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AVERAGE INDEPENDENT DOMINATION IN COMPLEMENTARY PRISMS

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ABSTRACT. Let G be a graph and $\overline{G}$ be the complement of G. The complementary prism $G\overline{G}$ of G is the graph formed from the disjoint union of G and $\overline{G}$ by adding the edges of a perfect matching between the corresponding vertices of G and $\overline{G}$. The average independent domination number $i_{av}(G)$ of a graph G is defined as $\frac{1}{|V|}\sum_{v\in V} i_v(G)$, where $i_v(G)$ is the minimum cardinality of a maximal independent set that contains v. In this paper, we consider the average independent domination in complementary prisms. We determine the average independent domination number of $G\overline{G}$ for specific graphs G and characterize the complementary prisms with small average independent domination numbers. We also present bounds on the average independent domination numbers of complementary prisms.

1. Introduction

Graph theoretic techniques provide a convenient tool for the investigation of networks. It is well-known that an interconnection network can be modeled by a graph with vertices representing sites of the network and edges representing links between sites of the network. Therefore various problems in networks can be studied by graph theoretical methods. The study of domination in graphs is an important research area, perhaps also the fastest-growing area within graph theory. The reason for the steady and rapid growth of this area may be the diversity of its applications to both theoretical and real-world problems. For instance, dominating sets in graphs are natural models for facility location problems in operations research. Research on domination in graphs has not only important theoretical

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signification, but also varied application in such fields as computer science, communication networks, ad hoc networks, biological and social networks, distributed computing, coding theory, and web graphs. Domination and its variations have been extensively studied [**1, 2, 5, 6, 3, 10, 11**]. In some sense, one could say that the domination based parameters reveal an underlying efficient communication network. Among the domination-type parameters that have been studied, the average independent domination number is one of the fundamental ones introduced by Henning [**6**].

In this paper, we consider finite undirected graphs without loops and multiple edges. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G)$ and edge set $E(G)$. The order of G, denoted by $|V(G)|$, is the number of vertices in G. The open neighborhood of v is $N(v) = \{u \in V(G) \,|uv \in E(G)\}$ and the closed neighborhood of v is $N[v] = \{v\} \cup N(v)$. For a set $S \subseteq V$, $N(S) = \bigcup_{v\in S} N(v)$ and $N[S] = N(S) \cup S$. The degree of a vertex v is $\deg_G(v) = |N(v)|$. A vertex of degree zero is an isolated vertex or an isolate. A leaf or an endvertex is a vertex of degree one and its neighbor is called a support vertex. The minimum degree of G is $\delta(G) = \min\{\deg_G(v) \,|v \in V(G)\}$. For $S \subseteq V(G)$, the subgraph of G induced by S is denoted by $G[S]$. A set S is a dominating set of G if $N[S] = V(G)$, or equivalently, every vertex in $V(G) - S$ is adjacent to a vertex of S. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. The independence number $\beta(G)$ of G is the maximum cardinality of an independent set in G, while the independent domination number $i(G)$ of G is the minimum cardinality of a maximal independent set of G [**12, 4, 8, 7**].

The independent domination number $i_v(G)$ of G relative to v is the minimum cardinality of a maximal independent set in G that contains v. A maximal independent set of cardinality $i_v(G)$ containing v is an $i_v(G)$-set. The average independent domination number of G is $i_{av}(G) = \frac{1}{|V|}\sum_{v\in V} i_v(G)$ [**6**].

Complementary prisms were first introduced by Haynes, Henning, Slater and Van der Merwe in [**9**]. For a graph G, its complementary prism, denoted by $G\overline{G}$, is formed from a copy of G and a copy of $\overline{G}$ by adding a perfect matching between corresponding vertices.
For each $v \in V(G)$, let $\overline{v}$ denote the vertex v in the copy of $\overline{G}$. Formally $G\overline{G}$ is formed from $G \cup \overline{G}$ by adding the edge $v\overline{v}$ for every $v \in V(G)$.

The corona of a graph G, denoted by $G \circ K_1$, is a graph constructed from a copy of the graph G where each vertex of $V(G)$ is adjacent to exactly one vertex of degree one.
We use $\lfloor x \rfloor$ to denote the largest integer not greater than x, and $\lceil x \rceil$ to denote the least integer not less than x. We let $x \equiv_l y$ mean $x \equiv y \pmod{l}$.

The paper is organized as follows. In section 2, some of the existing literature on average independent domination number is reviewed. The average independent domination number of the complementary prism $G\overline{G}$ when G is a specified family of graphs is computed. The independent domination number of $G\overline{G}$ when G is a specified family of graphs is given as an immediate result. The graphs G for which $i_{av}\left(G\overline{G}\right)$ is small are characterized. Sharp upper and lower bounds on the average independent domination number of the complementary prism $G\overline{G}$ of a graph G in terms of the order of G are presented. Finally, section 3 concludes the paper.

2. The average independent domination number

2.1. Known results.

THEOREM 2.1 ([**6**]). *For a vertex v of G,*

(a) $i(G) \leqslant i_v(G) \leqslant \beta(G)$.
(b) $i(G) \leqslant i_{av}(G) \leqslant \beta(G)$.

THEOREM 2.2 ([**6**]). *For any graph G of order n with independent domination number i and independence number β, $i_{av}(G) \leqslant \beta - i(\beta - i)/n$.*

THEOREM 2.3 ([**6**]). *If T is a tree of order $n \geqslant 2$, then $i_{av}(T) \leqslant n - 2 + 2/n$.*

2.2. Specific families.

OBSERVATION 2.1.

(a) *For $n \geqslant 1$, $\gamma(P_n) = \lceil n/3 \rceil$*
(b) *For $n > 3$, $\gamma(C_n) = \lceil n/3 \rceil$.*
(c) *For $n > 3$, $\gamma\left(\overline{C}_n\right) = 2$.*

THEOREM 2.4.

(a) *If $G = K_n$, then $i_{av}\left(G\overline{G}\right) = n$.*
(b) *If $G = tK_2$, then $i_{av}\left(G\overline{G}\right) = t + 1$.*
(c) *If $G = K_t \circ K_1$, then $i_{av}\left(G\overline{G}\right) = t + 1$.*
(d) *If $G = K_{1,n}$, then $i_{av}\left(G\overline{G}\right) = (n + 3)/2$.*
(e) *If $G = K_{m,n}$ where $2 \leqslant m \leqslant n$, then $i_{av}\left(G\overline{G}\right) = 1 + m + n/2 - mn/(m + n)$.*
(f) *If $G = C_n$ and $n > 3$, then $i_{av}\left(G\overline{G}\right) = \lceil (n + 4)/3 \rceil$.*
(g) *If $G = W_n$ and $n > 3$, then $i_{av}\left(G\overline{G}\right) = \lceil (n + 16)/3 \rceil/2$.*
(h) *If $G = P_n$ and $n \geqslant 2$, then*

$$i_{av}(G\overline{G}) = \begin{cases} (n+5)/3, & \text{if } n \equiv_3 1; \\ (n+10)/6 + ((n-2)/2n)(\lceil (n+1)/3 \rceil), & \text{otherwise.} \end{cases}$$

PROOF. To prove (a), for $G = K_n$, the complementary prism $G\overline{G}$ is the corona $K_n \circ K_1$. Let v be a vertex of $G\overline{G}$. If v is a support vertex, then any $i_v\left(G\overline{G}\right)$-set must contain each leaf of the other support vertices except v. If v is a leaf, then an $i_v\left(G\overline{G}\right)$-set must contain either each leaf or a support vertex which is not adjacent to v and each leaf of the other support vertices. Hence, for any $v \in V\left(G\overline{G}\right)$, $i_v\left(G\overline{G}\right) = n$. Since $\left|V\left(G\overline{G}\right)\right| = 2n$, $i_{av}\left(G\overline{G}\right) = \frac{1}{2n}\left(2n\,(n)\right) = n$.

To prove (b), label the $2t$ vertices of $V(G)$ as $u_i, v_i, 1 \leqslant i \leqslant t$, such that $u_iv_i \in E(G)$. The set $\{u_1, \ldots, u_t\} \cup \{\overline{v}_i\}$ is an $i_v\left(G\overline{G}\right)$-set of $G\overline{G}$ for $v = u_i$ or $\overline{v}_i$ with cardinality $t + 1$, and the set $\{v_1, \ldots, v_t\} \cup \{\overline{u}_i\}$ is an $i_v\left(G\overline{G}\right)$-set of $G\overline{G}$ for $v = \overline{u}_i$ or v_i with cardinality $t+1$. Therefore, for any $v \in V\left(G\overline{G}\right)$, $i_v\left(G\overline{G}\right) = t+1$. Since $\left|V\left(G\overline{G}\right)\right| = 4t$, $i_{av}\left(G\overline{G}\right) = \frac{1}{4t}\left(4t\,(t + 1)\right) = t + 1$.

To prove (c), let $G = K_t \circ K_1$. If $t = 1$, then $G = K_2$ and from (a), $i_{av}\left(G\overline{G}\right) = 2$. Assume that $t \geqslant 2$, and label the vertices of G as follows: let $A = \{a_i \,|1 \leqslant i \leqslant t\}$ be the set of t vertices that induce the K_t subgraph of G, and let $B = \{b_i \,|1 \leqslant i \leqslant t\}$ be the endvertices in G adjacent to the vertices in A such that $a_i b_i \in E(G)$. The set $\left\{\left\{a_i \cup \overline{b}_i\right\} \cup \{B \backslash b_i\}\right\}$ is an $i_v\left(G\overline{G}\right)$-set of $G\overline{G}$ for $v = a_i$ or $\overline{b}_i$ with cardinality $2 + (t-1) = t+1$. If $v = b_i$, then an $i_v\left(G\overline{G}\right)$-set of $G\overline{G}$ is composed of $\left\{b_i \cup a \cup \overline{b} \cup \{B \backslash \{b_i, b\}\}\right\}$ and if $v = \overline{a}_i$, then an $i_v\left(G\overline{G}\right)$-set of $G\overline{G}$ is composed of $\left\{\overline{a}_i \cup \overline{b}_i \cup a \cup \{B \backslash \{b_i, b\}\}\right\}$, both with cardinality $3 + (t-2) = t+1$, where $a \in A$ $(a \neq a_i)$ and $b \in B$ such that $ab \in E(G)$. Thus, for any $v \in V\left(G\overline{G}\right)$, $i_v\left(G\overline{G}\right) = t+1$. Since $\left|V\left(G\overline{G}\right)\right| = 4t$, $i_{av}\left(G\overline{G}\right) = \frac{1}{4t}\left(4t\,(t+1)\right) = t+1$.

To prove (d), since G is a star, the support vertex t in G is an isolated vertex $\overline{t}$ in $\overline{G}$ and a leaf in $G\overline{G}$. Denote the n leaves of G by $\{u_1, u_2, \ldots, u_n\}$. The leaves in G will form a complete graph on n vertices in $\overline{G}$. Let I be an $i_v\left(G\overline{G}\right)$-set of $G\overline{G}$. If $v = t$ or $v = \overline{u}_i$ where $1 \leqslant i \leqslant n$, then $I = \{t\} \cup \{\overline{u}_i\}$, and so $i_v\left(G\overline{G}\right) = 2$. If $v = \overline{t}$ or $v = u_i$ where $1 \leqslant i \leqslant n$, then $I = \left\{\overline{t}, u_1, \ldots, u_n\right\}$, and so $i_v\left(G\overline{G}\right) = n+1$. Consequently, for $\left|V\left(G\overline{G}\right)\right| = 2\,(n+1)$, $i_{av}\left(G\overline{G}\right) = \frac{1}{2(n+1)}\left((n+1)\,(2) + (n+1)\,(n+1)\right) = (n+3)/2$.

To prove (e), let $G = K_{m,n}$ $(2 \leqslant m \leqslant n)$, where R and S are the partite sets of G with cardinality m and n, respectively. Let $R = \{r_1, r_2, \ldots, r_m\}$ and $S = \{s_1, s_2, \ldots, s_n\}$. The vertices of R and S will form complete graphs K_m and K_n on m and n vertices, respectively, in $\overline{G}$.

If $r_i \in R$ where $1 \leqslant i \leqslant m$, then $N_{G\overline{G}}(r_i) = S \cup \{\overline{r}_i\}$. Let I be an $i_{r_i}\left(G\overline{G}\right)$-set of $G\overline{G}$. Since $G\overline{G}\left[\overline{S}\right] = K_n$, $I = \{\overline{s}\} \cup R$, and so $i_{r_i}\left(G\overline{G}\right) = 1 + m$ for $1 \leqslant i \leqslant m$.

If $s_i \in S$ where $1 \leqslant i \leqslant n$, then $N_{G\overline{G}}(s_i) = R \cup \{\overline{s}_i\}$. If I is an $i_{s_i}\left(G\overline{G}\right)$-set of $G\overline{G}$, then since $G\overline{G}\left[\overline{R}\right] = K_m$, $I = \{\overline{r}\} \cup S$. Therefore, $i_{s_i}\left(G\overline{G}\right) = 1 + n$ for $1 \leqslant i \leqslant n$.

If $\overline{r}_i \in \overline{R}$ where $1 \leqslant i \leqslant m$, then $N_{G\overline{G}}(\overline{r}_i) = \{r_i\} \cup \{\overline{r}_j \,|j \neq i \text{ and } 1 \leqslant j \leqslant m\}$. Let I be an $i_{\overline{r}_i}\left(G\overline{G}\right)$-set of $G\overline{G}$. We have $I = \{r_j \,|j \neq i \text{ and } 1 \leqslant j \leqslant m\} \cup \{\overline{r}_i\} \cup \{\overline{s}\}$ with cardinality $(m-1)+1+1 = m+1$. Therefore, for $1 \leqslant i \leqslant m$, $i_{\overline{r}_i}\left(G\overline{G}\right) = m+1$.

If $\overline{s}_i \in \overline{S}$ where $1 \leqslant i \leqslant n$, then $N_{G\overline{G}}(\overline{s}_i) = \{s_i\} \cup \{\overline{s}_j \,|j \neq i \text{ and } 1 \leqslant j \leqslant n\}$. If I is an $i_{\overline{s}_i}\left(G\overline{G}\right)$-set of $G\overline{G}$, then since $m \leqslant n$, we have $I = R \cup \{\overline{s}_i\}$. Thus, for $1 \leqslant i \leqslant n$, $i_{\overline{s}_i}\left(G\overline{G}\right) = m+1$.

As a consequence, being $\left|V\left(G\overline{G}\right)\right| = 2\,(m+n)$,

$$\begin{aligned} i_{av}(G\overline{G}) &= \frac{1}{2(m+n)}(m(1+m) + n(1+n) + m(m+1) + n(m+1)) \\ &= 1 + m + n/2 - mn/(m+n). \end{aligned}$$

To prove (f), let the vertices of $G = C_n$ be labeled sequentially $u_0, u_1, \ldots, u_{n-1}$. If I is an $i_{u_j}\left(G\overline{G}\right)$-set of $G\overline{G}$ where $0 \leqslant j \leqslant n-1$, then $u_j \in I$. We have $N_{G\overline{G}}(u_j) = \{u_{j-1}, u_{j+1}, \overline{u}_j\}$ where $j-1$ and $j+1$ are taken modulo n. Let

$S = V\left(G\overline{G}\right)\backslash N_{G\overline{G}}[u_j]$. Since for all $x \in G$, $y \in \overline{G}$, $\deg_{G\overline{G}[S]}(x) \leqslant \deg_{G\overline{G}[S]}(y)$, also $y \in I$.

Now, let $S = V\left(G\overline{G}\right) \setminus \{N_{G\overline{G}}[u_j] \cup N[y]\}$. It is easy to see that $G\overline{G}[S] = P_{n-2}$. Eventually, the dominating set of $G\overline{G}[S]$ that satisfies

$$\gamma\left(G\overline{G}[S]\right) = \gamma(P_{n-2}) = \lceil (n-2)/3 \rceil,$$

must be included in I. Therefore, $i_{u_j}\left(G\overline{G}\right) = 1 + 1 + \lceil (n-2)/3 \rceil = \lceil (n+4)/3 \rceil$ for $0 \leqslant j \leqslant n-1$.

Now, consider the vertices in $\overline{G} = \overline{C}_n$. If I is an $i_{\overline{u}_j}\left(G\overline{G}\right)$-set, then $\overline{u}_j \in I$ where $0 \leqslant j \leqslant n-1$. Let $S = V\left(G\overline{G}\right)\backslash N_{G\overline{G}}[\overline{u}_j]$. Then, we have $G\overline{G}[S] = C_{n+1}$. Therefore, the dominating set of $G\overline{G}[S]$ that satisfies $\gamma\left(G\overline{G}[S]\right) = \gamma(C_{n+1}) = \lceil (n+1)/3 \rceil$, must be included in I. Hence, $i_{\overline{u}_j}\left(G\overline{G}\right) = 1 + \lceil (n+1)/3 \rceil = \lceil (n+4)/3 \rceil$ for $0 \leqslant j \leqslant n-1$.

Consequently, for $\left|V\left(G\overline{G}\right)\right| = 2n$,

$$i_{av}\left(G\overline{G}\right) = \frac{1}{2n}\left(n\left(\lceil (n+4)/3 \rceil\right) + n\left(\lceil (n+4)/3 \rceil\right)\right) = \lceil (n+4)/3 \rceil.$$

To prove (g), let G be a wheel of order $n+1$ and consider $G\overline{G}$. Since the center vertex c of G is adjacent to every other vertex of G, it is an isolate in $\overline{G}$ and a leaf in $G\overline{G}$. Therefore, if I is an $i_c\left(G\overline{G}\right)$-set of $G\overline{G}$, then first $c \in I$. Since $W_n = C_n + K_1$, if the vertices of C_n are labeled sequentially $u_0, u_1, \ldots, u_{n-1}$ in $G\overline{G}$, then $N_{G\overline{G}}(c) = \{\overline{c}\} \cup \{u_i \,|0 \leqslant i \leqslant n-1\}$. Let $S = V\left(G\overline{G}\right)\backslash N_{G\overline{G}}[c]$. Eventually, $G\overline{G}[S] = \overline{C}_n$. Therefore, the dominating set of $\overline{C}_n$ that satisfies $\gamma\left(\overline{C}_n\right) = 2$, must be included in $i_c\left(G\overline{G}\right)$-set. Hence, $i_c\left(G\overline{G}\right) = 1 + 2 = 3$.

Now, consider the leaf $\overline{c}$ in $G\overline{G}$. Let$S = V\left(G\overline{G}\right)\backslash N_{G\overline{G}}[\overline{c}]$. Then $G\overline{G}[S] = C_n\overline{C}_n$. From (f), we know that $\forall v \in C_n\overline{C}_n$, $i_v\left(C_n\overline{C}_n\right) = \lceil (n+4)/3 \rceil$.

Thus, $i_{\overline{c}}\left(G\overline{G}\right) = 1 + \lceil (n+4)/3 \rceil = \lceil (n+7)/3 \rceil$.

Consider the vertices of C_n which are labeled sequentially $u_0, u_1, \ldots, u_{n-1}$ in $G\overline{G}$. Then, the proof is very similar to the proof of (f). Since $c \in N(u_j)$ where $0 \leqslant j \leqslant n-1$, the leaf $\overline{c}$ must also be included in the maximal independent set of $G\overline{G}$ containing u_j. Therefore, $i_{u_j}\left(G\overline{G}\right) = \lceil (n+4)/3 \rceil + 1 = \lceil (n+7)/3 \rceil$ for $0 \leqslant j \leqslant n-1$.

Consequently, consider the vertices of $\overline{C}_n$ in $G\overline{G}$. For a vertex $\overline{u}_j$ $(0 \leqslant j \leqslant n-1)$ of $G\overline{G}$, $\{\overline{u}_j\} \cup \{c\} \cup \{\overline{u}_i\}$ is a maximal independent set of $G\overline{G}$ containing $\overline{u}_j$ where $i = j-1$ or $i = j+1$ and $i = j-1$, $i = j+1$ are taken modulo n, so $i_{\overline{u}_j}\left(G\overline{G}\right) = 3$.

As a result, for $\left|V\left(G\overline{G}\right)\right| = 2n+2$,
$i_{av}\left(G\overline{G}\right) = \frac{1}{2n+2}\left(\lceil (n+7)/3 \rceil + 3 + n\left(\lceil (n+7)/3 \rceil\right) + (n)(3)\right) = \lceil (n+16)/3 \rceil/2.$

To prove (h), consider the vertices of $\overline{P}_n$ in $G\overline{G}$. If a vertex $\overline{v}$ of $\overline{P}_n$ in $G\overline{G}$ is an endvertex of P_n, then first an $i_{\overline{v}}\left(G\overline{G}\right)$-set must contain $\overline{v}$. Then, let $S = G\overline{G}\backslash N[\overline{v}]$.

Since $G\overline{G}[S] = P_n$, the dominating set of $G\overline{G}[S]$ satisfying $\gamma\left(G\overline{G}[S]\right) = \gamma(P_n) = \lceil n/3 \rceil$ must be included in $i_{\overline{v}}\left(G\overline{G}\right)$-set. Therefore, $i_{\overline{v}}\left(G\overline{G}\right) = 1 + \lceil n/3 \rceil$.

If a vertex $\overline{v}$ of $\overline{P}_n$ in $G\overline{G}$ is not an endvertex of P_n, then let $S = G\overline{G} \backslash N[\overline{v}]$, that is, $G\overline{G}[S] = P_{n+1}$, implying that, the dominating set of $G\overline{G}[S]$ satisfying $\gamma\left(G\overline{G}[S]\right) = \gamma(P_{n+1}) = \lceil (n+1)/3 \rceil$ must be included in a maximal independent set of $G\overline{G}$ containing $\overline{v}$. Therefore, $i_{\overline{v}}\left(G\overline{G}\right) = 1 + \lceil (n+1)/3 \rceil$.

Now consider the vertices of P_n in $G\overline{G}$.

Suppose $n \equiv_3 0$. Then there is a unique $\gamma(P_n)$-set satisfying $\gamma(P_n) = n/3$. Therefore, for a vertex v in $\gamma(P_n)$-set, $i_v\left(G\overline{G}\right) = n/3 + 1$. Now consider the other remaining vertices. For a vertex v that is not in $\gamma(P_n)$-set, $i_v\left(G\overline{G}\right) = n/3 + 2$. Hence,

$$\begin{aligned} i_{av}(G\overline{G}) = &\frac{1}{2n}(2(1+n/3) + (n-2)(1+\lceil (n+1)/3 \rceil) + (n/3)((n/3)+1) \\ &+ (2n/3)((n/3)+2)) \\ = &(n+10)/6 + ((n-2)/2n)(\lceil (n+1)/3 \rceil). \end{aligned}$$

Suppose $n \equiv_3 1$. For a vertex v of P_n in $G\overline{G}$, $i_v\left(G\overline{G}\right) = \lceil n/3 \rceil + 1$. Hence,

$$\begin{aligned} i_{av}(G\overline{G}) = &\frac{1}{2n}(2(1+\lceil n/3 \rceil) + (n-2)(1+\lceil (n+1)/3 \rceil) + n(\lceil n/3 \rceil + 1)) \\ = &(n+5)/3. \end{aligned}$$

Suppose $n \equiv_3 2$. Let the vertices of P_n be labeled sequentially $v_1, v_2, \ldots, v_n$. Then, $i_{v_j}\left(G\overline{G}\right) = \lceil n/3 \rceil + 2$, if $j = 3k$, $1 \leqslant k \leqslant (n-2)/3$, $1 \leqslant j \leqslant n$; $i_{v_j}\left(G\overline{G}\right) = \lceil n/3 \rceil + 1$, otherwise. Hence,

$$\begin{aligned} i_{av}(G\overline{G}) = &\tfrac{1}{2n}(2(1+\lceil n/3 \rceil) + (n-2)(1+\lceil (n+1)/3 \rceil) \\ &+((n-2)/3)(\lceil n/3 \rceil + 2) + ((2n+2)/3)(\lceil n/3 \rceil + 1)) \\ = &(n+10)/6 + ((n-2)/2n)(((\lceil ((n+1)/3 \rceil)). \end{aligned}$$

□

We next determine the independent domination numbers of the complementary prism $G\overline{G}$ when G is a specified family of graphs. Since for a graph G, $i(G) = \min\{i_v(G) \,|\, v \in V(G)\}$ [**3**], as an immediate corollary to Theorem 2.4, we have the following.

Corollary 2.1.

(a) *If* $G = K_n$, *then* $i\left(G\overline{G}\right) = n$.
(b) *If* $G = tK_2$, *then* $i\left(G\overline{G}\right) = t + 1$.
(c) *If* $G = K_t \circ K_1$, *then* $i\left(G\overline{G}\right) = t + 1$.
(d) *If* $G = K_{1,n}$, *then* $i\left(G\overline{G}\right) = 2$.
(e) *If* $G = K_{m,n}$ *where* $2 \leqslant m \leqslant n$, *then* $i\left(G\overline{G}\right) = m + 1$.
(f) *If* $G = C_n$ *and* $n > 3$, *then* $i\left(G\overline{G}\right) = \lceil (n+4)/3 \rceil$.
(g) *If* $G = W_n$ *and* $n > 3$, *then* $i\left(G\overline{G}\right) = 3$.
(h) *If* $G = P_n$ *and* $n \geqslant 2$, *then* $i\left(G\overline{G}\right) = \lceil n/3 \rceil + 1$.

2.3. Small values.

THEOREM 2.5. *For a graph G of order n and its complementary prism $G\overline{G}$,*

$$i_{av}\left(G\overline{G}\right) = 1 \quad \textit{if and only if} \quad n = 1.$$

PROOF. The sufficiency is immediate since if $|V(G)| = 1$, then $G\overline{G} = K_2$. Thus, $i_{av}\left(G\overline{G}\right) = 1$. Now, suppose that $i_{av}\left(G\overline{G}\right) = 1$. Since $G\overline{G}$ has order $2n$, this implies that $\sum_{v\in V\left(G\overline{G}\right)} i_v\left(G\overline{G}\right) = 2n$. For a vertex v of $G\overline{G}$ $i_{\overline{v}}\left(G\overline{G}\right) \geqslant 1$, and so $\sum_{v\in V\left(G\overline{G}\right)} i_v\left(G\overline{G}\right) \geqslant 2n$. Being $\left|V\left(G\overline{G}\right)\right| = 2n$, equality holds if and only if $i_{\overline{v}}\left(G\overline{G}\right) = 1$ for all $v \in V\left(G\overline{G}\right)$ implying that $n = 1$ and $G = K_1$. This establishes the necessity. □

THEOREM 2.6. *For a graph G of order n, if either G or $\overline{G}$ has diameter one, then $i_{av}\left(G\overline{G}\right) = n$.*

PROOF. If either G or $\overline{G}$ has diameter one, then $G\overline{G} = K_n \circ K_1$. Therefore from Theorem 2.4(a), the proof is immediate. □

2.4. Upper and lower bounds.

THEOREM 2.7. *Let G be a graph of order n. Then $i_{av}\left(G\overline{G}\right) \leqslant (3n-1)/2$, and this bound is sharp.*

PROOF. For vertices v and $\overline{v}$ in $G\overline{G}$, $\deg_{G\overline{G}}(v) = \deg_G(v)+1$ and $\deg_{G\overline{G}}(\overline{v}) = n - \deg_G(v)$. Since $G\overline{G}$ has order $2n$, we have $i_v\left(G\overline{G}\right) \leqslant 2n - (\deg_G(v) + 1)$, and $i_{\overline{v}}\left(G\overline{G}\right) \leqslant 2n - n + \deg_G(v) = n + \deg_G(v)$. Thus,

$$i_{av}\left(G\overline{G}\right) = \left(n\left(i_v\left(G\overline{G}\right)\right) + n\left(i_{\overline{v}}\left(G\overline{G}\right)\right)\right)/2n \leqslant (3n-1)/2.$$

Moreover, $G = K_n$ or $G = \overline{K}_n$ attains the upper bound. □

THEOREM 2.8. *Let G be a graph of order n. If $n \geqslant 2$ and G has a support vertex that dominates $V(G)$ or $\overline{G}$ has a support vertex that dominates $V\left(\overline{G}\right)$, then $2 \leqslant i_{av}\left(G\overline{G}\right) \leqslant n$, and both these bounds are sharp.*

PROOF. For a vertex v of $G\overline{G}$, $\delta(G)+1 \leqslant i_v\left(G\overline{G}\right) \leqslant n$ holds. Then, we obtain $\delta(G) + 1 \leqslant i_{av}\left(G\overline{G}\right) \leqslant n$. Since G has a support vertex, eventually G has at least one leaf. Therefore $\delta(G) = 1$. Hence, $2 \leqslant i_{av}\left(G\overline{G}\right) \leqslant n$. The conditions holds also true for $\overline{G}$.

The complete graph $G = K_2$ achieves both the lower and the upper bound. □

3. Conclusion

Complementary prisms are a deeply intriguing family of graphs and the study in complementary prisms is just beginning. In this paper, various results for the average independent domination of the complementary prism of specific families of graphs are shown. Upper and lower bounds for the average independent domination number of complementary prisms are proven and the sharp upper and lower bounds of Theorem 2.7 and Theorem 2.8 are noted.

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