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AN ESTIMATE OF THE REMAINDER IN TAYLOR'S PERTURBED FORMULA AND APPLICATIONS

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Abstract. In this paper, we provide a new estimation for the remainder in the Taylor's perturbed formula established by S. S. Dragomir published in his article [*New estimation of the remainder in Taylor's formula using Grüss type inequalities and applications,* Math. Inequal. Appl., **2**(2) 183-194, (1999)]. Our result improves other results existing in the litterature. Also, some applications for elementary mappings and for a Beta random variable are given.

1. Introduction

Let f be an n-times differentiable function in a neighborhood U of a point $a \in \mathbb{R}$. We recall that the Taylor polynomial of order n of f at a is the polynomial

$$T_n(f;a,x) := \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k \tag{1.1}$$

where, that $f^0 = f$ and $0! = 1$. Let $R_n(f;a,x) := f(x) - T_n(f;a,x)$ be the remainder term. $R_n(f;a,x)$ is called the n-th Taylor remainder of the function$f$ with center a. So, we have

$$f(x) = T_n(f;a,x) + R_n(f;a,x),$$

for all x the neighborhood U of the point a.

One of the first and fundamental results in mathematical analysis for Taylor remainder is given by well known Lagrange's formula.

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THEOREM 1.1 (Lagrange's formula for the remainder). *If f has an $(n+1)$th derivative in $[a,b]$ then there is some $a \leqslant \xi \leqslant b$ such that*

$$R_n(b) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(b-a)^{n+1}.$$

An interesting elementary proof of Theorem 1.1 was given in [**4**], by using the following theorem.

THEOREM 1.2. *Suppose that the function $f : [a,b] \to \mathbb{R}$ has an $(n+1)$th derivative in $[a,b]$ and that $\gamma \leqslant f^{(n+1)}(x) \leqslant \Gamma$ for every $a < x < b$. Then for any $a \leqslant x \leqslant b$, we have*

$$\frac{\gamma}{(n+1)!}(x-a)^{n+1} \leqslant R_n(x) \leqslant \frac{\Gamma}{(n+1)!}(x-a)^{n+1}. \tag{1.2}$$

Indeed, by using Darboux's intermediate value theorem and Theorem 1.1 above, the Lagrange's formula was obtained in [**4**]. Theorem 1.2 provides also estimates for the Taylor remainder in terms of bounds of the $(n+1)$-th derivative of f.

The following theorem (see [**2**]) is well known in the literature as Taylors formula or Taylor's theorem with the integral remainder.

THEOREM 1.3. *Let I be a closed intervalle of the real line $\mathbb{R}$. Let $a \in I$ and let n be a positive integer. Let $f : I \longrightarrow \mathbb{R}$ be a function such that $f^{(n)}$ is absolutely continuous, then for each $x \in I$, we have*

$$f(x) = T_n(f;a,x) + R_n(f;a,x), \tag{1.3}$$

where $T_n(f;a,x)$ is Taylor's polynomial, and the remainder is given by

$$R_n(f;a,x) := \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt. \tag{1.4}$$

This result can be proved by mathematical induction using the integration by parts formula. The expansion (1.3) is called Taylor's formula or Taylor's theorem with the integral remainder.

During the last two decades (see the references), many investigations concerning the Taylor remainder have been undertaken in two directions. The first one is to give generalizations of the expansion (1.3) using certain classes of special (harmonic and other) polynomials. The second one is to give more accurate estimates for these generalized Taylor formulas. Along these lines, the reader is invited to see [**12**], [**8**], [**3**] and the references therein.

Several remarkable integral inequalities involving the Taylor remainder are to be found in the papers [**1**], [**5**], [**11**] and [**16**].

In [**7**], S. S. Dragomir was the first author to introduce the following perturbed Taylor formula.

THEOREM 1.4 ([**7**]). *Let I be a closed interval of the real line $\mathbb{R}$. Let $a \in I$ and let n be a positive integer. Let $f : I \longrightarrow \mathbb{R}$ be a function such that $f^{(n)}$ is absolutely* *continuous, then for each $x \in I$, we have*

$$f(x) = T_n(f;a,x) + \frac{(x-a)^{n+1}}{(n+1)!}[f^{(n)};a,x] + G_n(f;a,x), \tag{1.5}$$

where $T_n(f;a,x)$ is Taylor's polynomial and the remaider $G_n(f;a,x)$ satisfies the estimation:

$$| G_n(f;a,x) | \leqslant \frac{1}{4}\frac{(x-a)^{n+1}}{n!}[\Gamma(x) - \gamma(x)], \tag{1.6}$$

where, $[f^{(n)};a,x]$ is the divided difference of $f^{(n)}$ in the points a and x, i.e.,

$$[f^{(n)};a,x] = \frac{f^{(n)}(x) - f^{(n)}(a)}{x-a}$$

and

$$\Gamma(x) := \sup_{t\in[a,x]} f^{(n+1)}(t), \quad \gamma(x) := \inf_{t\in[a,x]} f^{(n+1)}(t)$$

for all $x \geqslant a$, $x \in I$.

To prove his result, S. S. Dragomir used the following well known Grüss inequality (see for example [[**15**], p. 296]).

THEOREM 1.5. *Let $F, G : [a,b] \to \mathbb{R}$ be two integrable functions such that $m \leqslant F(x) \leqslant M$ and $\varphi \leqslant G(x) \leqslant \Phi$ for all $x \in [a,b]$ where m, M, φ and Φ are real constants. Then we have the inequality*

$$\left| \int_a^b F(x)G(x)\,dx - \frac{1}{b-a}\int_a^b F(x)\,dx \int_a^b G(x)\,dx \right| \leqslant \frac{b-a}{4}(M-m)(\Phi-\varphi)$$

and the inequality is sharp in the sense that the constant $\frac{1}{4}$ can not be replaced by a smaller one.

In [**5**], L. Bougoffa established some inequalities involving Taylor's remainder. By using the pre-Grüss inequality (see [**17**]), the following theorem was established in [**5**].

THEOREM 1.6. *Let $f : [a,b] \to \mathbb{R}$ be a function such that $f \in C^{n+1}([a,b])$ and $m \leqslant f^{(n+1)}(x) \leqslant M$ for each $x \in [a,b]$, where m and M are constants. Then*

$$\left| R_n(f;a,b) - \frac{f^{(n)}(b) - f^{(n)}(a)}{(n+1)!}(b-a)^n \right| \leqslant \frac{M-m}{2} . \frac{n}{(2n+1)^{\frac{1}{2}}} . \frac{(b-a)^{n+1}}{(n+1)!}. \tag{1.7}$$

One can see that inequality (1.7) is an improvement to the inequality (1.6).

The purpose of this paper is to establish a new estimation of the remainder $G_n(f;a,x)$ which improves both estimations given in (1.6) and (1.7) above. Our result is Theorem 2.2 (see Section 2). The basic tool used in the proof of this theorem is an inequality of Grüss type established by X. L. Cheng and J. Sun in [**6**] and by M. Matić in [**14**]. Applications (of Theorem 2.2) for certain elementary mappings are given in Section 3. In Section 4, we give an application for a Beta random variable.

This work may be considered as a continuation of the papers [**7**] and [**5**].

This paper has some natural relationships with the paper [**1**] where some integral inequalities involving Taylor's remainder were proved in connection with several integral inequalities established by H. Gauchman in the papers [**9**] and [**10**]. The remaminder of the Taylor's formula or its generalizations has a link with all papers listed in the references of this paper.

2. The result

Before we give our main result, we need to recall the following variant of the Grüss inequality which was established almost at the same time by X. L. Cheng and J. Sun in [**6**] as well as M. Matić in [**14**] respectively.

Theorem 2.1 (Cheng-Sun-Matić inequality). *Let $F, G : [a, b] \to \mathbb{R}$ be two integrable functions such that $\varphi \leqslant G(x) \leqslant \Phi$ for all $x \in [a, b]$, where φ and Φ are real constants, then we have:*

$$\begin{aligned}&\left|\int_a^b F(x)G(x)\,dx - \frac{1}{b-a}\int_a^b F(x)\,dx \int_a^b G(x)\,dx\right|\\ \leqslant\ &\frac{1}{2}\left(\int_a^b \left|F(x) - \frac{1}{b-a}\int_a^b F(y)\,dy\right|\,dx\right)(\Phi-\varphi)\end{aligned}$$

The main result reads as follows.

Theorem 2.2. *Let $f : I \longrightarrow \mathbb{R}$ be a function as above and $a \in I$. Then we have the Taylor's perturbed formula:*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{(x-a)^{n+1}}{(n+1)!}\frac{f^{(n)}(x)-f^{(n)}(a)}{x-a} + G_n(f;a,x); \tag{2.1}$$

the remainder $G_n(f;a,x)$ satisfies the following estimation:

$$|G_n(f;a,x)| \leqslant \frac{(x-a)^{n+1}}{(n-1)!(n+1)^{2+\frac{1}{n}}}[\Gamma(x)-\gamma(x)], \tag{2.2}$$

for all $x \geqslant a$, $x \in I$, where $\Gamma(x)$ and $\gamma(x)$ are given by

$$\Gamma(x) := \sup_{t\in[a,x]} f^{(n+1)}(t), \quad \gamma(x) := \inf_{t\in[a,x]} f^{(n+1)}(t), \quad \forall x \in I,\ x \geqslant a.$$

Proof. Let $x \in I$ with $x \geqslant a$. For all $t, \in [a, x]$ we set $F(t) = (x-t)^n$ and $G(x) = f^{(n+1)}(t)$. Then by using the Cheng-Sun-Matić inequality on the interval $[a, x]$, we have

$$\begin{aligned}&\left|\int_a^x (x-t)^n f^{(n+1)}(t)\,dt - \frac{1}{x-a}\int_a^x (x-t)^n\,dt. \int_a^x f^{(n+1)}(t)\,dt\right|\\ \leqslant\ &\frac{1}{2}\left(\int_a^x \left|(x-t)^n - \frac{1}{x-a}\int_a^x (x-s)^n\,ds\right|\,dt\right)[\Gamma(x)-\gamma(x)]\end{aligned} \tag{2.3}$$

and as

$$\int_a^x (x-t)^n\,dt = \frac{(x-a)^{n+1}}{n+1}, \qquad \int_a^x f^{(n+1)}(t)\,dt = f^{(n)}(x) - f^{(n)}(a)$$

we get

$$\begin{aligned}&\left|\int_a^x (x-t)^n f^{(n+1)}(t)\,dt - \frac{(x-a)^{n+1}}{n+1}.\frac{f^{(n)}(x)-f^{(n)}(a)}{x-a}\right| \\ &\quad \leqslant \frac{1}{2}\left(\int_a^x \left|(x-t)^n - \frac{(x-a)^n}{n+1}\right|\,dt\right)[\Gamma(x)-\gamma(x)]. \end{aligned} \tag{2.4}$$

To compute the integral in the right hand side of (2.4), we introduce the following function:

$$\psi(t) := (x-t)^n - \frac{(x-a)^n}{n+1}. \tag{2.5}$$

It is easy to see that ψ is strictly decreasing from $[a,x]$ onto $[\psi(x),\psi(a)]$, where $\psi(x) = -\frac{(x-a)^n}{n+1}$ and $\psi(a) = \frac{n(x-a)^n}{n+1}$. Let us set

$$t_n := x - \frac{x-a}{(n+1)^{\frac{1}{n}}}.$$

Then t_n is the unique point where the function ψ vanishes. It is easy to see that ψ is nonnegative on the interval $[a,t_n]$ and is negative on the interval $[t_n,x]$. Therefore, we have

$$\int_a^x |\psi(t)|\,dt = \int_a^{t_n} \psi(t)\,dt - \int_{t_n}^x \psi(t)\,dt := J_1 - J_2.$$

By easy computations, we get

$$J_1 = -J_2 = \frac{(x-a)^{n+1}}{(n+1)^{1+\frac{1}{n}}} - \frac{(x-t_n)^{n+1}}{n+1}. \tag{2.6}$$

However

$$(x-t_n)^{n+1} = \frac{(x-a)^{n+1}}{(n+1)^{\frac{n+1}{n}}}. \tag{2.7}$$

From (2.6) and (2.7), we get

$$\int_a^x |\psi(t)|\,dt = \frac{2(x-a)^{n+1}}{(n+1)^{\frac{n+1}{n}}}\left(1 - \frac{1}{n+1}\right) = \frac{2n(x-a)^{n+1}}{(n+1)^{\frac{2n+1}{n}}}. \tag{2.8}$$

Since

$$\int_a^x (x-t)^n f^{(n+1)}(t)\,dt = n!R_n(f;a,x)$$

then by (2.4) and (2.8) we get

$$\left|R_n(f;a,x) - \frac{(x-a)^{n+1}}{(n+1)!}.\frac{f^{(n)}(x)-f^{(n)}(a)}{x-a}\right| \leqslant \frac{(x-a)^{n+1}[\Gamma(x)-\gamma(x)]}{(n-1)!(n+1)^{\frac{2n+1}{n}}}. \tag{2.9}$$

Using the classical Taylor's formula (1.1) we deduce the expansion (2.1) and the estimation (2.2) for the remainder term $G_n(f;a,x)$. So the proof of the theorem is complete. □

Remarks. (i) The estimation (2.2) is actually an improvement of the estimate given in (1.6), since for every positive integer n, we have

$$\frac{n}{(n+1)^{\frac{2n+1}{n}}} < \frac{1}{4}.$$

(ii) The estimation (2.2) improves also the estimation (1.7), since for every positive integer n, we have

$$2\sqrt{2n+1} \leqslant (n+1)^{1+\frac{1}{n}}.$$

3. Applications for certain elementary mappings

3.1. Application for the exponential mapping.

Consider $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$. Then for each positive integer n, we have $f^{(n)} = f$ and then

$$T_n(f; a, x) = e^a \sum_{k=0}^{n} \frac{(x-a)^k}{k!}.$$

Using the estimation (2.2) we get

$$\left| e^x - e^a \sum_{k=0}^{n} \frac{(x-a)^k}{k!} - \frac{(e^x - e^a)(x-a)^n}{(n+1)!} \right| \leqslant \frac{(e^x - e^a)(x-a)^{n+1}}{(n-1)!(n+1)^{2+\frac{1}{n}}}, \tag{3.1}$$

for all $x \geqslant a$ and particularly:

$$\left| e^x - \sum_{k=0}^{n} \frac{x^k}{k!} - \frac{x^n(e^x - 1)}{(n+1)!} \right| \leqslant \frac{x^{n+1}(e^x - 1)}{(n-1)!(n+1)^{2+\frac{1}{n}}}, \tag{3.2}$$

for all $x \geqslant 0$.

3.2. Application for the logarithmic mapping.

Consider $f : (0, \infty) \to \mathbb{R}$, $f(x) = \ln(x)$. Then for each positive integer n, we have

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{x^n}, \quad x > 0$$

and then

$$T_n(f; a, x) = \ln(a) + \sum_{k=1}^{n} \frac{(-1)^{k-1}(x-a)^k}{ka^k}, \quad a > 0.$$

We have:

$$f^{(n)}(x) - f^{(n)}(a) = \frac{(-1)^n(n-1)!(x^n - a^n)}{a^n x^n},$$

$$\Gamma(x) - \gamma(x) == \frac{n!(x^{n+1} - a^{n+1})}{a^{n+1}x^{n+1}}.$$

Using the estimation (2.2) we get:

$$\left| \ln(x) - \ln(a) - \sum_{k=1}^{n} \frac{(-1)^{k-1}(x-a)^k}{ka^k} - \frac{(-1)^n(x^n - a^n)(x-a)^n}{n(n+1)a^n x^n} \right|$$

$$\leqslant \frac{n}{(n+1)^{2+\frac{1}{n}}} \cdot \frac{(x-a)^{n+1}(x^{n+1}-a^{n+1})}{a^{n+1}x^{n+1}}, \quad x \geqslant a \tag{3.3}$$

which is equivalent with:

$$\left| \ln(\frac{x}{a}) - \sum_{k=1}^{n} \frac{(-1)^{k-1}(x-a)^k}{ka^k} - \frac{(-1)^n(x^n-a^n)(x-a)^n}{n(n+1)a^n x^n} \right|$$
$$\leqslant \frac{n(x-a)^{n+1}(x^{n+1}-a^{n+1})}{(n+1)^{2+\frac{1}{n}} a^{n+1} x^{n+1}}, \tag{3.4}$$

for all $x \geqslant a$ and, particularly,

$$\left| \ln(x+1) - \sum_{k=1}^{n} \frac{(-1)^{k-1}x^k}{k} - \frac{(-1)^n x^n((x+1)^n-1)}{n(n+1)(x+1)^n} \right|$$
$$\leqslant \frac{nx^{n+1}((x+1)^{n+1}-1)}{(n+1)^{2+\frac{1}{n}}(x+1)^{n+1}}, \tag{3.5}$$

for all $x \geqslant 0$.

3.3. Application for the mapping x^m.

Let $m \in \mathbb{R}$ and consider $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^m$. Then for each positive integer n, we have

$$f^{(n)}(x) = M(n)x^{m-n} := m(m-1)\dots(m-n+1)x^{m-n}, \quad x > 0$$

and then

$$T_n(f; a, x) = a^m + \sum_{k=1}^{n} \frac{M(k)a^{m-k}}{k!}(x-a)^k, \quad a > 0.$$

We have:

$$f^{(n)}(x) - f^{(n)}(a) = M(n)[x^{m-n} - a^{m-n}].$$

For each integer $n \geqslant m-1$, we have

$$\Gamma(x) - \gamma(x) = |M(n+1)| \cdot \frac{x^{n+1-m} - a^{n+1-m}}{a^{n+1-m}x^{n+1-m}}, \quad x \geqslant a.$$

Using the estimation (2.2), for every integer $n \geqslant m-1$, we get:

$$\left| x^m - a^m - \sum_{k=1}^{n} \frac{M(k)a^{m-k}}{k!}(x-a)^k - \frac{M(n)(x^{m-n}-a^{m-n})}{(n+1)!}(x-a)^n \right|$$
$$\leqslant \frac{|M(n+1)|}{(n-1)!(n+1)^{2+\frac{1}{n}}} \frac{(x-a)^{n+1}(x^{n+1-m}-a^{n+1-m})}{a^{n+1-m}x^{n+1-m}}, \tag{3.6}$$

for all $x \geqslant a$ and, particularly,

$$\left| x^m - 1 - \sum_{k=1}^{n} \frac{M(k)}{k!}(x-1)^k - \frac{M(n)(x^{m-n}-1)}{(n+1)!}(x-1)^n \right|$$
$$\leqslant \frac{|M(n+1)|}{(n-1)!(n+1)^{2+\frac{1}{n}}} \frac{(x-1)^{n+1}(x^{n+1-m}-1)}{x^{n+1-m}}, \tag{3.7}$$

for all $x \geqslant 1$. As a consequence, we have

$$\left|(x+1)^m - 1 - \sum_{k=1}^{n} \frac{M(k)}{k!}x^k - \frac{M(n)x^n((x+1)^{m-n}-1)}{(n+1)!}\right|$$
$$\leqslant \frac{|M(n+1)|}{(n-1)!(n+1)^{2+\frac{1}{n}}} \frac{x^{n+1}((x+1)^{n+1-m}-1)}{(x+1)^{n+1-m}}, \tag{3.8}$$

for all $x \geqslant 0$, and all positive inetger $n \geqslant m-1$.

4. Application for a Beta random variable

Let Ω be the set

$$\Omega := \{(p,q) : p,q > 0\}.$$

A Beta random variable X, with parameters $(p,q) \in \Omega$ has the probability density function

$$f(x;p,q) := \frac{x^{p-1}(1-x)^{q-1}}{B(p,q)}; \quad 0 < x < 1.$$

We observe that the density $f(.;p,q)$ is not bounded when $0 < p < 1$ or $0 < q < 1$. Assume that $p > 1$ and $q > 1$. Then $f(.;p,q)$ is defined and absolutely continuous on the interval $[0,1]$, and we have

$$\frac{df(x;p,q)}{dx} = -(p+q-2)\frac{x^{p-2}(1-x)^{q-2}}{B(p,q)}[x-x_0],$$

where $x_0 := \frac{p-1}{p+q-2}$. Consequently, for every $a, x \in [0,1]$ such that $a \leqslant x$, we have

$$\Phi(x) - \phi(x) \leqslant f(x_0;p,q) = \frac{(p-1)^{p-1}(q-1)^{q-1}}{B(p,q)(p+q-2)^{p+q-2}},$$

where

$$\Phi(x) := \sup_{t\in[a,x]} f(t;p,q), \quad \phi(x) := \inf_{t\in[a,x]} f(t;p,q).$$

By applying Theorem 2.2 to the cumulative distribution function $F(x) := Pr(X \leqslant x)$ of the variable X, we get the following proposition:

Proposition 4.1. *Let X be a Beta random variable with parameters (p,q), such that $p,q > 1$. For all $a, x \in [0,1]$ such that $x \geqslant a$, we have the inequality:*

$$\left|Pr(a \leqslant X \leqslant x) - (x-a)f(a;p,q) - \frac{(x-a)}{2}[f(x;p,q) - f(a;p,q)]\right|$$
$$\leqslant \frac{(x-a)^2(p-1)^{p-1}(q-1)^{q-1}}{8B(p,q)(p+q-2)^{p+q-2}}. \tag{4.1}$$

In particular, we have

$$\left|Pr(X \leqslant x) - \frac{x^p(1-x)^{q-1}}{2B(p,q)}\right| \leqslant \frac{x^2(p-1)^{p-1}(q-1)^{q-1}}{8B(p,q)(p+q-2)^{p+q-2}}. \tag{4.2}$$

As a consequence, we obtain the following inequality

$$B(p,q) \leqslant \frac{(p-1)^{p-1}(q-1)^{q-1}}{8(p+q-2)^{p+q-2}}, \tag{4.3}$$

for all $p, q > 1$.

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