (\gamma - 1) c_2 \int_0^t \omega(s) ds\right]^{\frac{1}{1-\gamma}}.$$

PROOF. Define

$$w(t) := c_2 \int_0^t \omega(s) z^\gamma(s) ds.$$

Clearly, $w(0) = 0$ and by 3.1 one can write

$$w'(t) = c_2\omega(t) z^\gamma(t) \leqslant c_2\omega(t)(c_1 + w(t))^\gamma.$$

So that the last inequality becomes

$$\frac{dw}{(c_1 + w)^\gamma} \leqslant c_2\omega(t) dt. \tag{3.2}$$

Since w is nonnegative and $w(0) = 0$, the integration of 3.2 from 0 to t yields

$$\frac{1}{1-\gamma}\left[(c_1 + w(t))^{1-\gamma} - (c_1)^{1-\gamma}\right] \leqslant c_2 \int_0^t \omega(s) ds.$$

However, rearranging the last inequality we arrive at

$$z(t) \leqslant c_1 + w(t) \leqslant \left[c_1^{1-\gamma} - (\gamma - 1) c_2 \int_0^t \omega(s) ds\right]^{\frac{1}{1-\gamma}}.$$

□

For convenience, we set

$$x^*(t) := \{|x(s)|, \ 0 \leqslant s \leqslant t\}.$$

Then $x^*(t)$ is nonnegative continuous function on $[0, T]$.

As a first case, we consider the following nonlinear equation with delay

$$\frac{dx(t)}{dt} = \beta |x(t-\tau(t))|^{\alpha} x(t) + f(t, u(t-\sigma(t))) + p(t), \ x \in C_T, \ t \in \mathbb{R}, \tag{3.3}$$

where for all $t \in \mathbb{R}$

$$p(t+T) = p(t), \tag{3.4}$$

and

$$\tau(t+T) = \tau(t), \ \ \sigma(t+T) = \sigma(t). \tag{3.5}$$

Assume that the function $f(t,z)$ is continuous and periodic in t of period T. That is

$$f(t+T, z) = f(t,z), t \in \mathbb{R}, z \in C_T. \tag{3.6}$$

Suppose further that there exist a continuous functions $\hat{f}$ and a positive constant l so that $|z| \leqslant l$ implies that

$$|f(t,z)| \leqslant \hat{f}(t) \ \text{ on } \ [0,T]. \tag{3.7}$$

Lemma 3.2. *Assume (3.4)–(3.7) hold. Suppose that in (1.1) the following condition hold*

$$(M_0 + M_1)^{-\alpha} > \alpha\beta T.$$

where

$$M_1 := \sup_{0 \leqslant t \leqslant T} \int_0^t \left[\hat{f}(s) + |p(s)|\right] ds \ \text{ and } M_0 := \sup_{0 \leqslant t \leqslant T} \left(\frac{\left|\hat{f}(t)\right| + |p(t)|}{\beta} \right)^{\frac{1}{\alpha+1}}.$$

Then, the equation (1.1) has at least one T–periodic solution.

Proof. In order to apply Lemma 2.1, we take

$$C_T = Z := \{x \in C(\mathbb{R}, \mathbb{R}) \ | \ x(t+T) = x(t)\},$$

endowed with the norm

$$\|x\| = \sup_{t \in \mathbb{R}} |x(t)| = \sup_{t \in [0,T]} |x(t)|.$$

Clearly, C_T and Z are Banach spaces with such a norm $\|\cdot\|$. Set

$$Lx(t) = \dot{x}(t), \ x \in C_T, \ t \in \mathbb{R},$$

$$N(x(t), \lambda) = \beta |x(t-\tau(t))|^{\alpha} x(t) + \lambda f(t, u(t-\tau(t))) + \lambda p(t),$$
$$\text{for all } x \in C_T \text{ and } t \in \mathbb{R},$$

and

$$Py = Qy = \frac{1}{T}\int_0^T y(t)\,dt, \ y \in C_T.$$

Obviously, $\ker L = \{x \mid x \in C_T, \ x = \xi, \ \xi \in \mathbb{R}\}$, $ImL = \{y \mid y \in C_T, \ \int_0^T y(t)\,dt = 0\}$ are closed in C_T and $\dim\ker L = co\dim ImL$. Hence, L is a Fredholm mapping of index zero. Furthermore, the generalized inverse (to L) $K_P : ImL \to \ker P \cap DomL$ has the form

$$K_P(x) = \int_0^t x(s)\,ds - \frac{1}{T}\int_0^T \int_0^t x(s)\,dsdt.$$

One has

$$\begin{aligned}&(QN)(x,\lambda)\\ &= \frac{1}{T}\int_0^T \left[\beta\left|x(t-\tau(t))\right|^{\alpha} x(t) + \lambda f(t, x(t-\sigma(t))) + \lambda p(t)\right] dt,\end{aligned}$$

and

$$\begin{aligned}&K_P(I-Q)N(x,\lambda)\\ &= -\frac{1}{T}\int_0^T \left[\beta x^{\alpha}(t-\tau(t))\, x(t) + \lambda f(t, u(t-\tau(t))) + \lambda p(t)\right] dt\\ &+ \frac{1}{T}\int_0^T \int_0^t \left[\beta\left|x(s-\tau(s))\right|^{\alpha} x(s) + \lambda f(s, u(s-\tau(s))) + \lambda p(s)\right] dsdt\\ &+ \left(\frac{t}{T} - \frac{1}{2}\right)\int_0^T \left[\beta\left|x(t-\tau(t))\right|^{\alpha} x(t) + \lambda f(t, u(t-\tau(t))) + \lambda p(t)\right] dt.\end{aligned}$$

Clearly, QN and $K_P(I-Q)N$ are continuous. Moreover, $QN\left(\bar{\Omega}\times[0,1]\right)$ and $K_P(I-Q)N\left(\bar{\Omega}\times[0,1]\right)$ are relatively compact for any open bounded set $\Omega \subset C_T$. Hence, N is L–compact on $\bar{\Omega}$. The corresponding differential equation for the operator $Lx = \lambda N(x(t),\lambda)$, $\lambda \in (0,1)$, takes the form

$$\dot{x}(t) = \lambda\beta\left|x(t-\tau(t))\right|^{\alpha} x(t) + \lambda^2 f(t, u(t-\sigma(t))) + \lambda^2 p(t). \tag{3.8}$$

Let $x \in C_T$ be a solution of (3.8) for some $\lambda \in (0,1)$. By integrating (3.8) over the interval $[0,T]$, we obtain

$$\int_0^T \left[\lambda\beta\left|x(t-\tau(t))\right|^{\alpha} x(t) + \lambda^2 f(t, u(t-\sigma(t))) + \lambda^2 p(t)\right] dt = 0.$$

It follows, from the mean value theorem, that there is a point $\xi \in [0,T]$ such that

$$\beta\left|x(\xi-\tau(\xi))\right|^{\alpha} x(\zeta) + \lambda f(\xi, u(\xi-\sigma(\xi))) + \lambda p(\xi) = 0.$$

Moreover, the fact that $\beta > 0$ ensures that for any $u \in C_T$ we have

$$\left|x(\zeta)\right|\left|x(\xi-\tau(\xi))\right|^{\alpha} \leqslant \frac{\left|f(\xi, u(\xi-\sigma(\xi)))\right| + \left|p(\xi)\right|}{\beta}.$$

Now, $\xi - \tau(\xi) = t_0 + mT$, for some $t_0 \in [0,T]$ and some integer m. Then,

$$\left|x(\xi-\tau(\xi))\right| = \left|x(t_0+mT)\right| = \left|x(t_0)\right|.$$

Let $\tilde{x} := \min\{|x(t_0)|, |x(\xi)|\}$. So, making use of (3.7), we obtain

$$\begin{aligned}(\tilde{x})^{1+\alpha} &\leqslant |x(\zeta)| \, |x(\xi-\tau(\xi))|^{\alpha} \leqslant \frac{|f(\xi, x(\xi-\sigma(\xi)))| + |p(\xi)|}{\beta} \\ &\leqslant \frac{\left|\hat{f}(\xi)\right| + |p(\xi)|}{\beta}.\end{aligned}$$

Thus,

$$\tilde{x} \leqslant \left(\frac{\left|\hat{f}(\xi)\right| + |p(\xi)|}{\beta}\right)^{\frac{1}{\alpha+1}} \leqslant M_0.$$

So, let $t^* \in \{\xi, t_0\}$. Remembering the definition of $\tilde{x}$, we can have

$$|x(t^*)| \leqslant M_0.$$

Consequently,

$$\begin{aligned}|x(t)| &\leqslant |x(t^*)| + \int_{t^*}^{t} |\dot{x}(t)| \, dt \\ &\leqslant M_0 + \int_0^t |\beta \, |x(s-\tau(s))|^{\alpha} x(s) + f(s, x(s-\sigma(s))) + p(s)| \, ds.\end{aligned}$$

Also

$$|x(t)| \leqslant x^*(t) \leqslant \beta \int_0^t (x^*)^{\alpha+1}(s) + M_0 + \sup \int_0^t \left[\hat{f}(s) + |p(s)|\right] ds.$$

By using Lemma 2.2 with $z(t) := x^*(t)$, $c_2 = \beta$, $\varphi \equiv 1$, $\gamma = 1+\alpha$ and $c_1 := M_0 + M_1$ we obtain

$$\begin{aligned}|x(t)| \leqslant x^*(t) &\leqslant \left[(M_0+M_1)^{-\alpha} - \alpha\beta t\right]^{\frac{-1}{\alpha}} \\ &\leqslant \left[(M_0+M_1)^{-\alpha} - \alpha\beta t\right]^{\frac{-1}{\alpha}}, \; 0 \leqslant t \leqslant T,\end{aligned}$$

we deduce that for $t \in [0, T]$

$$\begin{aligned}\|x(t)\| &\leqslant \left[(M_0+M_1)^{-\alpha} - \alpha\beta t\right]^{\frac{-1}{\alpha}} \\ &\leqslant \left[(M_0+M_1)^{-\alpha} - \alpha\beta T\right]^{\frac{-1}{\alpha}} =: J. \qquad (3.9)\end{aligned}$$

For $0 < \alpha < 1$, in view of (3.9), we found a constant $J > 0$ which is independent of u and α such that

$$\|x\| \leqslant J \text{ for all } x \in C_T \text{ and } u \in C_T.$$

Now take

$$\Omega_1 = \{x \in C_T \mid \|x\| \leqslant J\}.$$

Clearly, Ω_1 is a closed convex bounded subset of a Banach space. So Ω_1 satisfies the condition (a) in Lemma 2.1. When $x \in \partial\Omega_1 \cap \ker L = \partial\Omega_1 \cap \mathbb{R}$, x is a constant in $\mathbb{R}$ with $\|x\| = J$. Consequently,

$$\begin{aligned}(QN)(x,0) &= -\frac{\beta}{T}\int_0^T |x(t-\tau(t))|^\alpha x(t)\,dt \\ &= -\frac{\beta}{T}\int_0^T J^\alpha(\pm J)\,dt = \pm\beta J^{\alpha+1} \neq 0.\end{aligned}$$

Finally, consider the mapping

$$\Psi(x,\mu) = \mu x + (QN)(x,0),\ \ \mu \in [0,T].$$

For every $\mu \in [0,1]$ and x belonging to the intersection of $\ker L$ and $\partial\Omega_1$, we have

$$x\Psi(x,\mu) = \mu x^2 + \frac{(1-\mu)}{T}x\int_0^T \beta|x(t-\tau(t))|^\alpha x(t)\,dt > 0.$$

It follows from the property of invariance under a homotopy that

$$\begin{aligned}\deg\{QN(x,0),\Omega_1\cap\ker L,0\} &= \deg\{-x,\Omega_1\cap\ker L,0\} \\ &= \deg\{-x,\Omega_1\cap\mathbb{R},0\} \neq 0.\end{aligned}$$

Since Ω_1 verifies all the requirements of Lemma 2.1 we conclude that (1.1) has at least one T–periodic solution $x \in \Omega_1$. The proof is complete. □

In fact equation (1.1) has a T–periodic solution for all T–periodic function $u \in C_T$ with $\|u\| \leqslant l$. So in this connection we offer existence criteria for the periodic solutions of the (1.2).

Next we return for the problem of existence T–periodic solution with feedback control system (1.1)–(1.2). From the results of the previous sections we derive what follows. Consider the equation (1.2)

$$\frac{du(t)}{dt} = a(t)g(u(t)) + G(t,x(t-\tau(t)),u(t-\sigma(t))).$$

Define

$$\Omega_2 := \{\varphi \in C_T : \|\varphi\| \leqslant J_2\},$$

where $J_2 \in (0,l]$.

In order to simplify notation, we let

$$K(t,s) := \frac{\exp\left(-\int_t^s a(u)\,du\right)}{1-\mu}, \tag{3.10}$$

where μ is as

$$\mu := \exp\left(-\int_0^T a(u)\,du\right). \tag{3.11}$$

Assume that

$$a(t+T) = a(t),\ G(t+T,x,y) = G(t,x,y) \text{ and } \sigma(t+T) = \sigma(t). \tag{3.12}$$

We assume further that

$$\rho = \sup_{0 \leqslant t \leqslant T} \{|a(t)|,\ 0 \leqslant t \leqslant T\},\quad \int_0^T a(u)\,du > 0,$$

$$\lambda = \sup_{0 \leqslant t \leqslant T} \left\{ \int_t^{t+T} |K(t,s)\,a(s)|\,ds \right\}, \tag{3.13}$$

and

$$K_0 = \sup_{0 \leqslant t \leqslant T} \left\{ \int_t^{t+T} |K(t,s)|\,ds \right\}.$$

Throughout this section we assume that there exists constants $k_1, k_2 > 0$ such that for x, u, y and $v \in C_T$ we have

$$|G(t,x,y) - G(t,u,v)| \leqslant k_1 \|x-u\| + k_2 \|y-v\|. \tag{3.14}$$

The following lemma is fundamental to our results.

Lemma 3.3. *Suppose the hypotheses (3.11)–(3.14) hold. Then, u is a T-periodic solution of (1.2) if and only if u is a solution of the integral equation*

$$\begin{aligned} u(t) &= \int_t^{t+T} K(t,s)\,a(s)\,(Hu)(s)\,ds \\ &\quad - \int_t^{t+T} K(t,s)\,G(s, x(s-\tau(s)), u(s-\sigma(s)))\,ds. \end{aligned} \tag{3.15}$$

Proof. In the proof we may assume that $x \in C_T$ and we choose $H(u) = u - g(u)$ so that (1.2) may be written as

$$u'(t) - a(t)\,u(t) = -a(t)\,(Hu)(t) + G(t, x(t-\tau(t)),, u(t-\sigma(t))).$$

Multiply both sides of the above equation by $e^{\int_t^{+\infty} a(s)ds}$ to obtain

$$\begin{aligned} &\frac{d}{dt}\left(u(t)\,e^{\int_t^{+\infty} a(s)ds}\right) \\ &= -a(t)\,e^{\int_t^{+\infty} a(s)ds}\,(Hu)(t) + e^{\int_t^{+\infty} a(s)ds} G(t, x(t-\tau(t)),, u(t-\sigma(t))). \end{aligned}$$

Upon integration from t to $t+T$ we get

$$\begin{aligned} u(t) &= \int_t^{t+T} K(t,s)\,a(s)\,(Hu)(s)\,ds \\ &\quad - \int_t^{t+T} K(t,s)\,G(s, x(s-\tau(s)), u(s-\sigma(s)))\,ds. \end{aligned}$$

□

Define on C_T the operators P_1 and P_2 as follows

$$(P_1 u)(t) := \int_t^{t+T} K(t,s)\,a(s)\,(Hu)(s)\,ds, \tag{3.16}$$

and

$$(P_2 u)(t) := -\int_t^{t+T} K(t,s)\, G(s, x(s-\tau(s)), u(s-\sigma(s)))\, ds. \tag{3.17}$$

It is clear in view of (3.15), (3.16), (3.17) and the above analysis that the existence of periodic solutions for (1.1) is equivalent to the existence of solutions for the operator equation

$$P_1 u + P_2 u = u \text{ in } \Omega_2. \tag{3.18}$$

LEMMA 3.4. *Let P_1 defined in (3.16). Assume that the hypotheses (**H1**), (**H2**) and (**H3**) hold. If*

$$\lambda\eta \leqslant 1, \tag{3.19}$$

where η satisfies (ii) in Lemma 2.2. Then, $P_1 : \Omega_2 \to \Omega_2$ is a large contraction.

PROOF. Obviously $P_1\varphi$ is continuous whenever φ is such. It is also easy to check that $P_1\varphi \in C_T$. Now, for any $\varphi \in C_T$ we have

$$\begin{aligned} |(P_1\varphi)(t)| &= \left| \int_t^{t+T} G(t,s)\, a(s) [\varphi(s) - g(\varphi(s))]\, ds \right| \\ &\leqslant \sup_{0\leqslant t\leqslant T} \left| \int_t^{t+T} |G(t,s)\, a(s)|\, ds \right| \|\varphi(t) - g(\varphi(t))\| \\ &\leqslant \lambda \|\varphi - g(\varphi)\| = \lambda \|H\varphi\|. \end{aligned}$$

Since $\|\varphi\| \leqslant J_2$ and H is a large contraction on Ω_2 from Lemma (2.2), then we have

$$|(P_1\varphi)(t)| \leqslant \lambda \|H\varphi\| \leqslant \lambda\eta \|\varphi\| \leqslant J_2.$$

Thus, $P_1\varphi \in \Omega_2$. Consequently, P_1 maps Ω_2 into itself. That is $P_1 : \Omega_2 \to \Omega_2$. Now, let $\varepsilon \in (0,1)$ be given. and let with . Since H is a large contraction on Ω_2, then by using condition (ii) in Lemma (2.2) one can choose, for $\varphi, \phi \in \Omega_2$ with $\|\varphi - \phi\| \geqslant \varepsilon$, $0 < \delta < 1$ such that

$$\begin{aligned} |(P_1\varphi)(t) - (P_1\phi)(t)| &\leqslant \lambda |(H\varphi)(t) - (H\phi)(t)| \leqslant \lambda\delta \|\varphi - \phi\| \\ &\leqslant \delta \|\varphi - \phi\|. \end{aligned}$$

Then, $\|P_1\varphi - P_1\phi\| \leqslant \delta \|\varphi - \phi\|$. Consequently, P_1 is a large contraction. □

LEMMA 3.5. *Assume the conditions of Lemma 3.3, hold. Suppose also that conditions (3.4)–(3.6) hold. Then, $P_2 : C_T \to C_T$ and the image of P_2 is contained in a compact set, where P_2 is defined by (3.17).*

PROOF. Let P_2 be defined by (3.17). A simple change of variables shows that $P_2\varphi$ is periodic i.e., $(P_2\varphi)(t+T) = (P_2\varphi)(t)$. To see that P_2 is continuous on C_T it suffices to show that for all φ, ϕ in C_T such that $\|\varphi - \phi\| \leqslant \xi$ implies $\|P_2\varphi - P_2\phi\| \leqslant$

ε. So, let $\varphi, \phi \in C_T$. For $\varepsilon > 0$ arbitrary we define $\xi = \frac{\varepsilon}{N}$ with $N := k_2 K_0$. Now, if $\|\varphi - \phi\| \leqslant \xi$ we observe that

$$\begin{aligned}
&\left|(P_2\varphi)(t) - (P_2\phi)(t)\right| \\
&\leqslant \int_t^{t+T} |K(t,s)\left[G(s, x(s-\tau(s)), \varphi(s-\sigma(s)))\right. \\
&\left. - G(s, x(s-\tau(s)), \phi(s-\sigma(s)))\right]| \, ds \\
&\leqslant \int_t^{t+T} |K(t,s)| \left\{k_1 |x(s-\tau(s)) - x(s-\tau(s))|\right. \\
&\left. + k_2 |\varphi(s-\sigma(s)) - \phi(s-\sigma(s))|\right\} ds \\
&\leqslant k_2 \int_t^{t+T} |K(t,s)| \, |\varphi(s-\sigma(s)) - \phi(s-\sigma(s))| \, ds \\
&\leqslant k_2 \sup_{0\leqslant t\leqslant T} \left\{\int_t^{t+T} |K(t,s)| \, ds\right\} \|\varphi - \phi\| \\
&\leqslant k_2 K_0 \|\varphi - \phi\| \leqslant \varepsilon. \qquad (3.20)
\end{aligned}$$

To show that the image of P_2 is contained in a compact set, we calculate $\frac{d}{dt}(P_2\varphi_n)(t)$ and show that it is uniformly bounded. For that end, we take the derivative in (3.17) and obtain

$$\begin{aligned}
\frac{d}{dt}(P_2\varphi_n)(t) = &- \int_t^{t+T} \left(\frac{\partial}{\partial t} K(t,s)\right) G(s, x(s-\tau(s)), \varphi(s-\sigma(s))) \, ds \\
&- K(t, t+T) \, G(t+T, x(t+T-\tau(t+T)), \varphi(t+T-\sigma(t+T))) \\
&+ K(t,t) \, G(t, x(t-\tau(t)), \varphi(t-\sigma(t))) \\
= &-a(t) \int_t^{t+T} K(t,s) \, G(s, x(s-\tau(s)), \varphi(s-\sigma(s))) \, ds \\
&+ G(t, x(t-\tau(t)), \varphi(t-\sigma(t))), \qquad (3.21)
\end{aligned}$$

where $K(t,s)$ is given in (3.10). Let $G_0 := \sup_{0\leqslant t\leqslant T} |G(t,0,0)|$. By (3.19)we have

$$|G(t,x,y) - G(t,0,0)| + |G(t,0,0)| \leqslant k_1 \|x\| + k_2 \|y\| + G_0 \qquad (3.22)$$

For that, by making use of (3.21)–(3.22) Thus the above expression yields

$$\begin{aligned}
\left\|\frac{d}{dt}(P_2\varphi_n)\right\| &= \left[\frac{\rho T}{1-\mu} + 1\right] \left[k_1 \|x\| + k_2 \|\varphi\| + G_0\right] \\
&= \left[\frac{\rho T}{1-\mu} + 1\right] \left[k_1 J_1 + J_2 k_2 + G_0\right] \\
&=: D,
\end{aligned}$$

for some positive constant D. Thus the sequence $(P_2\varphi_n)$ is uniformly bounded and equicontinuous. By Ascoli-Arzela's theorem the set $\{P_2\varphi : \varphi \in C_T\}$ is equicontinuous. So, P_2 is a compact operator on C_T. Also, form (3.20) $P_2 : C_T \to C_T$. Consequently, $\{P_2\varphi : \varphi \in C_T\}$ is contained in a compact subset of C_T. □

Lemma 3.6. *Under the hypotheses of lemmas 3.4, 3.5, if*

$$[\lambda\eta + K_0 k_2] J_2 + K_0 [k_1 J + G_0] \leqslant J_2,$$

then, equation (1.2) has a $T-$periodic solution u in the subset Ω_2 for any $x \in \Omega_1$.

Proof. From lemma 3.5 the operator $P_2 : \Omega_2 \to \Omega_2$ is continuous and compact. Also, from lemma 3.4, the operator $P_1 : \Omega_2 \to \Omega_2$ is a large contraction. Moreover, for any $\varphi, \phi \in \Omega_2$, we see that $|(P_1\varphi)(t)| \leqslant \lambda \|H\varphi\| \leqslant \lambda\eta \|\varphi\|$, and

$$\begin{aligned}
\|P_1\varphi + P_2\phi\| &\leqslant \|P_1\varphi\| + \|P_2\phi\| \\
&\leqslant \lambda\eta \|\varphi\| + K_0 [k_1 \|x\| + k_2 \|\phi\| + G_0] \\
&\leqslant \lambda\eta J_2 + K_0 [k_1 J + k_2 J_2 + G_0] \\
&\leqslant [\lambda\eta + K_0 k_2] J_2 + K_0 k_1 J + K_0 G_0 \\
&\leqslant J_2.
\end{aligned}$$

Thus $P_1\varphi + P_2\phi \in \Omega_2$. Clearly, all the hypotheses of the Theorem 2.1 are satisfied. Thus there exists a fixed point $u \in \Omega_2$ such that $P_1 u + P_2 u = u$ for $x \in \Omega_1$ □

Theorem 3.1. *Assume that all hypotheses of lemmas 3.2 and 3.6 hold true. Then, system (1.1)–(1.2) has at least one T–periodic solution $(x, u) \in \Omega_1 \times \Omega_2$.*

Example 3.1. Let $\alpha = 0.01$ and $\beta = 0.8$. Consider the following the neutral differential system of equations

$$\frac{dx(t)}{dt} = \beta |x(t - \tau(t))|^{\alpha} x(t) + f(t, u(t - \sigma(t))) + p(t), \tag{3.23}$$

$$\frac{du(t)}{dt} = a(t) g(u(t)) + G(t, x(t - \tau(t)), u(t - \sigma(t))), \tag{3.24}$$

where

$$\tau(t) = \cos^2(t), \ \sigma(t) = \sin^2(t).$$

Suppose that the functions a, p, f,g and G are defined as follows

$$a(t) = \frac{1}{4}\sin^2(t), \ p(t) = \frac{2 + \cos(2t)}{10}, \ g(x) = \frac{1}{4}\arctan(x), \ f(t, u) = \frac{1}{10} e^{-u^2},$$

and

$$G(t, x, u) = \cos(2t) \left\{ \log(|x| + 1) + \frac{1}{16 e^{-1/8}} \sin(u) \right\}.$$

Then the system (3.23)–(3.24) has a π–periodic solution.

Proof. Notice first that $|f(t, u)| \leqslant \hat{f} \equiv \frac{1}{2}$ for $u \in \mathbb{R}$. A simple calculation yields $G_0 = G(t, 0, 0) = 0$, $\mu = e^{-\frac{\pi}{8}}$

$$\lambda = \frac{1}{(1-\mu)} \sup_{0 \leqslant t \leqslant \pi} \int_t^{t+\pi} \frac{1}{4} \sin^2(s) e^{-\frac{1}{4}\int_t^s \sin^2(v) dv} = 1.$$

Also

$$\frac{1}{(1-\mu)} \sup_{0\leqslant t\leqslant \pi} \int_t^{t+\pi} e^{-\int_t^s \frac{1}{4}\sin^2(v)dv} = \frac{1}{(1-\mu)} e^{\frac{1}{16}} \sup_{0\leqslant t\leqslant T} e^{\frac{1}{8}t-\frac{1}{16}\sin 2t} \int_t^{t+\pi} e^{-\frac{1}{8}s} ds$$
$$\leqslant \frac{1}{(1-\mu)} 8e^{\frac{1}{16}} \sup_{0\leqslant t\leqslant \pi} e^{-\frac{1}{16}\sin 2t}\left(1-e^{-\frac{\pi}{8}}\right)$$
$$\leqslant \frac{1}{(1-\mu)} 8e^{\frac{1}{8}}(1-\mu)$$
$$= 8e^{1/8} = K_0.$$

Furthermore,

$$M_1 = \frac{\pi+2}{10} \text{ and } M_0 = \left(\frac{\frac{1}{2}+\frac{2+1}{10}}{0.8}\right)^{\frac{1}{1.01}}. \tag{3.25}$$

In fact $\alpha = 0.01$ and $\beta = 0.8$ and from (3.25) it is easy to show that

$$\begin{aligned}(M_0+M_1)^{-\alpha} - \alpha\beta T &= \frac{1}{\left(\frac{\pi+2}{10}+\left(\frac{8}{(0.8)10}\right)^{\frac{1}{1.01}}\right)^{0.01}} - (0.01)(0.8)\pi \\ &= 0.99586 - 0.02513 = 0.970730,\end{aligned}$$

that is

$$(M_0+M_1)^{-\alpha} > \alpha\beta T.$$

It follows

$$\left[(M_0+M_1)^{-\alpha} - \alpha\beta T\right]^{\frac{-1}{\alpha}} = (0.970730)^{\frac{-1}{0.01}} = 19.505 \leqslant 20 := J.$$

Moreover, for all $x, y, v, u \in \mathbb{R}$, the calculations show that

$$|G(t,x,y) - G(t,v,u)| \leqslant k_1 |x-v| + k_2 |y-u|,$$

where $k_1 = 1$ and $k_2 = \frac{1}{16e^{-1/8}}$.

On the other hand, since the function $g(u)$ is a strictly increasing on $\mathbb{R}$ and

$$0 \leqslant g'(u) = \frac{1}{4}\frac{1}{u^2+1} \leqslant \frac{1}{4} < 1 \text{ for all } u \in \mathbb{R}.$$

Consequently, for any positive number J, we can choose $J_2 = l$ so that

$$J_2 = l \geqslant 4K_0 k_1 J.$$

and

$$0 < \frac{1}{4}\frac{1}{l^2+1} < g'(u) = \frac{1}{4}\frac{1}{u^2+1} \leqslant \frac{1}{4} < 1 \text{ for all } u \in [-l, l].$$

Let $\Omega_1 = [-J, J]$ and $\Omega_2 = [-J_2, J_2]$. We find

$$\lambda\eta < 1.$$

and

$$\begin{aligned}
[\lambda\eta + K_0k_2]\, J_2 + K_0k_1J + K_0G_0 &= [\lambda\eta + K_0k_2]\, J_2 + K_0k_1J \\
&\leqslant \left[\frac{1}{4} + \frac{1}{16e^{-1/8}}8e^{1/8}\right] J_2 + K_0k_1J \\
&\leqslant \frac{3}{4}J_2 + \frac{1}{4}J_2 = J_2.
\end{aligned}$$

Thus, under these hypotheses the system (3.23)–(3.24) satisfies all the conditions of Theorem 3.1. Hence, the system (3.23)–(3.24) has at least one positive π–periodic solution $(x, u) \in \Omega_1 \times \Omega_2$. □

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