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EXISTENCE RESULTS FOR SYSTEM OF ITERATIVE AND CONFORMABLE TYPE FRACTIONAL ORDER BOUNDARY VALUE PROBLEMS

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ABSTRACT. In this paper, we focus on the existence of at least one positive solution for iterative system of boundary value problem involving conformable fractional order derivative by implementing the properties of the Green functions and the fixed point theorem on cone in a Banach space.

1. Introduction

Fractional calculus has been used to model many physical and engineering procedures that are best depicted by differential equations of fractional order. The conventional mathematical models of integer-order derivatives, along with nonlinear models, often do not perform appropriately. It is a useful and effective tool for modeling such nonlinear systems. In the past couple of decades, fractional calculus has played a very significant role in various areas like those of mechanics, chemistry, control structures, dynamic procedures, viscoelasticity, etc. [**23, 14, 21, 12, 15, 6, 11**].

Differential equations (DEqs) of fractional order combined with initial or boundary conditions have become substantial and serve a leading role in branches of applied mathematics. Foremost established industries namely automotive, biotechnology, chemistry, electronics and communications depend on boundary value problems (BVPs) to simulate diverse phenomenon at different intervals as well as to design and produce high-tech products. In these applicable settings, positive solutions appear to have an impact. In fact, applications in the disciplines of economics, physics, and biology have been discovered in mathematical models in the type of

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the system of DEqs based on various boundary conditions, see [**2, 24, 7, 16, 17, 18, 20**].

The definition of the fractional order derivative used is now the type Riemann–Liouville as well as the type Caputo, which includes an integral expression and gamma function. A new definition has been developed and the conformable fractional order derivative has been named, see [**1, 8, 9, 10, 13**]. In [**4**] authors studied the existence of at least one positive solution to the fractional order boundary value problem (FBVP), which includes this new definition, and used a compression-expansion functional fixed point theorem. Recently, Prasad and Krushna [**19**] developed sufficient conditions for the existence of multiple positive solutions to the iterative system of BVPs concerning conformable fractional order derivative by implementing six functionals fixed point theorem. In this article we are concerned with a coupled system of iterative type fractional order DEqs

$$D_{n_1} D_{m_1} w_1(t) + f_1\big(t, w_1(t), w_2(t)\big) = 0, \ t \in (0,1), \tag{1.1}$$

$$D_{n_2} D_{m_2} w_2(t) + f_2\big(t, w_1(t), w_2(t)\big) = 0, \ t \in (0,1), \tag{1.2}$$

coupled with the Sturm–Liouville type conditions

$$\begin{cases} \xi_{11} w_1(0) - \xi_{12} D_{m_1} w_1(0) = 0, \\ \xi_{21} w_1(1) + \xi_{22} D_{m_1} w_1(1) = 0, \end{cases} \tag{1.3}$$

$$\begin{cases} \xi_{11} w_2(0) - \xi_{12} D_{m_2} w_2(0) = 0, \\ \xi_{21} w_2(1) + \xi_{22} D_{m_2} w_2(1) = 0, \end{cases} \tag{1.4}$$

where $\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}$ are positive real numbers and $0 < m_i, n_i < 1$, D_{m_i}, D_{n_i}, for $i = 1, 2$ are the conformable fractional order derivatives.

We assume that the conditions given below stands hold throughout the paper:

(H1) $\xi_{11}, \xi_{12}, \xi_{21}, \xi_{22}$ are positive real numbers such that either $\xi_{11}^2 + \xi_{12}^2 > 0$ or $\xi_{21}^2 + \xi_{22}^2 > 0$,

(H2) $\Delta_1 = \xi_{21}\xi_{12} + \xi_{11}\xi_{22} + \dfrac{\xi_{11}\xi_{21}}{m_1} > 0$,

(H3) $\Delta_2 = \xi_{21}\xi_{12} + \xi_{11}\xi_{22} + \dfrac{\xi_{11}\xi_{21}}{m_2} > 0$,

(H4) $f_1, f_2 : [0,1] \times \mathbb{R}^2 \to \mathbb{R}^+$ are continuous.

According to a positive solution for the system of the problem (1.1)-(1.4), we indicate

$$\Big(w_1(t), w_2(t)\Big) \in \Big(C^{m_1+n_1}[0,1] \times C^{m_2+n_2}[0,1]\Big)$$

satisfying (1.1)-(1.4) with

$$w_1(t) \geqslant 0, w_2(t) \geqslant 0, \text{ for all } t \in [0,1] \text{ and } (w_1, w_2) \neq (0,0).$$

The rest of this article is organized as follows. Section 2 consists some auxiliary results. The main theorems are presented in Section 3, and in Section 4, as an application, we demonstrate our results with an example.

2. Auxiliary Results

The Green functions for the homogeneous FBVPs are constructed and the bounds for all of these Green functions are calculated, that are required to determine the key results.

Definition 2.1. ([**10**]) Let $g : [0, \infty) \to \mathbb{R}, \alpha \in (0, 1)$ and $t > 0$. The conformable fractional derivative of g is defined by

$$D_\alpha g(t) = \lim_{\varepsilon \to 0} \left[\frac{g(t + \varepsilon t^{1-\alpha}) - g(t)}{\varepsilon} \right],$$

for $t > 0$ and the conformable fractional derivative at 0 is defined as

$$D_\alpha g(0) = \lim_{t \to 0^+} D_\alpha g(t).$$

If g is differentiable then $D_\alpha g(t) = t^{1-\alpha} g'(t)$.

Definition 2.2. ([**10**]) Let $\alpha \in (0, 1)$. The conformable fractional integral of a function $g : [0, \infty) \to \mathbb{R}$ of order α is denoted by $I_\alpha g(t)$ and is defined as

$$I_\alpha g(t) = \int_0^t s^{\alpha - 1} g(s) ds.$$

Lemma 2.1 ([**1**]). *Let $g : (0, \infty) \to \mathbb{R}$ be differentiable and $0 < \alpha \leqslant 1$. Then for all $t > 0$, we have*

$$I_\alpha D_\alpha g(t) = g(t) - g(0).$$

Let $\mathcal{G}_1(t, p)$ be the Green's function for the homogeneous DEqs

$$-D_{n_1} D_{m_1} w_1(t) = 0, \ t \in (0, 1), \tag{2.1}$$

satisfying the boundary conditions (1.3).

Lemma 2.2 ([**4**]). *Suppose the condition $(H1)$ is fulfilled. If $h_1(t) \in C[0, 1]$, then the fractional order DEqs*

$$D_{n_1} D_{m_1} w_1(t) + h_1(t) = 0, \quad t \in (0, 1), \tag{2.2}$$

satisfying the conditions (1.3) *has a unique solution,*

$$w_1(t) = \int_0^1 \mathcal{G}_1(t, p) h_1(p) dp,$$

where

$$\mathcal{G}_1(t, p) = \begin{cases} \dfrac{1}{\Delta_1} \Big[\xi_{12} + \dfrac{\xi_{11}}{m_1} p^{m_1} \Big] \Big[\xi_{22} + \dfrac{\xi_{21}}{m_1} (1 - t^{m_1}) \Big], & p \leqslant t, \\ \dfrac{1}{\Delta_1} \Big[\xi_{12} + \dfrac{\xi_{11}}{m_1} t^{m_1} \Big] \Big[\xi_{22} + \dfrac{\xi_{21}}{m_1} (1 - p^{m_1}) \Big], & t \leqslant p. \end{cases} \tag{2.3}$$

Lemma 2.3. *Suppose the conditions $(H1)$ and $(H2)$ are fulfilled. The Green's function $\mathcal{G}_1(t, p)$ given in* (2.3) *is non-negative, for all $(t, p) \in [0, 1] \times [0, 1]$.*

Let us define

$$\begin{cases} \Phi_1(t) = \xi_{12} + \dfrac{\xi_{11}}{m_1} t^{m_1}, \\ \Phi_2(t) = \xi_{22} + \dfrac{\xi_{21}}{m_1}(1 - t^{m_1}). \end{cases}$$

LEMMA 2.4 ([**19**]). *For $t, p \in I = \left[\frac{1}{4}, \frac{3}{4}\right]$, then the Green's function $\mathcal{G}_1(t,p)$ given in* (2.3) *satisfies the following inequality*

$$m_2^* \mathcal{G}_1(p,p) \leqslant \mathcal{G}_1(t,p) \leqslant \mathcal{G}_1(p,p), \tag{2.4}$$

where

$$m_2^* = \min\left\{ \frac{\Phi_1(\frac{1}{4})}{\Phi_1(\frac{3}{4})}, \frac{\Phi_2(\frac{3}{4})}{\Phi_2(\frac{1}{4})} \right\}.$$

In a similar way, we construct the Green's function $\mathcal{G}_2(t,p)$ for the homogeneous fractional order DEq

$$-D_{n_2} D_{m_2} w_2(t) = 0, \ t \in (0,1), \tag{2.5}$$

satisfying the boundary conditions (1.4).

LEMMA 2.5 ([**4**]). *Suppose the condition* $(H1)$ *is fulfilled. If $h_2(t) \in C[0,1]$, then the fractional order DEqs*

$$D_{n_2} D_{m_2} w_2(t) + h_2(t) = 0, \ t \in (0,1), \tag{2.6}$$

satisfying the boundary conditions (1.4) *has a unique solution,*

$$w_2(t) = \int_0^1 \mathcal{G}_2(t,p) h_2(p) dp,$$

where

$$\mathcal{G}_2(t,p) = \begin{cases} \dfrac{1}{\Delta_2}\Big[\xi_{12} + \dfrac{\xi_{11}}{m_2} p^{m_2}\Big]\Big[\xi_{22} + \dfrac{\xi_{21}}{m_2}(1 - t^{m_2})\Big], & p \leqslant t, \\ \dfrac{1}{\Delta_2}\Big[\xi_{12} + \dfrac{\xi_{11}}{m_2} t^{m_2}\Big]\Big[\xi_{22} + \dfrac{\xi_{21}}{m_2}(1 - p^{m_2})\Big], & t \leqslant p. \end{cases} \tag{2.7}$$

LEMMA 2.6. *Suppose the conditions* $(H1)$ *and* $(H3)$ *are fulfilled. The Green's function $\mathcal{G}_2(t,p)$ given in* (2.7) *is non-negative, for all $(t,p) \in [0,1] \times [0,1]$.*

Let us define

$$\begin{cases} \Phi_1^*(t) = \xi_{12} + \dfrac{\xi_{11}}{m_2} t^{m_2}, \\ \Phi_2^*(t) = \xi_{22} + \dfrac{\xi_{21}}{m_2}(1 - t^{m_2}). \end{cases}$$

LEMMA 2.7 ([**19**]). *For $t, p \in I = \left[\frac{1}{4}, \frac{3}{4}\right]$, then the Green's function $\mathcal{G}_2(t,p)$ given in* (2.7) *satisfies the inequality*

$$m_2^{**} \mathcal{G}_2(p,p) \leqslant \mathcal{G}_2(t,p) \leqslant \mathcal{G}_2(p,p), \tag{2.8}$$

where

$$m_2^{**} = \min\left\{ \frac{\Phi_1^*(\frac{1}{4})}{\Phi_1^*(\frac{3}{4})}, \frac{\Phi_2^*(\frac{3}{4})}{\Phi_2^*(\frac{1}{4})} \right\}.$$

Property P1. Let P be a cone in a Banach space E and Ω be a bounded open subset of E with $0 \in \Omega$. Then a continuous functional $\alpha : P \to [0, \infty)$ is said to satisfy Property **P1** if one of the following conditions hold:

(a) α is convex, $\alpha(0) = 0, \alpha(x) \neq 0$ if $x \neq 0$ and $\inf\limits_{x \in P \cap \partial\Omega} \alpha(x) > 0$,

(b) α is sublinear, $\alpha(0) = 0, \alpha(x) \neq 0$ if $x \neq 0$ and $\inf\limits_{x \in P \cap \partial\Omega} \alpha(x) > 0$,

(c) α is concave and unbounded.

Property P2. Let P be a cone in a Banach space E and Ω be a bounded open subset of E with $0 \in \Omega$. Then a continuous functional $\beta : P \to [0, \infty)$ is said to satisfy Property **P2** if one of the following conditions hold:

(a) β is convex, $\beta(0) = 0, \beta(x) \neq 0$ if $x \neq 0$,

(b) β is sublinear, $\beta(0) = 0, \beta(x) \neq 0$ if $x \neq 0$,

(c) $\beta(x + y) \geqslant \beta(x) + \beta(y)$ for all $x, y \in P$, $\beta(0) = 0, \beta(x) \neq 0$ if $x \neq 0$.

In getting the existence criteria for iterative system of FBVP (1.1)-(1.4), the following fixed point theorem of cone expansion and compression of functional type due to Avery, Henderson and O'Regan [**5**], which generalizes the fixed point theorems of Anderson–Avery [**3**] and Sun–Zhang [**22**].

Theorem 2.1 ([**5**]). *Let Ω_1, Ω_2 be two bounded open sets in a Banach Space E such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$ in E. Suppose $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is completely continuous operator, α and β are non-negative continuous functional on P, and one of the two conditions:*

(i) *α satisfies* Property P1 *with $\alpha(Tx) \geqslant \alpha(x)$, for all $x \in P \cap \partial\Omega_1$ and β satisfies* Property P2 *with $\beta(Tx) \leqslant \beta(x)$, for all $x \in P \cap \partial\Omega_2$*

(ii) *β satisfies* Property P2 *with $\beta(Tx) \leqslant \beta(x)$, for all $x \in P \cap \partial\Omega_1$ and α satisfies* Property P1 *with $\alpha(Tx) \geqslant \alpha(x)$, for all $x \in P \cap \partial\Omega_2$, is satisfied.*

Then T has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Main Results

We consider the Banach space $\mathcal{B} = \mathcal{E} \times \mathcal{E}$, where $\mathcal{E} = \{w_1 : w_1 \in C[0, 1]\}$ endowed with the norm $\|(w_1, w_2)\| = \|w_1\|_0 + \|w_2\|_0$, for $(w_1, w_2) \in \mathcal{B}$ and we denote the norm,

$$\|w_1\|_0 = \max_{t \in [0,1]} |w_1(t)|.$$

Define a cone $\mathcal{P} \subset \mathcal{B}$ by

$$\mathcal{P} = \Big\{(w_1, w_2) \in \mathcal{B} : w_1(t), w_2(t) \ \text{ are non} - \text{negative and increasing on } [0, 1] \\ \text{and } \min_{t \in I} \big[w_1(t) + w_2(t)\big] \geqslant \eta \|(w_1, w_2)\|\Big\},$$

where $I = \left[\dfrac{1}{4}, \dfrac{3}{4}\right]$ and

$$\eta = \min\{m_2^*, m_2^{**}\} \tag{3.1}$$

Let

$$\Theta_1 = \min\left\{ \int_0^1 \mathcal{G}_1(p,p)dp, \int_0^1 \mathcal{G}_2(p,p)dp \right\}$$

and

$$\Theta_2 = \max\left\{ \int_{\frac{1}{4}}^{\frac{3}{4}} \eta\mathcal{G}_1(p,p)dp, \int_{\frac{1}{4}}^{\frac{3}{4}} \eta\mathcal{G}_2(p,p)dp \right\},$$

where η is given in (3.1).

Now let us define continuous functionals α and β on the cone $\mathcal{P}$ by

$$\alpha(w_1, w_2) = \min_{t\in I} \left\{ \big|w_1\big| + \big|w_2\big| \right\} \quad \text{and}$$
$$\beta(w_1, w_2) = \max_{t\in[0,1]} \left\{ \big|w_1\big| + \big|w_2\big| \right\} = w_1(1) + w_2(1) = \big\|(w_1, w_2)\big\|.$$

For all $(w_1, w_2) \in \mathcal{P}$, it is evident that $\alpha(w_1, w_2) \leqslant \beta(w_1, w_2)$.

The operators $\mathcal{A}_1$ and $\mathcal{A}_2$ are denoted by $\mathcal{A}_1 : \mathcal{P} \to \mathcal{E}$, $\mathcal{A}_2 : \mathcal{P} \to \mathcal{E}$ and are defined as

$$\begin{cases} \mathcal{A}_1\big(w_1, w_2\big)(t) = \displaystyle\int_0^1 \mathcal{G}_1(t,p) f_1\big(p, w_1(p), w_2(p)\big)dp, \\ \mathcal{A}_2\big(w_1, w_2\big)(t) = \displaystyle\int_0^1 \mathcal{G}_2(t,p) f_2\big(p, w_1(p), w_2(p)\big)dp. \end{cases}$$

THEOREM 3.1. *Assume condition* $(H4)$ *is satisfied. Suppose there exist positive real numbers* ρ, Ψ *with* $\rho < \eta\Psi$ *such that* f_j, $j = 1, 2$ *satisfies the conditions:*

$$(C1)\ \ f_j\big(t, w_1, w_2\big) \geqslant \frac{1}{2}\frac{\rho}{\eta\Theta_2}, \ \forall\ t \in I \text{ and } \big(w_1, w_2\big) \in \big[\rho, \Psi\big],$$
$$(C2)\ \ f_j\big(t, w_1, w_2\big) \leqslant \frac{1}{2}\frac{\Psi}{\Theta_1}, \ \forall\ t \in [0,1] \text{ and } \big(w_1, w_2\big) \in \big[0, \Psi\big].$$

Then the system of FBVP (1.1)-(1.4) *has at least one positive and nondecreasing solution,* $(w_1^\bullet, w_2^\bullet)$ *satisfying* $\rho \leqslant \alpha(w_1^\bullet, w_2^\bullet)$ *with* $\beta(w_1^\bullet, w_2^\bullet) \leqslant \Psi$.

PROOF. The completely continuous operator $\mathcal{A} : \mathcal{P} \to \mathcal{B}$ is simply described as

$$\mathcal{A}(w_1, w_2) = \big(\mathcal{A}_1(w_1, w_2), \mathcal{A}_2(w_1, w_2)\big).$$

It can be evident that a fixed point of $\mathcal{A}$ is the solution of the FBVP (1.1)-(1.4). We seek a fixed point of $\mathcal{A}$. First, we show that $\mathcal{A} : \mathcal{P} \to \mathcal{P}$. Let $(w_1, w_2) \in \mathcal{P}$. Clearly, $\mathcal{A}_1(w_1, w_2)(t) \geqslant 0$ and $\mathcal{A}_2(w_1, w_2)(t) \geqslant 0$ for $t \in [0,1]$. Also, for $(w_1, w_2) \in \mathcal{P}$,

$$\begin{cases} \|\mathcal{A}_1(w_1, w_2)\|_0 \leqslant \displaystyle\int_0^1 \mathcal{G}_1(p,p) f_1\big(p, w_1(p), w_2(p)\big)dp, \\ \|\mathcal{A}_2(w_1, w_2)\|_0 \leqslant \displaystyle\int_0^1 \mathcal{G}_2(p,p) f_2\big(p, w_1(p), w_2(p)\big)dp, \end{cases}$$

and

$$\begin{aligned}\min_{t\in I}\mathcal{A}_1(w_1,w_2)(t) =&\min_{t\in I}\left[\int_0^1 \mathcal{G}_1(t,p)f_1\big(p,w_1(p),w_2(p)\big)dp\right]\\ \geqslant& m_2^*\int_0^1 \mathcal{G}_1(p,p)f_1\big(p,w_1(p),w_2(p)\big)dp\\ \geqslant&\eta\|\mathcal{A}_1(w_1,w_2)\|_0.\end{aligned}$$

Similarly $\min_{t\in I}\mathcal{A}_2(w_1,w_2)(t)\geqslant\eta\|\mathcal{A}_2(w_1,w_2)\|_0$. Therefore

$$\begin{aligned}\min_{t\in I}\Big\{\mathcal{A}_1(w_1,w_2)(t)+\mathcal{A}_2(w_1,w_2)(t)\Big\}&\geqslant\eta\|\mathcal{A}_1(w_1,w_2)\|_0+\eta\|\mathcal{A}_2(w_1,w_2)\|_0\\ &=\eta\|(\mathcal{A}_1(w_1,w_2),\mathcal{A}_2(w_1,w_2))\|\\ &=\eta\|\mathcal{A}(w_1,w_2)\|.\end{aligned}$$

Thus $\mathcal{A}(w_1,w_2)\in\mathcal{P}$ which implies that $\mathcal{A}:\mathcal{P}\to\mathcal{P}$. Moreover the operator $\mathcal{A}$ is a completely continuous. Let $\Omega_1=\big\{(w_1,w_2):\alpha(w_1,w_2)<\rho\big\}$ and $\Omega_2=\big\{(w_1,w_2):\beta(w_1,w_2)<\Psi\big\}$. It is easy to see that $0\in\Omega_1$, and Ω_1, Ω_2 are bounded open subsets of $\mathcal{B}$. Let $(w_1,w_2)\in\Omega_1$, then we have

$$\rho>\alpha(w_1,w_2)=\min_{t\in I}\sum_{i=1}^{2}\Big[w_i(t)\Big]\geqslant\eta\sum_{i=1}^{2}\Big\|w_i\Big\|=\eta\beta(w_1,w_2).$$

Thus $\Psi>\dfrac{\rho}{\eta}>\beta(w_1,w_2)$, i.e., $(w_1,w_2)\in\Omega_2$, so $\Omega_1\subseteq\Omega_2$.

Claim 1: $\alpha\big(\mathcal{A}(w_1,w_2)\big)\geqslant\alpha(w_1,w_2)$, for $(w_1,w_2)\in\mathcal{P}\cap\partial\Omega_1$. To show this let $(w_1,w_2)\in\mathcal{P}\cap\partial\Omega_1$ then $\Psi=\beta(w_1,w_2)\geqslant\sum_{i=1}^{2}\Big[w_i(p)\Big]\geqslant\alpha(w_1,w_2)=\rho$, for $p\in I$. Thus it follows from $(C1)$, Lemma 2.4 and Lemma 2.7 that

$$\begin{aligned}\alpha\Big(\mathcal{A}\big(w_1,w_2\big)(t)\Big)&=\min_{t\in I}\sum_{j=1}^{2}\left[\int_0^1\mathcal{G}_j(t,p)f_j\big(p,w_1(p),w_2(p)\big)dp\right]\\ &\geqslant\sum_{j=1}^{2}\left[\int_{\frac14}^{\frac34}\eta\mathcal{G}_j(p,p)f_j\big(p,w_1(p),w_2(p)\big)dp\right]\\ &\geqslant\frac12\frac{\rho}{\eta\Theta_2}\int_\eta^1\eta\mathcal{G}_1(p,p)dp+\frac12\frac{\rho}{\eta\Theta_2}\int_{\frac14}^{\frac34}\eta\mathcal{G}_2(p,p)dp\\ &=\frac{\rho}{2}+\frac{\rho}{2}=\rho=\alpha(w_1,w_2).\end{aligned}$$

Claim 2: $\beta\big(\mathcal{A}(w_1,w_2)\big)\leqslant\beta(w_1,w_2)$, for $(w_1,w_2)\in\mathcal{P}\cap\partial\Omega_2$. To show this let $(w_1,w_2)\in\mathcal{P}\cap\partial\Omega_2$ then $\sum_{i=1}^{2}\Big[w_i(p)\Big]\leqslant\beta(w_1,w_2)=\Psi$, for $p\in[0,1]$. Thus it follows from $(C2)$, Lemma 2.4 and Lemma 2.7 yields

$$\begin{aligned}\beta\Big(\mathcal{A}\big(w_1,w_2\big)(t)\Big) &= \max_{t\in[0,1]}\sum_{j=1}^{2}\Big[\int_0^1 \mathcal{G}_j(t,p)f_j\big(p,w_1(p),w_2(p)\big)dp\Big]\\ &\leqslant \sum_{j=1}^{2}\Big[\int_0^1 \mathcal{G}_j(p,p)f_j\big(p,w_1(p),w_2(p)\big)dp\Big]\\ &\leqslant \frac{1}{2}\frac{\Psi}{\Theta_1}\int_0^1 \mathcal{G}_1(p,p)dp+\frac{1}{2}\frac{\Psi}{\Theta_1}\int_0^1 \mathcal{G}_2(p,p)dp\\ &\leqslant \frac{\Psi}{2}+\frac{\Psi}{2}=\Psi=\beta(w_1,w_2).\end{aligned}$$

Evidently, α meets Property $\mathbf{P1}(c)$ and β meets Property $\mathbf{P2}(a)$. Therefore the condition (i) of Theorem 2.1 is fulfilled and thus $\mathcal{A}$ has at least one fixed point $(w_1^\bullet,w_2^\bullet)\in\mathcal{P}\cap(\overline{\Omega}_2\backslash\Omega_1)$. Hence the system of FBVP (1.1)-(1.4) has at least one positive and nondecreasing solution $(w_1^\bullet,w_2^\bullet)$ satisfying $\rho\leqslant\alpha(w_1^\bullet,w_2^\bullet)$ with $\beta(w_1^\bullet,w_2^\bullet)\leqslant\Psi$. □

THEOREM 3.2. *Assume condition* $(H4)$ *is satisfied. Suppose there exist positive real numbers* $\rho,\ \Psi$ *with* $\rho<\Psi$ *such that* $f_j,\quad j=1,2$ *satisfies the conditions:*

$$(C3)\ f_j\big(t,w_1,w_2\big)\leqslant\frac{1}{2}\frac{\rho}{\Theta_2},\ \forall\ t\in[0,1]\ \textit{and}\ \big(w_1,w_2\big)\in\big[0,\rho\big],$$

$$(C4)\ f_j\big(t,w_1,w_2\big)\geqslant\frac{1}{2}\frac{\Psi}{\eta\Theta_1},\ \forall\ t\in I\ \textit{and}\ \big(w_1,w_2\big)\in\Big[\Psi,\frac{\Psi}{\eta}\Big].$$

Then the system of FBVP (1.1)-(1.4) *has at least one positive and nondecreasing solution,* $(w_1^\bullet,w_2^\bullet)$ *satisfying* $\rho\leqslant\beta(w_1^\bullet,w_2^\bullet)$ *with* $\alpha(w_1^\bullet,w_2^\bullet)\leqslant\Psi$.

PROOF. Let $\Omega_3=\{(w_1,w_2):\beta(w_1,w_2)<\rho\}$ and $\Omega_4=\{(w_1,w_2):\alpha(w_1,w_2)<\Psi\}$. We have $0\in\Omega_3$ and $\Omega_3\subseteq\Omega_4$ with Ω_3 and Ω_4 are bounded open subsets of $\mathcal{B}$.

Claim 1: $\beta\big(\mathcal{A}(w_1,w_2)\big)\leqslant\beta(w_1,w_2)$, $(w_1,w_2)\in\mathcal{P}\cap\partial\Omega_3$. To establish this let $(w_1,w_2)\in\mathcal{P}\cap\partial\Omega_3$ then $\sum_{i=1}^{2}\Big[w_i(p)\Big]\leqslant\beta(w_1,w_2)=\rho$, for $p\in[0,1]$, and so it follows from the condition $(C3)$, Lemma 2.4 and Lemma 2.7 that yields

$$\begin{aligned}\beta\Big(\mathcal{A}\big(w_1,w_2\big)(t)\Big) &= \max_{t\in[0,1]}\sum_{j=1}^{2}\Big[\int_0^1 \mathcal{G}_j(t,p)f_j\big(p,w_1(p),w_2(p)\big)dp\Big]\\ &\leqslant \sum_{j=1}^{2}\Big[\int_0^1 \mathcal{G}_j(p,p)f_j\big(p,w_1(p),w_2(p)\big)dp\Big]\\ &\leqslant \frac{1}{2}\frac{\rho}{\Theta_2}\int_0^1 \mathcal{G}_1(p,p)dp+\frac{1}{2}\frac{\rho}{\Theta_2}\int_0^1 \mathcal{G}_2(p,p)dp\\ &= \frac{\rho}{2}+\frac{\rho}{2}=\rho=\beta(w_1,w_2).\end{aligned}$$

Claim 2: If $(w_1, w_2) \in \mathcal{P} \cap \partial\Omega_4$ then $\alpha\big(\mathcal{A}(w_1, w_2)\big) \geqslant \alpha(w_1, w_2)$. To see this let $(w_1, w_2) \in \mathcal{P} \cap \partial\Omega_4$ then $\dfrac{\Psi}{\eta} = \dfrac{\alpha(w_1, w_2)}{\eta} \geqslant \beta(w_1, w_2) \geqslant \displaystyle\sum_{i=1}^{2} \Big[w_i(p)\Big] \geqslant \alpha(w_1, w_2) =$ Ψ, for $p \in I$. Thus it follows from $(C4)$, Lemma 2.4 and Lemma 2.7 that

$$\begin{aligned}
\alpha\Big(\mathcal{A}(w_1, w_2)(t)\Big) &= \min_{t\in I} \sum_{j=1}^{2} \Bigg[\int_0^1 \mathcal{G}_j(t,p) f_j\big(p, w_1(p), w_2(p)\big)dp\Bigg] \\
&\geqslant \sum_{j=1}^{2} \Bigg[\int_{\frac{1}{4}}^{\frac{3}{4}} \eta \mathcal{G}_j(p,p) f_j\big(p, w_1(p), w_2(p)\big)dp\Bigg] \\
&\geqslant \frac{1}{2}\frac{\Psi}{\eta\Theta_1} \int_{\frac{1}{4}}^{\frac{3}{4}} \eta\mathcal{G}_1(p,p)dp + \frac{1}{2}\frac{\Psi}{\eta\Theta_1} \int_{\frac{1}{4}}^{\frac{3}{4}} \eta\mathcal{G}_2(p,p)dp \\
&= \frac{\Psi}{2} + \frac{\Psi}{2} = \Psi = \alpha(w_1, w_2).
\end{aligned}$$

Thus it is verified that α fulfills Property **P1**(c) and β fulfills Property **P2**(a). The condition (ii) of Theorem 2.1 is therefore satisfied and hence $\mathcal{A}$ has at least one fixed point $(w_1^\bullet, w_2^\bullet) \in \mathcal{P} \cap (\overline{\Omega}_4 \backslash \Omega_3)$, i.e., the system of FBVP (1.1)-(1.4) has at least one positive and nondecreasing solution $(w_1^\bullet, w_2^\bullet)$ satisfying $\rho \leqslant \beta(w_1^\bullet, w_2^\bullet)$ with $\alpha(w_1^\bullet, w_2^\bullet) \leqslant \Psi$. □

4. Example

We present a relevant example to demonstrate the use of Theorem 3.1. Consider the system of FBVP,

$$D_{0.7}D_{0.8}w_1(t) + f_1(t, w_1, w_2) = 0,\ \ t \in (0,1), \tag{4.1}$$

$$D_{0.8}D_{0.7}w_2(t) + f_2(t, w_1, w_2) = 0,\ \ t \in (0,1), \tag{4.2}$$

$$\begin{cases} 13w_1(0) - 7D_{0.8}w_1(0) = 0, \\ 15w_1(1) + 8D_{0.8}w_1(1) = 0, \end{cases} \tag{4.3}$$

$$\begin{cases} 13w_2(0) - 7D_{0.7}w_2(0) = 0, \\ 15w_2(1) + 8D_{0.7}w_2(1) = 0, \end{cases} \tag{4.4}$$

where

$$\begin{cases} f_1(t, w_1, w_2) = \dfrac{13\sin t}{12} + \dfrac{(t^2+2)(w_1+w_2)}{9} + \dfrac{63}{65}, \\[2ex] f_2(t, w_1, w_2) = \dfrac{381t^2(w_1+w_2)}{2500} + \dfrac{233e^{-(w_1+w_2)^2}}{250}. \end{cases}$$

By means of straightforward computations, we obtain $\eta = 0.0643$, $\Theta_1 = 15.3125$

and $\Theta_2 = 1534.6023$. If we choose $\rho = 7$, $\Psi = 120$ and then $\rho < \eta\Psi$ and f_j, for $j = 1, 2$ satisfies

$$\circ\ f_j\big(t, w_1, w_2\big) \geqslant 0.035467 = \frac{1}{2}\frac{\rho}{\eta\Theta_2},\ t \in \big[0.25, 1\big] \text{ and } \big(w_1, w_2\big) \in \big[7, 120\big],$$

$$\circ\ f_j\big(t, w_1, w_2\big) \leqslant 3.918367 = \frac{1}{2}\frac{\Psi}{\Theta_1},\ t \in [0, 1] \text{ and } \big(w_1, w_2\big) \in \big[0, 120\big].$$

Therefore all the conditions of Theorem 3.1 are fulfilled. Thus by Theorem 3.1, the system of FBVP (4.1)-(4.4) has at least one positive and nondecreasing solution.

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