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RECIPROCAL DEGREE DISTANCE OF SOME DERIVED GRAPHS

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Abstract. A molecular graph is a graphical description of the structural formula of a chemical compound with the help of graph theory. In the construction of a chemical graph, atoms are represented by vertices and bonds between the atoms are represented by edges. The reciprocal degree distance (RDD), which is a weight version of the Harary index, defined for a connected graph G as vertex-degree-weighted sum of the reciprocal distances. In this paper, we have determined expressions for RDD index of some derived graphs in terms of the parameters of the parent graph. Especially,we establish expressions for the RDD index of subdivision graph, vertex-semitotal graph, edge-semitotal graph and total graph.

1. Introduction

The word *topological index* is used to characterize some graphical properties of the molecule. It is the property that is used to preserve isomorphism of a graph. A molecular graph is a graphical description of the structural formula of a chemical compound with the help of graph theory. In the construction of a chemical graph, atoms are represented by vertices and bonds between the atoms are represented by edges [**5**]. In all major fields of chemistry, chemical graphs are used for many different purposes [**2**]. As has already been proved in [**4**], the said indices are useful for characterizing alkanes by an integer. Although the computed expression does not have direct application, we believe that it can be used to compute the said indices for chemical and molecular graphs, which are useful for characterizing alkanes by an integer.

Dobrynin and Kochetova [**4**] and Gutman [**7**] independently proposed a vertex-degree-weighted version of Wiener index called *degree distance*, which is defined for

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a connected graph G as

$$DD(G) = \frac{1}{2} \sum_{i,j \in V(G)} (d_G(i) + d_G(j)) d_G(i,j).$$

The *reciprocal degree distance*(RDD) of a connected graph G is defined in [**1**] as

$$RDD(G) = \frac{1}{2} \sum_{i,j \in V(G)} \frac{(d_G(i) + d_G(j))}{d_G(i,j)}.$$

Hua and Zhang [**8**] have obtained lower and upper bounds for the reciprocal degree distance of graph in terms of other graph invariants. The mathematical behavior of reciprocal degree distance of some composite graphs are obtained in [**9, 10**]. In this paper, we have determined expressions for a RDD index of some derived graphs in terms of the parameters of the parent graph. Especially, we establish expressions for the RDD index of subdivision graph, vertex-semitotal graph, edge-semitotal graph and total graph.

The Zagreb indices have been introduced more than thirty years ago by Gutman and Trinajestic [**6**]. They are defined as

$$M_1(G) = \sum_{i \in V(G)} d_G(i)^2 \text{ and } M_2(G) = \sum_{ij \in E(G)} d_G(i) d_G(j).$$

As the sums involved run over the edges of the complement of G, such quantities were called Zagreb coindices. More formally, the first and second Zagreb coindices of a graph G are defined as

$$\overline{M}_1(G) = \sum_{ij \notin E(G)} [d_G(i) + d_G(j)] \text{ and } \overline{M}_2(G) = \sum_{ij \notin E(G)} d_G(i) d_G(j).$$

2. Preliminaries

The degree of vertex i in G is denoted by $d_G(i)$ and is defined as the number of edges incident with vertex i. Also, the degree of an edge $e = ij$ in G is denoted by $d_G(e)$ and is defined as the number of edges incident to both its end vertices, i and j, that is, $d_G(e) = d_G(i) + d_G(j) - 2$.

Let H and K be two subgraphs of G, such that $V(H) \cap V(K) = \phi$. Let $i \in V(H)$ and $j \in V(K)$ be the vertices such that

$$d_G(i,j) = min\left\{d_G(u,v) : u \in V(H), v \in V(K)\right\},$$

then i and j are called the terminal vertices of subgraphs H and K in G. In graph G, the distance between vertex c of H and the vertex f of K is 1, which is the minimum of all vertices of H and K. So, c and f are known as terminal vertices of H and K in G.

Harold Wiener introduced the Wiener index [**11**], which correlates to the boiling point and structure of the molecule of paraffins. The Wiener index is the oldest topological index and is defined as,

$$W(G) = \sum_{i,j \subseteq V(G)} d_G(i,j),$$

where $d_G(i,j)$ is the distance in G.

The edge version of the Wiener index was introduced in 2010 [**3**], and is defined as,

$$W_e(G) = \sum_{\{e,f\} \subseteq E(G)} [d_G(e,f) + 1],$$

where $d_G(e,f)$ denotes the distance between the edges $e = xy$ and $f = ij$ in G and defined as;

$$d_G(e,f) = min\left\{(d_G(x,i), d_G(x,j), d_G(y,i), d_G(y,j)\right\}.$$

The edge version of the Harary index is defined as,

$$H_e(G) = \sum_{\{e,f\} \subseteq E(G)} \frac{1}{d_G(e,f) + 1},$$

where $d_G(e,f)$ denotes the distance between the edges $e = xy$ and $f = ij$ in G.

The degree distance index of a connected graph G is defined as

$$DD(G) = \sum_{i,j \subseteq V(G)} (d_G(i) + d_G(j)) d_G(i,j).$$

The edge version of degree distance index of a connected graph G is defined as

$$DD_e(G) = \sum_{\{e,f \subseteq E(G)\}} [d_G(e) + d_G(f)][d_G(e,f) + 1],$$

where distance between the edge $e = xy$ and a vertex i is defined as,

$$d_G(e,i) = min\left\{d_G(i,x), d_G(i,y)\right\}.$$

The edge version of reciprocal degree distance index is defined as

$$RDD_e(G) = \sum_{\{e,f \subseteq E(G)\}} \frac{[d_G(e) + d_G(f)]}{d_G(e,f) + 1}.$$

3. Main Results

First we define four graphs related to a connected graph G.

(i) $S(G)$ (subdivision graph) is the graph obtained by inserting an additional vertex in each edge of G. Equivalently, each edge of G is replaced by a path of length 2. Therefore,

$$|V(S)| = |V(G)| + |E(G)| = n + m \text{ and } |E(S)| = 2\,|E(G)| = 2m.$$

(ii) $R(G)$ is obtained from G by adding a new vertex corresponding to each edge of G, then joining each new vertex to the end vertices of the corresponding edge . Thus

$$|V(R(G))| = |V(G)| + |E(G)| = n + m$$

and

$$|E(R(G))| = |E(S)| + |E(G)| = 2m + m = 3m.$$

(iii) $Q(G)$ is obtained from G by inserting a new vertex in to each edge of G, then joining with edges those pairs of new vertices on adjacent edges of G. We have

$$|V(Q(G))| = |E(G)| + |V(G)| = n+m \text{ and } |E(Q(G))| = |E(S)| + |E(L)| = m + \frac{M_1}{2}.$$

(iv) $T(G)$(total graph) has as its vertices the edges and vertices of G. Adjacency in $T(G)$ is defined as adjacency or incidence for the corresponding elements of G. We have

$$|V(T(G))| = |V(G)| + |E(G)| = n + m$$

and

$$|E(T(G))| = |E(G)| + |E(S)| + |E(L)| = 2m + \frac{M_1}{2}.$$

In the above graphs, we consider the following edge partitions: $E_1 = \{v_1v_2 : v_1, v_2 \in V(G)\}$, $E_2 = \{e_1e_2 : e_1, e_2 \in E(G)\}$ and $E_3 = \{v_1e_1 : v_1 \in V(G), e_1 \in E(G)\}$.

THEOREM 3.1. *Let $S(G)$ be a subdivision graph of G. Then*

$$RDD(S(G)) = \frac{1}{2}RDD(G) + 2H_e(G) + \alpha,$$

where

$$\alpha = \sum_{v_1 \in V(G), e_1 \in E(G)} \frac{(d_G(v_1) + 2)}{(2d_G(v_1, e_1) + 1)}.$$

PROOF. For any vertex v_i of $S(G)$, we have

$$d_{S(G)}(v_i) = \begin{cases} d_G(v_i), v_i \in V(G) \\ 2, \; \textit{otherwise}. \end{cases}$$

Moreover,

$d_{S(G)}(v_1, v_2) = 2d_G(v_1, v_2);\; v_1, v_2 \in V(G)$

$d_{S(G)}(e_1, e_2) = 2[d_G(e_1, e_2) + 1];\; e_1, e_2 \in E(G)$

$d_{S(G)}(v_1, e_1) = 2d_G(v_1, e_1) + 1;\; v_1 \in V(G), e_1 \in E(G)$.

By taking the definition of reciprocal degree distance, we have

$$\begin{aligned} RDD(S(G)) &= \sum_{\{u,v\} \subseteq V(S(G))} \frac{d_S(u) + d_S(v)}{d_S(u,v)} \\ &= \sum_{\{v_1,v_2\} \subseteq V(S(G))} \frac{d_S(v_1) + d_S(v_2)}{d_S(v_1, v_2)} \\ &+ \sum_{\{e_1,e_2\} \subseteq V(S(G))} \frac{d_S(e_1) + d_S(e_2)}{d_S(e_1, e_2)} \\ &+ \sum_{\{v_1,e_1\} \subseteq V(S(G))} \frac{d_S(v_1) + d_S(e_1)}{d_S(v_1, e_1)} \end{aligned}$$

and

$$\begin{aligned}
RDD(S(G)) &= \sum_{\{v_1,v_2\}\subseteq V(G)} \frac{(d_G(v_1)+d_G(v_2))}{2d_G(v_1,v_2)} \\
&+ \sum_{\{e_1,e_2\}\subseteq E(G)} \frac{(2+2)}{2(d_G(e_1,e_2)+1)} \\
&\quad + \sum_{v_1\in V(G),e_1\in E(G)} \frac{(d_G(v_1)+2)}{(2d_G(v_1,e_1)+1)} \\
&= \frac{1}{2}RDD(G)+2H_e(G)+\sum_{v_1\in V(G),e_1\in E(G)} \frac{(d_G(v_1)+2)}{(2d_G(v_1,e_1)+1)}.
\end{aligned}$$

□

THEOREM 3.2. *If $R(G)$ is the edge-semitotal graph G, then*

$$RDD(R(G)) \leqslant \frac{1}{4}[RDD(G)+M_1(G)+\bar{M}_1(G)]+RDD_e(G)+\beta,$$

where

$$\beta = \sum_{v_1\in V(G),e_1\in E(G)} \frac{(d_G(v_1)+d_G(v_2)+2)}{(d_G(v_1,e_1)+1)}.$$

PROOF. For any vertex v_i of $R(G)$, we get

$$d_{R(G)}(v_i) = \begin{cases} d_G(v_i), v_i \in V(G) \\ d_G(v_i)+2, otherwise. \end{cases}$$

Also,
$d_{R(G)}(v_1,v_2) = d_G(v_1,v_2)+1;\ v_1,v_2 \in V(G)$,
$d_{R(G)}(e_1,e_2) = [d_G(e_1,e_2)+1];\ e_1,e_2 \in E(G)$
$d_{R(G)}(v_1,e_1) = d_G(v_1,e_1)+1;\ v_1 \in V(G), e_1 \in E(G)$.
By using the definition of RDD, we obtain

$$\begin{aligned}
RDD(R(G)) &= \sum_{\{u,v\}\subseteq V(R(G))} \frac{d_{R(G)}(u)+d_{R(G)}(v)}{d_{R(G)}(u,v)} \\
&= \sum_{\{v_1,v_2\}\subseteq V(R(G))} \frac{d_{R(G)}(v_1)+d_{R(G)}(v_2)}{d_{R(G)}(v_1,v_2)} \\
&+ \sum_{\{e_1,e_2\}\subseteq V(R(G))} \frac{d_{R(G)}(e_1)+d_{R(G)}(e_2)}{d_{R(G)}(e_1,e_2)} \\
&+ \sum_{\{v_1,e_1\}\subseteq V(d_{R(G)})} \frac{d_{R(G)}(v_1)+d_{R(G)}(e_1)}{d_{R(G)}(v_1,e_1)}
\end{aligned}$$

and

$$\begin{aligned}
RDD(R(G)) &= \sum_{\{v_1,v_2\}\subseteq V(G)} \frac{[d_G(v_1)+d_G(v_2)]}{d_G(v_1,v_2)+1} \\
&+ \sum_{\{e_1,e_2\}\subseteq E(G)} \frac{[d_G(v_1)+d_G(v_2)+4]}{[d_G(e_1,e_2)+1]} \\
&+ \sum_{v_1\in V(G),e_1\in E(G)} \frac{[d_G(v_1)+d_G(v_2)+2]}{[d_G(v_1,e_1)+1]} \\
&= \sum_{\{v_1,v_2\}\subseteq V(G)} \frac{[d_G(v_1)+d_G(v_2)]}{d_G(v_1,v_2)+1} \\
&+ \sum_{\{e_1,e_2\}\subseteq E(G)} \frac{[d_G(e_1)+d_G(e_2)]}{[d_G(e_1,e_2)+1]} \\
&\quad + \sum_{v_1\in V(G),e_1\in E(G)} \frac{[d_G(v_1)+d_G(v_2)+2]}{[d_G(v_1,e_1)+1]}
\end{aligned}$$

From Jensen's inequality, we have,

$$\frac{1}{d_G(u_1,v_1)+d_G(u_2,v_2)} \leqslant \frac{1}{4d_G(u_1,v_1)} + \frac{1}{4d_G(u_2,v_2)},$$

with equality if and only if $d_G(u_1,v_1)+1=1$. Therefore,

$$\begin{aligned}
&RDD(R(G)) \leqslant \frac{1}{4}\Bigg[\sum_{\{v_1,v_2\}\subseteq V(G)} \frac{[d_G(v_1)+d_G(v_2)]}{d_G(v_1,v_2)} \\
+ &\sum_{\{v_1,v_2\subseteq E(G)\}} (d_G(v_1)+d_G(v_2))] \\
&+RDD_e(G) + \sum_{v_1\in V(G),e_1\in E(G)} \frac{[d_G(v_1)+d_G(v_2)+2]}{[d_G(v_1,e_1)+1]} \\
= &\frac{1}{4}\Bigg[\sum_{\{v_1,v_2\}\subseteq V(G)} \frac{[d_G(v_1)+d_G(v_2)]}{d_G(v_1,v_2)} + \sum_{v_1v_2\in E(G)} (d_G(v_1)+d_G(v_2)) \\
+ &\sum_{v_1v_2\notin E(G)} (d_G(v_1)+d_G(v_2))\Bigg] + RDD_e(G) \\
+ &\sum_{v_1\in V(G),e_1\in E(G)} \frac{[d_G(v_1)+d_G(v_2)+2]}{[d_G(v_1,e_1)+1]} \\
= &\frac{1}{4}[RDD(G)+M_1(G)+\bar{M}_1(G)] + RDD_e(G) \\
+ &\sum_{v_1\in V(G),e_1\in E(G)} \frac{(d_G(v_1)+d_G(v_2)+2)}{(d_G(v_1,e_1)+1)}.
\end{aligned}$$

$\square$

THEOREM 3.3. *If $Q(G)$ is the vertex-semi total graph G, then*

$$RDD(Q(G)) \leqslant 2RDD(G) + H_e(G) + \frac{m(m-1)}{2} + 2\gamma,$$

where

$$\gamma = \sum_{v_1 \in V(G), e_1 \in E(G)} \frac{(d_G(v_1)+1)}{(d_G(v_1,e_1)+1)}.$$

PROOF. Note that for any vertex v_i of $Q(G)$, we have

$$d_{Q(G)}(v_i) = \begin{cases} 2d_G(v_i), v_i \in V(G) \\ 2, \ otherwise. \end{cases}$$

Also,
$d_{Q(G)}(v_1, v_2) = d_G(v_1, v_2);\ v_1, v_2 \in V(G)$
$d_{Q(G)}(e_1, e_2) = [d_G(e_1, e_2) + 2];\ e_1, e_2 \in E(G)$
$d_{Q(G)}(v_1, e_1) = d_G(v_1, e_1) + 1;\ v_1 \in V(G), e_1 \in E(G)$
Hence by the definition of RDD, we have

$$\begin{aligned}
RDD(Q(G)) &= \sum_{\{u,v\} \subseteq V(Q(G))} \frac{d_{Q(G)}(u) + d_{Q(G)}(v)}{d_{Q(G)}(u,v)} \\
&= \sum_{\{v_1,v_2\} \subseteq V(Q(G))} \frac{d_{Q(G)}(v_1) + d_{Q(G)}(v_2)}{d_{Q(G)}(v_1,v_2)} \\
&+ \sum_{\{e_1,e_2\} \subseteq V(Q(G))} \frac{d_{Q(G)}(e_1) + d_{Q(G)}(e_2)}{d_{Q(G)}(e_1,e_2)} \\
&+ \sum_{\{v_1,e_1\} \subseteq V(Q(G))} \frac{d_{Q(G)}(v_1) + d_{Q(G)}(e_1)}{d_{Q(G)}(v_1,e_1)} \\
&= \sum_{\{v_1,v_2\} \subseteq V(G)} \frac{[2d_G(v_1) + 2d_G(v_2)]}{d_G(v_1,v_2)} + \sum_{\{e_1,e_2\} \subseteq E(G)} \frac{(2+2)}{[d_G(e_1,e_2)+2]} \\
&+ \sum_{v_1 \in V(G), e_1 \in E(G)} \frac{[2d_G(v_1+2]}{[d_G(v_1,e_1)+1]} \\
&= 2 \sum_{\{v_1,v_2\} \subseteq V(G)} \frac{[d_G(v_1) + d_G(v_2)]}{d_G(v_1,v_2)} + 4 \sum_{\{e_1,e_2\} \subseteq E(G)} \frac{1}{[d_G(e_1,e_2)+2]} \\
&+ \sum_{v_1 \in V(G), e_1 \in E(G)} \frac{[2d_G(v_1)+2]}{[d_G(v_1,e_1)+1]}.
\end{aligned}$$

From Jensen's inequality, we have,

$$\frac{1}{d_G(u,v) + d_G(u_2,v_2)} \leqslant \frac{1}{4d_G(u_1,v_1)} + \frac{1}{4d_G(u_2,v_2)},$$

with equality if and only if $d_G(u_1, v_1) + 1 = 1$, also we have

$$\sum_{\{e_1,e_2\}\subseteq E(G)} (1) = \frac{m(m-1)}{2}.$$

Therefore,

$$\begin{aligned}
RDD(Q(G)) &\leqslant 2RDD(G) + \sum_{\{e_1,e_2\}\subseteq E(G)} \frac{1}{[d_G(e_1,e_2)+1]} + \sum_{\{e_1,e_2\}\subseteq E(G)} (1) \\
&+ 2\sum_{v_1\in V(G),e_1\in E(G)} \frac{[d_G(v_1)+1]}{[d_G(v_1,e_1)+1]} \\
&= 2RDD(G) + H_e(G) + \frac{m(m-1)}{2} + 2\sum_{v_1\in V(G),e_1\in E(G)} \frac{[d_G(v_1)+1]}{[d_G(v_1,e_1)+1]}.
\end{aligned}$$

□

THEOREM 3.4. *If $T(G)$ is the total graph G, then*

$$RDD(T(G)) = 2RDD(G) + RDD_e(G) + \sum_{v_1\in V(G),e_1\in E(G)} \frac{[2d_G(v_1) + d_G(v_2) + 2]}{[d_G(v_1,e_1)+1]}.$$

PROOF. For any vertex v_i of $T(G)$, we get

$$d_{T(G)}(v_i) = \begin{cases} 2d_G(v_i), v_i \in V(G) \\ d_G(v_i) + 2, \ \textit{otherwise}. \end{cases}$$

Also,
$d_{T(G)}(v_1, v_2) = d_G(v_1, v_2);\ v_1, v_2 \in V(G)$
$d_{T(G)}(e_1, e_2) = [d_G(e_1, e_2) + 1];\ e_1, e_2 \in E(G)$
$d_{T(G)}(v_1, e_1) = d_G(v_1, e_1) + 1;\ v_1 \in V(G), e_1 \in E(G)$.
We have reciprocal degree distance,

$$\begin{aligned}
RDD(T(G)) &= \sum_{\{u,v\}\subseteq V(T(G))} \frac{d_{T(G)}(u) + d_{T(G)}(v)}{d_{T(G)}(u,v)} \\
&= \sum_{\{v_1,v_2\}\subseteq V(T(G))} \frac{d_{T(G)}(v_1) + d_{T(G)}(v_2)}{d_T(v_1,v_2)} \\
&+ \sum_{\{e_1,e_2\}\subseteq V(T(G))} \frac{d_{T(G)}(e_1) + d_{T(G)}(e_2)}{d_{T(G)}(e_1,e_2)} \\
&+ \sum_{\{v_1,e_1\}\subseteq V(T(G))} \frac{d_{T(G)}(v_1) + d_{T(G)}(e_1)}{d_{T(G)}(v_1,e_1)}
\end{aligned}$$

and

$$\begin{aligned}
RDD(T(G)) &= \sum_{\{v_1,v_2\}\subseteq V(G)} \frac{[2d_G(v_1)+2d_G(v_2)]}{d_G(v_1,v_2)} \\
&+ \sum_{\{e_1,e_2\}\subseteq E(G)} \frac{[d_G(e_1)+d_G(e_2)+4]}{[d_G(e_1,e_2)+1]} \\
&+ \sum_{v_1\in V(G),e_1\in E(G)} \frac{[2d_G(u_1)+d_G(v_2)+2]}{[d_G(v_1,e_1)+1]} \\
&= 2RDD(G)+RDD_e(G) \\
&+ \sum_{v_1\in V(G),e_1\in E(G)} \frac{[2d_G(v_1)+d_G(v_2)+2]}{[d_G(v_1,e_1)+1]}.
\end{aligned}$$

□

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