TRI-IDEALS OF SEMIRINGS

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Abstract. In this paper, we introduce the notion of a tri-ideal as a generalization of bi-quasi-interior ideal, quasi-interior ideal, bi-interior ideal, bi-quasi ideal, quasi ideal, interior ideal, left(right) ideal and ideal of a semiring. Then, we study the properties of tri-ideals of a semiring and characterize the tri-simple semiring using tri-ideals of a semiring.

1. Introduction

The algebraic structure play a prominent role in mathematics with wide range of applications. Generalization of ideals of algebraic structures and ordered algebraic structure plays a very remarkable role and also necessary for further advance stuctionties and application of various algebraic structures. During 1950-1980, the concepts of bi-ideals, quasi ideals and interior ideals were studied by many mathematicians and during 1950-2019, the applications of these ideals only studied by mathematicians.

Between 1980 and 2016 there have been no new generalization of these ideals of algebraic structures. Then the author [22, 23, 24, 21, 29, 25, 30, 27, 26] introduced and studied bi quasi ideals, bi-interior ideals, bi quasi interior ideals, quasi interior ideals and weak interior ideals of \( \Gamma \)-semirings, semirings, \( \Gamma \)-semigroups,semigroups as a generalization of bi-ideal, quasi ideal and interior ideal of algebraic structures and characterized regular algebraic structures as well as simple algebraic structures using these ideals. The notion of a semiring was introduced by Vandiver [34] in 1934, but semirings had appeared in earlier studies on the theory of ideals of rings. Semiring is a generalization of ring but also of

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a generalization of distributive lattice. Semirings are structurally similar to semigroups than to rings. Semiring theory has many applications in other branches of mathematics.


Many mathematicians introduced various generalizations of concept of ideals in algebraic structures, proved important results and characterizations of regular algebraic structures using bi-ideals, quasi ideals and simple algebraic structures using interior ideals. Henriksen [4] and Shabir and Batod [32] studied ideals in semirings. We know that the notion of a one sided ideal of any algebraic structure is a generalization of notion of an ideal. The quasi ideals are generalization of left ideal and right ideal whereas the bi-ideals are generalization of quasi ideals.

In 1952, the concept of bi-ideals was introduced by Good and Hughes [3] for semigroups. The notion of bi-ideals in rings and semirings were introduced by Lajos and Szasz [11, 12]. Bi-ideal is a special case of (m-n) ideal. Steinfeld [33] first introduced the notion of quasi ideals for semigroups and then for rings. Iseki [7, 5, 6, 8] introduced the concept of quasi ideal for a semiring. In 1995, M. Murali Krishna Rao [16, 17, 18, 20] introduced the notion of $\Gamma$-semiring as a generalization of $\Gamma$- ring, ternary semiring and semiring. Murali Krishna Rao and Venkateswarlu [19, 30, 27, 26] studied regular $\Gamma$-incline, field $\Gamma$-semiring and derivations. Quasi ideals, bi-ideals in $\Gamma$-semirings studied by Jagtap and Pawar [9, 10]. Murali Krishna Rao [20, 22, 23, 24, 21, 29, 25] introduced the notion of left (right) bi-quasi ideal, the notion of bi-interior ideal and the notion of bi quasi-interior ideal of $\Gamma$-semiring as a generalization of ideal of $\Gamma$-semiring, studied their properties and characterized the simple $\Gamma$-semiring and regular $\Gamma$-semiring using these ideals.

In this paper, we introduce the notion of tri-ideals as a generalization of quasi ideal, bi-ideal, interior ideal, left(right) ideal and ideal of semiring and study the properties of tri-ideals of a semiring.

2. Preliminaries

In this section we will recall some of the fundamental concepts and definitions, which are necessary for this paper.

**Definition 2.1.** ([1]) A set $M$ together with two associative binary operations called addition and multiplication (denoted by $+$ and $\cdot$ respectively) will be called semiring provided

(i) addition is a commutative operation.
(ii) multiplication distributes over addition both from the left and from the right.
(iii) there exists $0 \in M$ such that $x + 0 = x$ and $x \cdot 0 = 0 \cdot x = 0$ for all $x \in M$.

**Example 2.1.** Let $M$ be the set of all natural numbers. Then $(M, \text{max}, \text{min})$ is a semiring.

**Definition 2.2.** Let $M$ be a semiring. If there exists $1 \in M$ such that $a \cdot 1 = 1 \cdot a = a$, for all $a \in M$, is called an unity element of $M$ then $M$ is said to be semiring with unity.

**Definition 2.3.** An element $a$ of a semiring $M$ is called a regular element if there exists an element $b$ of $M$ such that $a = aba$.

**Definition 2.4.** A semiring $M$ is called a regular semiring if every element of $M$ is a regular element.

**Definition 2.5.** An element $a$ of a semiring $M$ is called a multiplicatively idempotent (an additively idempotent) element if $aa = a(a + a = a)$.

**Definition 2.6.** A semiring $M$ is called an inverse element of $a$ of $M$ if $ab = ba = 1$.

**Definition 2.7.** A semiring $M$ is called a division semiring if for each non-zero element of $M$ has multiplication inverse.

**Definition 2.8.** A non-empty subset $A$ of a semiring $M$ is called
(i) a subsemiring of $M$ if $(A, +)$ is a subsemigroup of $(M, +)$ and $AA \subseteq A$.
(ii) a quasi ideal of $M$ if $A$ is a subsemiring of $M$ and $AM \cap MA \subseteq A$.
(iii) a bi-ideal of $M$ if $A$ is a subsemiring of $M$ and $AMA \subseteq A$.
(iv) an interior ideal of $M$ if $A$ is a subsemiring of $M$ and $MAM \subseteq A$.
(v) a left (right) ideal of $M$ if $A$ is a subsemiring of $M$ and $MA \subseteq A$.
(vi) a left (right) ideal of $M$ if $A$ is a subsemiring of $M$ and $MA \subseteq A$.
(vii) a $k$-ideal if $A$ is a subsemiring of $M$, $AM \subseteq A$ and $MA \subseteq A$.
(viii) a bi-interior ideal of $M$ if $A$ is a subsemiring of $M$ and $MBM \cap BMB \subseteq B$.
(ix) a left bi-quasi ideal (right bi-quasi ideal) of $M$ if $A$ is a subsemiring of $M$ and $MA \cap AMA \subseteq A$.
(x) a left quasi-interior ideal (right quasi-interior ideal) of $M$ if $A$ is a subsemiring of $M$ and $MAM \subseteq A$.
(xi) a bi-quasi-interior ideal of $M$ if $A$ is a subsemiring of $M$ and $BMB \subseteq B$.

**Definition 2.9.** A semiring $M$ is called a left bi-quasi simple semiring if $M$ has no left bi-quasi ideal other than $M$ itself.

### 3. Tri-ideals of semirings

In this section, we introduce the notion of tri-ideal as a generalization of bi-ideal, quasi-ideal and interior ideal of a semiring and study the properties of tri-ideal of a semiring. Throughout this paper $M$ is a semiring with unity element.
Definition 3.1. A non-empty subset $B$ of a semiring $M$ is said to be right tri-ideal of $M$ if $B$ is a subsemiring of $M$ and $BBMB \subseteq B$.

Definition 3.2. A non-empty subset $B$ of a semiring $M$ is said to be left tri-ideal of $M$ if $B$ is a subsemiring of $M$ and $BMBB \subseteq B$.

Definition 3.3. A non-empty subset $B$ of a semiring $M$ is said to be tri-ideal of $M$ if $B$ is a subsemiring of $M$ and $B$ is a left and a right tri-ideal of $M$.

Remark: A tri-ideal of a semiring $M$ need not be quasi-ideal, interior ideal, bi-interior ideal, and bi-quasi ideal of a semiring $M$.

Example 3.1. (i) If $M = \left\{ \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} \mid a, b, c \in Q \right\}$, then $M$ is a semiring with respect to usual addition of matrices and ternary operation is defined as usual matrix multiplication and $A = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}$. Then $A$ is not a left tri-ideal of semiring $M$.

(ii) If $M = \left\{ \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \mid a, b, c \in Q \right\}$ then $M$ is a semiring with respect to usual addition of matrices and usual matrix multiplication and $A = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}$. Then $A$ is not a bi-ideal and $A$ is a left tri-ideal of the semiring $M$.

(iii) If $M = \left\{ \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} \mid a, b, c \in Q \right\}$ then $M$ is a semiring with respect to usual addition of matrices and ternary operation is defined as usual matrix multiplication and $A = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid 0 \neq a, 0 \neq b \in Q \right\}$. Then $A$ is not a bi-ideal and $A$ is a left tri-ideal of the semiring $M$.

In the following theorem, we mention some important properties and we omit the proofs since they are straightforward.

Theorem 3.1. Let $M$ be a semiring. Then the following are hold.

1. Every left ideal is a tri-ideal of $M$.
2. Every right ideal is a tri-ideal of $M$.
3. Every quasi ideal is a tri-ideal of $M$.
4. Every ideal is a tri-ideal of $M$.
5. If $L$ is a left ideal and $R$ is a right ideal of $M$ then $B = R \cap L$ is a tri-ideal of $M$.
6. If $L$ is a left ideal and $R$ is a right ideal of a semiring $M$ then $B = RL$ is a tri-ideal of $M$.
7. Let $M$ be a semiring and $B$ be a subsemiring of $M$. If $MMBM \subseteq B$ and $BMMM \subseteq B$ then $B$ is a tri-ideal of $M$.
8. Let $M$ be a semiring and $B$ be a subsemiring of $M$. If $MMM \subseteq B$ then $B$ is a left tri-ideal of $M$.

Theorem 3.2. If $B$ be a left bi-quasi ideal of a semiring $M$, then $B$ is a tri-ideal of $M$.
Proof. Suppose $B$ is a left bi-quasi ideal of the semiring $M$. Then $BMB \subseteq MB$. We have $BMBB \subseteq BMB$. Therefore $BMB \subseteq MB \cap BMB \subseteq B$. Hence $B$ is a left tri-ideal of $M$. Similarly we can show that $B$ is a right tri-ideal of $M$. Hence $B$ is a tri-ideal of $M$. □

Corollary 3.1. If $B$ be a right bi-quasi ideal of a semiring $M$, then $B$ is a tri-ideal of $M$.

Corollary 3.2. If $B$ be a bi-quasi ideal of a semiring $M$, then $B$ is a tri-ideal of $M$.

Theorem 3.3. If $B$ be a bi-interior ideal of a semiring $M$, then $B$ is a left tri-ideal of $M$.

Proof. Suppose $B$ is a bi-interior ideal of the semiring $M$. Then

$$MBM \cap BMB \subseteq B,$$

and $BMB \subseteq MBM \cap BMB \subseteq B$.

Hence $B$ is a left tri-ideal of $M$. □

Corollary 3.3. If $B$ be a bi-interior ideal of a semiring $M$, then $B$ is a right tri-ideal of $M$.

Corollary 3.4. If $B$ be a bi-interior ideal of a semiring $M$, then $B$ is a tri-ideal of $M$.

Theorem 3.4. If $B$ be a subsemiring of a semiring $M$ and $MBB \subseteq B$, then $B$ is a left tri-ideal of $M$.

Theorem 3.5. Every bi-ideal of a semiring $M$ is a left tri-ideal of a semiring $M$.

Proof. Let $B$ be a bi-ideal of the semiring $M$. Then $BMBB \subseteq BMB \subseteq B$. Therefore $BMBB \subseteq B$. Hence every bi-ideal of a semiring $M$ is a left tri-ideal of the semiring $M$. □

Corollary 3.5. Every bi-ideal of a semiring $M$ is a right tri-ideal of $M$.

Corollary 3.6. Every bi-ideal of a semiring $M$ is a tri-ideal of $M$.

Theorem 3.6. Every bi-quasi interior ideal of a semiring $M$ is a left tri-ideal of a semiring $M$.

Proof. Let $B$ be a bi-quasi interior ideal of the semiring $M$. Then $BMBMB \subseteq B$. Therefore $BMB \subseteq B$. Hence $BMB \subseteq B$. This completes the proof. □

Corollary 3.7. Every bi-quasi interior ideal of a semiring $M$ is a right tri-ideal of a semiring $M$.

Corollary 3.8. Every bi-quasi interior ideal of a semiring $M$ is a tri-ideal of a semiring $M$.

Theorem 3.7. Every interior ideal of a semiring $M$ is a left tri-ideal of $M$. 
Proof. Let $I$ be an interior ideal of the semiring $M$. Then $IMI \subseteq MIM \subseteq I$. Hence $I$ is a left tri-ideal of the semiring $M$. \hfill $\square$

Corollary 3.9. Every interior ideal of a semiring $M$ is a right tri-ideal of $M$.

Corollary 3.10. Every interior ideal of a semiring $M$ is a tri-ideal of $M$.

Theorem 3.8. Let $M$ be a semiring and $B$ be a subsemiring of $M$ and $B = BB$. Then $B$ is a left tri-ideal of $M$ if and only if there exist left ideal $L$ and a right ideal $R$ such that $RL \subseteq B \subseteq R \cap L$.

Proof. Suppose $B$ is a tri-ideal of the semiring $M$. Then $BMBB \subseteq B$. Let $R = BM$ and $L = MB$. Then $R$ and $L$ are a right ideal and a left ideal of $M$ respectively. Therefore $RL \subseteq B \subseteq R \cap L$.

Conversely suppose that there exist $R$ and $L$ are a right ideal and a left ideal of $M$ respectively such that $RL \subseteq B \subseteq R \cap L$. Then

$$BMBB \subseteq (R \cap L)M(R \cap L)(R \cap L) \subseteq RL \subseteq B.$$ 

Hence $B$ is a left tri-ideal of $M$. \hfill $\square$

Corollary 3.11. Let $M$ be a semiring and $B$ be a subsemiring of $M$ and $B = BB$. Then $B$ is a right tri-ideal of $M$ if and only if there exist right ideal $R$ and a left ideal $L$ such that $RL \subseteq B \subseteq R \cap L$.

Corollary 3.12. Let $M$ be a semiring and $B$ be a subsemiring of $M$ and $B = BB$. Then $B$ is a tri-ideal of $M$ if and only if there exist left ideal $L$ and a right ideal $R$ of $M$ such that $RL \subseteq B \subseteq R \cap L$.

Theorem 3.9. The intersection of a left tri-ideal $B$ of a semiring $M$ and a right ideal $A$ of $M$ is always a left tri-ideal of $M$.

Proof. Suppose $C = B \cap A$. Then $CMCC \subseteq BMBB \subseteq B$ and $CMCC \subseteq AMAA \subseteq A$. Since $A$ is a left ideal of $M$, we have $CMCC \subseteq B \cap A = C$. Hence the intersection of a left tri-ideal $B$ of the semiring $M$ and a left ideal $A$ of $M$ is always a left tri-ideal of $M$. \hfill $\square$

Corollary 3.13. The intersection of a right tri-ideal $B$ of a semiring $M$ and a right ideal $A$ of $M$ is always a right tri-ideal of $M$.


Theorem 3.10. Let $A$ and $C$ be left tri-ideals of a semiring $M$, $B = AC$ and $B$ is an additively subsemigroup of $M$. If $AA = A$ then $B$ is a left tri-ideal of $M$.

Proof. Let $A$ and $C$ be left tri-ideals of the semiring $M$ and $B = AC$. Then

$$BM = ACAC = ACAAC \subseteq AMAC \subseteq AC = B.$$ 

Therefore $B = AC$ is a subsemiring of $M$ and

$$BMBB = ACMACAC \subseteq AMAC \subseteq AC = B.$$ 

Hence $B$ is a left tri-ideal of $M$. \hfill $\square$
Theorem 3.11. Let $A$ and $C$ be subsemirings of a semiring $M$ and $B = AC$ and $B$ is additively subsemigroup of $M$. If $A$ is the left ideal of $M$, then $B$ is a tri-ideal of $M$.

Proof. Let $A$ and $C$ be subsemirings of $M$ and $B = AC$. Suppose $A$ is the left ideal of $M$. Then $BB = ACAC \subseteq AC = B$. Thus $BMBB = ACMACAC \subseteq AC = B$. Hence $B$ is a left tri-ideal of $M$. \(\Box\)

Corollary 3.15. Let $A$ and $C$ be subsemirings of a semiring $M$ and $B = AC$ and $B$ is additively subsemigroup of $M$. If $C$ is a right ideal then $B$ is a right tri-ideal of $M$.

Theorem 3.12. Let $M$ be a semiring and $T$ be a non-empty subset of $M$. If subsemiring $B$ of $M$ containing $TMTT$ and $B \subseteq T$, then $B$ is a left tri-ideal of semiring $M$.

Proof. Let $B$ be a subsemiring of $M$ containing $TMTT$. Then $BMB \subseteq TMTT \subseteq B$. Therefore $BMBB \subseteq B$. Hence $B$ is a left tri-ideal of $M$. \(\Box\)

Theorem 3.13. Let $B$ be a tri-ideal of a semiring $M$ and $I$ be an interior ideal of $M$. Then $B \cap I$ is a left tri-ideal of $M$.

Proof. Suppose $B$ is the tri-ideal of $M$ and $I$ is an interior ideal of $M$. Obviously $B \cap I$ is subsemiring of $M$. Then

$$\begin{align*}
(B \cap I)M(B \cap I)(B \cap I) & \subseteq BMBB \subseteq B \\
(B \cap I)M(B \cap I)(B \cap I) & \subseteq IIMI 
\end{align*}$$

Therefore $(B \cap I)M(B \cap I)(B \cap I) \subseteq B \cap I$. Hence $B \cap I$ is a left tri-ideal of $M$. \(\Box\)

Corollary 3.16. Let $B$ be a tri-ideal of a semiring $M$ and $I$ be an interior ideal of $M$. Then $B \cap I$ is a right tri-ideal of $M$.

Corollary 3.17. Let $B$ be a tri-ideal of a semiring $M$ and $I$ be an interior ideal of $M$. Then $B \cap I$ is a tri-ideal of $M$.

Theorem 3.14. Let $M$ be a semiring and $T$ be a subsemiring of $M$. Then every subsemiring of $T$ containing $TMTT$ is a left tri-ideal of $M$.

Proof. Let $C$ be a subsemiring of $T$ containing $TMTT$. Then $CMMC \subseteq TMTT \subseteq C$.

Hence $C$ is a left tri-ideal of $M$. \(\Box\)

Theorem 3.15. The intersection of left tri-ideals $\{B_\lambda \mid \lambda \in A\}$ of a semiring $M$ is a left tri-ideal of $M$.

Proof. Let $B = \bigcap_{\lambda \in A} B_\lambda$. Then $B$ is a subsemiring of $M$. Since $B_\lambda$ is a left tri-ideal of $M$, we have $B_\lambda MB_\lambda B_\lambda \subseteq B_\lambda$, for all $\lambda \in A$. Then $\cap B_\lambda MB_\lambda B_\lambda \cap B_\lambda \subseteq \cap B_\lambda$ and thus $BMBB \subseteq B$. Hence $B$ is a left tri-ideal of $M$. \(\Box\)
Theorem 3.16. Let \( B \) be a left tri-ideal of a semiring \( M \), \( e \in B \), \( eB \subseteq B \) and \( e \) be \( \beta \)-idempotent. Then \( eB \) is a left tri-ideal of \( M \).

Proof. Let \( B \) be a left tri-ideal of the semiring \( M \). Suppose \( x \in B \cap eM \). Then \( x \in B \) and \( x = ey, y \in M \). Thus \( x = ey = eey = ex \in eB \). Therefore \( B \cap eM \subseteq eB, eB \subseteq B \) and \( eB \subseteq eM \). Thus \( eB \subseteq B \cap eM \) and \( eB = B \cap eM \). Hence \( eB \) is a left tri-ideal of \( M \).

Corollary 3.18. Let \( M \) be a semiring and \( e \) be idempotent. Then \( eM \) and \( Me \) are left tri-ideal and right tri-ideal of \( M \) respectively.

Theorem 3.17. Let \( M \) be a semiring. If \( M = Ma \), for all \( a \in M \). Then every left tri-ideal of \( M \) is a quasi ideal of \( M \).

Proof. Let \( B \) be a left tri-ideal of the semiring \( M \) and \( a \in B \). Then \( Ma \subseteq MB \) and \( M \subseteq MB \subseteq M \). Thus \( MB = M \) and \( BM = BMB \subseteq BMBB \subseteq B \). So, \( MB \cap BM \subseteq M \cap BM \subseteq BM \subseteq B \). Therefore \( B \) is a quasi ideal of \( M \). Hence the theorem.

4. Tri-simple semiring, regular semiring and minimal tri-ideals of a semiring

In this section, we introduce the notion of left tri-simple semiring and characterize the left tri-simple semiring using left tri-ideals of semiring and study the properties of minimal left tri-ideals of a semiring.

Definition 4.1. A semiring \( M \) is a left (right) simple semiring if \( M \) has no proper left (right) ideals of \( M \).

Definition 4.2. A semiring \( M \) is said to be simple semiring if \( M \) has no proper ideals of \( M \).

Definition 4.3. A semiring \( M \) is said to be bi-simple semiring if \( M \) has no proper bi-ideals of \( M \).

Definition 4.4. A semiring \( M \) is said to be left(right) tri-simple semiring if \( M \) has no left(right) tri-ideal other than \( M \) itself.

Definition 4.5. A semiring \( M \) is said to be tri-simple semiring if \( M \) has no tri-ideal other than \( M \) itself.

Theorem 4.1. If \( M \) is a division semiring then \( M \) is a tri-simple semiring.

Proof. Let \( B \) be a proper left tri-ideal of the division semiring \( M \), \( x \in M \) and \( 0 \neq a \in B \). Since \( M \) is a division semiring, there exists \( b \in M \) such that \( ab = 1 \). Then \( abx = x = xab \). Therefore \( x \in BM \) and \( M \subseteq BM \). We have \( BM \subseteq M \). Hence \( M = BM \). Similarly we can prove \( MB = M \).

\[
M = MB = BMB = BMBB \subseteq B, M \subseteq B.
\]

Therefore \( M = B \) and \( BM = BBMB \subseteq B, M \subseteq B \). Therefore \( M = B \). Hence division semiring \( M \) has no proper -tri-ideals.
Theorem 4.2. Let $M$ be a left simple semiring. Every left tri-ideal of $M$ is a right ideal of $M$.

Proof. Let $M$ be a left simple semiring and $B$ be a left tri-ideal of $M$. Then $BMBB \subseteq B$ and $MB$ is a left ideal of $M$. Since $M$ is a left simple semiring, we have $MB = M$. Therefore $BMBB \subseteq B$. Thus $BM \subseteq B$. Hence the theorem. □

Corollary 4.1. Let $M$ be a right simple semiring. Every right tri-ideal is a left ideal of $M$.

Corollary 4.2. Let $M$ be a left and a right simple semiring. Every tri-ideal is an ideal of $M$.

Theorem 4.3. Let $M$ be a semiring. $M$ is a left tri-simple semiring if and only if $< a > \subseteq M$, for all $a \in M$ where $< a >$ is the smallest left tri-ideal generated by $a$.

Proof. Let $M$ be a semiring. Suppose $M$ is the left tri-simple semiring, $a \in M$ and $B = Ma$. Then $B$ is a left ideal of $M$. Therefore, by Theorem 4.1, $B$ is a left tri-ideal of $M$. Hence $Ma = M$, for all $a \in M$. Then $Ma \subseteq< a > \subseteq M$ and $M \subseteq< a > \subseteq M$. Therefore $M =< a >$.

Suppose $< a >$ is the smallest left tri-ideal of $M$ generated by $a$, $< a > \subseteq M$, $A$ is the left tri-ideal and $a \in A$. Then $< a > \subseteq A \subseteq M$ and $M \subseteq A \subseteq M$. Therefore $A = M$. Hence $M$ is a left tri-ideal simple semiring.

Theorem 4.4. If semiring $M$ is a left simple semiring then every left tri-ideal of $M$ is a right ideal of $M$.

Proof. Let $B$ be a left tri-ideal of the left simple semiring $M$. Then $MB$ is a left ideal of $M$ and $MB \subseteq M$. Therefore $MB = M$. Then $BMBB \subseteq B$ and $BMB \subseteq B$. Thus $BM \subseteq B$. Hence every left tri-ideal is a right ideal of $M$. □

Corollary 4.3. If semiring $M$ is right simple semiring then every right tri-ideal of $M$ is a left ideal of $M$.

Corollary 4.4. Every tri-ideal of left and right simple semiring $M$ is an ideal of $M$.

Theorem 4.5. Let $M$ be a semiring and $B$ be a left tri-ideal of $M$. Then $B$ is a minimal left tri-ideal of $M$ if and only if $B$ is a left tri-simple subsemiring of $M$.

Proof. Let $B$ be a minimal left tri-ideal of the semiring $M$ and $C$ be a left tri-ideal of $B$. Then $CBCC \subseteq C$ and $CBCC$ is a left tri-ideal of $M$. Since $C$ is a tri-ideal of $B$, we have $CBCC = B$ and $B = CBCC \subseteq C$. Thus $B = C$.

Conversely suppose that $B$ is a left tri-simple subsemiring of $M$. Let $C$ be a left tri-ideal of $M$ and $C \subseteq B$. Then $CBCC \subseteq CMCC \subseteq BMBB \subseteq B$. Therefore $C$ is a left tri-ideal of $B$. Thus $B = C$ since $B$ is a left tri-simple subsemiring of $M$. Hence $B$ is a minimal left tri-ideal of $M$.

Corollary 4.5. Let $M$ be a semiring and $B$ be a right tri-ideal of $M$. Then $B$ is a minimal right tri-ideal of $M$ if and only if $B$ is a right tri-simple subsemiring of $M$. □
Corollary 4.6. Let $M$ be a semiring and $B$ be a tri-ideal of $M$. Then $B$ is a minimal tri-ideal of $M$ if and only if $B$ is a tri- simple subsemiring of $M$.

Theorem 4.6. Let $M$ be a commutative idempotent semiring. Then $M$ is a regular semiring if and only if $BMBB = B$ for all tri- ideals $B$ of $M$.

Proof. Suppose $M$ is a regular commutative idempotent semiring, $B$ is a tri-ideal of $M$ and $x \in B$. Then $BMBB \subseteq B$ and there exists $y \in M, x = yxx \in BMBB$. Therefore $x \in BMBB$. Hence $BMBB = B$.

Conversely suppose that $BMBB = B$ for all tri- ideals $B$ of $M$. Let $B = R \cap L$, where $R$ and $L$ are ideals of $M$. Then $B$ is tri- ideal of $M$. Therefore

$$(R \cap L)M(R \cap L)(R \cap L) = R \cap L$$

and $R \cap L = (R \cap L)M(R \cap L)(R \cap L)$. Thus $RMRL \subseteq RL \subseteq R \cap L$ since $RL \subseteq L$ and $RL \subseteq R$. Therefore $R \cap L = RL$. Hence $M$ is a regular semiring.

5. Conclusion

As a further generalization of ideals, we introduced the notion of a tri-ideal of semiring as a generalization of ideal, left ideal, right ideal, bi-ideal, quasi ideal, bi-quasi ideal, bi-interior ideal, bi-quasi interior ideal and interior ideal of semiring and studied some of their properties. We introduced the notion of tri- simple semiring and characterized the tri-simple semiring of semiring. We proved every bi-quasi ideal of semiring and bi-interior ideal of semiring are tri-ideals and studied some of the properties of bi-interior ideals of semiring. In continuity of this paper, we study prime tri-ideals, prime, maximal and minimal tri-ideals of semiring.

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